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弱 Berwald 双挠积 Finsler 度量

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摘 要

本文主要研究了双挠积 Finsler 度量的平均 Berwald 曲率和迷向平均 Berwald 曲率，给出了双挠积 Finsler 度量是弱 Berwald 度量的充要条件，证明了在一定条件下具有迷向平均 Berwald 曲率的双挠积 Finsler 度量是弱 Berwald 度量。

关键词

Finsler 度量，双挠积，弱 Berwald 度量，迷向平均 Berwald 曲率

Weakly Berwald Doubly-Twisted Product Finsler Metrics

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Abstract

This paper mainly studies the mean Berwald curvature and isotropic mean Berwald curvature doubly-twisted product of Finsler metrics. The necessary and sufficient conditions for the doubly-twisted product of Finsler metrics are weakly Berwald metrics. It is proved that under certain conditions the doubly-twisted product of Finsler metrics with isotropic mean Berwald curvature is weakly Berwald metrics.

Keywords

Finsler Metrics, Doubly Twisted Product, Weakly Berwald Metrics, Isotropic Mean Berwald Curvature

1. 引言

挠积是一种构造特殊度量的有效方法. 1981 年, Chen 给出了 Riemann 度量挠积的概念[1]. 2006 年, Kozma, Peter 和 Shimada 将挠积推广到了 Finsler 几何中, 研究了挠积 Finsler 度量的 Cartan 联络, 测地线及其完备性[2]. 近几年, 挠积 Finsler 度量得到了一些学者的关注和研究[3][4].

在 Finsler 几何中, 弱 Berwald 度量是一类重要的 Finsler 度量. Berwald 首先提出了 Berwald 曲率的概念[5][6]. 1986 年, Matsumoto 给出了 Berwald 度量的定义[7]. 2001 年, 沈忠民证明了具有消失 Berwald 曲率的 Finsler 度量是 Berwald 度量. Berwald 曲率的迹被称为平均 Berwald 曲率, 且具有消失平均 Berwald 曲率的 Finsler 度量为弱 Berwald 度量[8]. 2005 年, 程新跃和沈忠民给出了迷向平均 Berwald 曲率的定义[9], 它是 Berwald 曲率的推广. 2013 年, Peyghan 和 Tayebi 证明了具有迷向平均 Berwald 曲率的挠积 Finsler 度量是弱 Berwald 度量[3]. 2020 年, 杨曌、何勇和张晓玲证明了双扭曲积 Finsler 流形具有迷向平均 Berwald 曲率当且仅当它是弱 Berwald 流形[10].

本文将挠积 Finsler 度量推广为双挠积 Finsler 度量, 主要研究双挠积 Finsler 度量的平均 Berwald 曲率和迷向平均 Berwald 曲率, 尝试给出双挠积 Finsler 度量为弱 Berwald 度量的充要条件. 且受到文献[3]和[10]的启发, 本文将探索具有迷向平均 Berwald 曲率的双挠积 Finsler 度量与弱 Berwald 度量之间的关系.

2. 预备知识

设 M 为 n 维光滑流形, M 上的局部坐标为 $(x^1,\cdots,x^n)$. 设 TM 是 M 的切丛, 其诱导的局部坐标为 $(x,y)=(x^1,\cdots,x^n,y^1,\cdots,y^n)$, M 上的 Finsler 度量定义如下.

定义1. [11]光滑流形 M 上的 Finsler 度量是一连续函数 F: $TM\to[0,\infty)$, 满足

(i) 正则性: F 在 $TM^\circ=TM\setminus\{0\}$ 上是 C^∞ 函数;

(ii) 正齐次性: $F(x,\lambda y)=\lambda F(x,y)$, $\forall\lambda>0$;

(iii) 强凸性: $n\times n$ 的 Hessian 矩阵 $(g_{ij}):=(\frac{1}{2}\frac{\partial^2F^2}{\partial y^i\partial y^j})$ 在 TM° 上是正定的.

定义2. [12]设 F 是 Finsler 度量, 则由 F 诱导的喷射系数为

$$\mathbb{G}^i:=\frac{1}{4}g^{il}(\frac{\partial^2F^2}{\partial y^l\partial x^k}y^k-\frac{\partial F^2}{\partial x^l}); \tag{2.1}$$

根据 $\mathbb{G}^i$ 关于 y 的二阶齐次性和 Euler 定理有

$$y^j\frac{\partial\mathbb{G}^i}{\partial y^j}=2\mathbb{G}^i. \tag{2.2}$$

定义3. [8]设 F 是 Finsler 度量. Berwald 曲率的系数为

$$\mathbb{B}^i_{jkl}:=\frac{\partial^3\mathbb{G}^i}{\partial y^j\partial y^k\partial y^l}; \tag{2.3}$$

平均 Berwald 曲率的系数为

$$\mathbb{E}_{jk}:=\frac{1}{2}\mathbb{B}^m_{jkm}. \tag{2.4}$$

若 $\mathbb{B}^i_{jkl}=0$, 称 F 为 Berwald 度量; 若 $\mathbb{E}_{jk}=0$, 称 F 为弱 Berwald 度量.

定义4. [9]设 F 是 Finsler 度量. F 诱导的平均 Berwald 曲率满足

$$\mathbb{E}_{ij}=\frac{1}{2}(n+1)cF_{y^iy^j}, \tag{2.5}$$

称 F 具有迷向平均 Berwald 曲率, 其中 $c=c(x)$ 是 M 上的标量函数.

本文中, 设 (G^{AB}) 是矩阵 (G_{BC}) 的逆矩阵, 使得 $\mathrm{G}^{AB}\mathrm{G}_{BC}=\delta^A_C$.

设 M_1 和 M_2 分别是 n_1 维和 n_2 维光滑流形, M_1 和 M_2 上的局部坐标分别为 $(x^1,\cdots,x^{n_1})$ 和 $(u^1,\cdots,u^{n_2})$. TM_1 和 TM_2 上的局部坐标分别为 $(x^1,\cdots,x^{n_1},y^1,\cdots,y^{n_1})$ 和 $(u^1,\cdots,u^{n_2},v^1,\cdots,v^{n_2})$. $M=M_1\times M_2$ 为 M_1 和 M_2 的乘积流形, 维数为 n_1+n_2. 设 $\pi_1:M\to M_1$ 和 $\pi_2:M\to M_2$ 是自然投影, 设 $d\pi_1:TM\to TM_1$ 和 $d\pi_2:TM\to TM_2$ 分别是由 π_1 和 π_2 诱导的切映射. 令 $X=(x,u)\in M$, $Y=(y,v)\in T_XM$, 且 $T_XM=T_xM_1\oplus T_uM_1$.

定义5. 设 (M_1,F_1) 和 (M_2,F_2) 是两个 Finsler 流形, 且 f_1 和 $f_2:M_1\times M_2\to\mathbf{R}^+$ 是两个光滑实值函数. F_1 和 F_2 的双挠积 Finsler度量是在乘积流形 $M=M_1\times M_2$ 上按如下方式定义的 Finsler 度

量 $F:TM\to\mathbf{R}^+$

$$\begin{aligned}F^2(X,Y)=&f_2^2(\pi_1(X),\pi_2(X))F_1^2(\pi_1(X),d\pi_1(Y))\\&+f_1^2(\pi_1(X),\pi_2(X))F_2^2(\pi_2(X),d\pi_2(Y)),\end{aligned}\tag{2.6}$$

其中 f_1 和 f_2 被称为挠函数. 很明显 F 是 M 上的一个 Finsler 度量.

如果 $f_1\equiv 1$ 与 $f_2\equiv 1$ 有且仅有一个成立, 则 F 为挠积 Finsler 度量. 如果 $f_1\equiv 1$ 且 $f_2\equiv 1$, 则 F 为乘积 Finsler 度量. 如果 f_1 和 f_2 都不恒等于常数, 则称 F 为非平凡的双挠积 Finsler 度量.

记

$$(i)\quad g_{ij}:=\frac{1}{2}\frac{\partial^2F_1^2}{\partial y^i\partial y^j},\qquad (ii)\quad g_{\alpha\beta}:=\frac{1}{2}\frac{\partial^2F_2^2}{\partial v^\alpha\partial v^\beta}.\tag{2.7}$$

则 F 的基本张量矩阵为

$$(\mathrm{G}_{AB})=(\frac{1}{2}\frac{\partial^2F^2}{\partial Y^A\partial Y^B})=\begin{pmatrix}f_2^2g_{ij} & 0\\ 0 & f_1^2g_{\alpha\beta}\end{pmatrix},\tag{2.8}$$

其逆矩阵为

$$(\mathrm{G}^{BA})=\begin{pmatrix}{f_2}^{-2}g^{ji} & 0\\ 0 & {f_1}^{-2}g^{\beta\alpha}\end{pmatrix}.\tag{2.9}$$

本文约定, 小写拉丁字母指标, 如 i,j 等, 变化范围从 1 到 n_1; 小写希腊字母指标, 如 α,β 等, 变化范围从 1 到 n_2; 大写拉丁字母指标, 如 A,B 等, 变化范围从 1 到 n_1+n_2. 与光滑流行 (M_1,F_1) 和 (M_2,F_2) 有关的几何量, 分别在其正上方加指标 1 和 2, 如 $\overset{1}{\mathbb{G}}{}^i$ 和 $\overset{2}{\mathbb{G}}{}^\alpha$ 分别表示由 F_1 和 F_2 诱导的喷射系数.

3. 双挠积 Finsler 度量的平均 Berwald 曲率

本节主要推导由双挠积 Finsler 度量诱导的平均 Berwald 曲率的系数.

命题3.1. *设 F 是 Finsler 度量 F_1 和 F_2 的双挠积. 那么, F 诱导的 Berwald 曲率的系数 $\mathbb{B}^A_{BCD}$ 为:*

$$\begin{aligned}\mathbb{B}^k_{ijl}=&\overset{1}{\mathbb{B}}{}^k_{ijl}-\frac{1}{4f_2^2}(\frac{\partial^2g^{kh}}{\partial y^j\partial y^l}\frac{\partial F_1^2}{\partial y^i}+\frac{\partial^2g^{kh}}{\partial y^i\partial y^l}\frac{\partial F_1^2}{\partial y^j}+\frac{\partial^2g^{kh}}{\partial y^i\partial y^j}\frac{\partial F_1^2}{\partial y^l})\frac{\partial f_2^2}{\partial x^h}\\&-\frac{1}{2f_2^2}(\frac{\partial g^{kh}}{\partial y^i}g_{jl}+\frac{\partial g^{kh}}{\partial y^j}g_{il}+\frac{\partial g^{kh}}{\partial y^l}g_{ij}+g^{kh}\frac{\partial g_{ij}}{\partial y^l})\frac{\partial f_2^2}{\partial x^h}\\&-\frac{1}{4f_2^2}\frac{\partial^3g^{kh}}{\partial y^i\partial y^j\partial y^l}(\frac{\partial f_2^2}{\partial x^h}F_1^2+\frac{\partial f_1^2}{\partial x^h}F_2^2),\end{aligned}\tag{3.1}$$

$$\mathbb{B}^k_{i\beta l}=\mathbb{B}^k_{il\beta}=\mathbb{B}^k_{\beta il}=-\frac{1}{4f_2^2}\frac{\partial^2g^{kh}}{\partial y^i\partial y^l}\frac{\partial f_1^2}{\partial x^h}\frac{\partial F_2^2}{\partial v^\beta},\tag{3.2}$$

$$\mathbb{B}^k_{\alpha\beta l}=\mathbb{B}^k_{\alpha l\beta}=\mathbb{B}^k_{l\alpha\beta}=-\frac{1}{2f_2^2}\frac{\partial g^{kh}}{\partial y^l}\frac{\partial f_1^2}{\partial x^h}g_{\alpha\beta}, \tag{3.3}$$

$$\mathbb{B}^k_{\alpha\beta\lambda}=-\frac{1}{f_2^2}g^{kh}\frac{\partial f_1^2}{\partial x^h}\mathbb{C}_{\alpha\beta\lambda}, \tag{3.4}$$

$$\mathbb{B}^\gamma_{ijl}=-\frac{1}{f_1^2}g^{\gamma\mu}\frac{\partial f_2^2}{\partial u^\mu}\mathbb{C}_{ijl}, \tag{3.5}$$

$$\mathbb{B}^\gamma_{i\beta l}=\mathbb{B}^\gamma_{\beta il}=\mathbb{B}^\gamma_{il\beta}=-\frac{1}{2f_1^2}\frac{\partial g^{\gamma\mu}}{\partial v^\beta}\frac{\partial f_2^2}{\partial u^\mu}g_{il}, \tag{3.6}$$

$$\mathbb{B}^\gamma_{\alpha\beta l}=\mathbb{B}^\gamma_{\alpha l\beta}=\mathbb{B}^\gamma_{l\alpha\beta}=-\frac{1}{4f_1^2}\frac{\partial^2 g^{\gamma\mu}}{\partial v^\alpha\partial v^\beta}\frac{\partial f_2^2}{\partial u^\mu}\frac{\partial F_1^2}{\partial y^l}, \tag{3.7}$$

$$\begin{aligned}\mathbb{B}^\gamma_{\alpha\beta\lambda}=&\overset{2}{\mathbb{B}}{}^\gamma_{\alpha\beta\lambda}-\frac{1}{4f_1^2}(\frac{\partial^2 g^{\gamma\mu}}{\partial v^\beta\partial v^\lambda}\frac{\partial F_2^2}{\partial v^\alpha}+\frac{\partial^2 g^{\gamma\mu}}{\partial v^\alpha\partial v^\lambda}\frac{\partial F_2^2}{\partial v^\beta}+\frac{\partial^2 g^{\gamma\mu}}{\partial v^\alpha\partial v^\beta}\frac{\partial F_2^2}{\partial v^\lambda})\frac{\partial f_1^2}{\partial u^\mu}\\&-\frac{1}{2f_1^2}(\frac{\partial g^{\gamma\mu}}{\partial v^\alpha}g_{\beta\lambda}+\frac{\partial g^{\gamma\mu}}{\partial v^\beta}g_{\alpha\lambda}+\frac{\partial g^{\gamma\mu}}{\partial v^\lambda}g_{\alpha\beta}+g^{\gamma\mu}\frac{\partial g_{\alpha\beta}}{\partial v^\lambda})\frac{\partial f_1^2}{\partial u^\mu}\\&-\frac{1}{4f_1^2}\frac{\partial^3 g^{\gamma\mu}}{\partial v^\alpha\partial v^\beta\partial v^\lambda}(\frac{\partial f_2^2}{\partial u^\mu}F_1^2+\frac{\partial f_1^2}{\partial u^\mu}F_2^2).\end{aligned} \tag{3.8}$$

证明. 根据 (2.1) 得

$$\mathbb{G}^A=\frac{1}{4}G^{AB}(\frac{\partial^2 F^2}{\partial Y^B\partial X^C}Y^C-\frac{\partial F^2}{\partial X^B}), \tag{3.9}$$

令 (3.9) 中的 $A=i$, 则

$$\mathbb{G}^i=\frac{1}{4}G^{ih}(\frac{\partial^2 F^2}{\partial y^h\partial x^j}y^j+\frac{\partial^2 F^2}{\partial y^h\partial u^\alpha}v^\alpha-\frac{\partial F^2}{\partial x^h})+\frac{1}{4}G^{i\mu}(\frac{\partial^2 F^2}{\partial v^\mu\partial x^j}y^j+\frac{\partial^2 F^2}{\partial v^\mu\partial u^\alpha}v^\alpha-\frac{\partial F^2}{\partial u^\mu}), \tag{3.10}$$

由 (2.6) 直接计算有

$$\frac{\partial F^2}{\partial y^h}=f_2^2\frac{\partial F_1^2}{\partial y^h},\quad \frac{\partial F^2}{\partial x^h}=\frac{\partial f_2^2}{\partial x^h}F_1^2+f_2^2\frac{\partial F_1^2}{\partial x^h}+\frac{\partial f_1^2}{\partial x^h}F_2^2, \tag{3.11}$$

$$\frac{\partial^2 F^2}{\partial y^h\partial u^\alpha}=\frac{\partial f_2^2}{\partial u^\alpha}\frac{\partial F_1^2}{\partial y^h},\quad \frac{\partial^2 F^2}{\partial y^h\partial x^j}=\frac{\partial f_2^2}{\partial x^j}\frac{\partial F_1^2}{\partial y^h}+f_2^2\frac{\partial^2 F_1^2}{\partial y^h\partial x^j}, \tag{3.12}$$

把 (2.9), (3.11) 和 (3.12) 代入 (3.10), 可得

$$\mathbb{G}^i=\overset{1}{\mathbb{G}}{}^i+\frac{1}{4f_2^2}g^{ih}[(\frac{\partial f_2^2}{\partial x^j}y^j+\frac{\partial f_2^2}{\partial u^\alpha}v^\alpha)\frac{\partial F_1^2}{\partial y^h}-\frac{\partial f_2^2}{\partial x^h}F_1^2-\frac{\partial f_1^2}{\partial x^h}F_2^2], \tag{3.13}$$

(3.13) 两边同时关于 y^j 微分可得

$$\begin{aligned}\frac{\partial\mathbb{G}^i}{\partial y^j}=&\frac{\partial\overset{1}{\mathbb{G}}{}^i}{\partial y^j}+\frac{1}{4f_2^2}\frac{\partial g^{ih}}{\partial y^j}[(\frac{\partial f_2^2}{\partial x^k}y^k+\frac{\partial f_2^2}{\partial u^\alpha}v^\alpha)\frac{\partial F_1^2}{\partial y^h}-\frac{\partial f_2^2}{\partial x^h}F_1^2-\frac{\partial f_1^2}{\partial x^h}F_2^2]\\&+\frac{1}{4f_2^2}g^{ih}[\frac{\partial f_2^2}{\partial x^j}\frac{\partial F_1^2}{\partial y^h}+(\frac{\partial f_2^2}{\partial x^k}y^k+\frac{\partial f_2^2}{\partial u^\alpha}v^\alpha)\frac{\partial^2 F_1^2}{\partial y^h\partial y^j}-\frac{\partial f_2^2}{\partial x^h}\frac{\partial F_1^2}{\partial y^j}],\end{aligned} \tag{3.14}$$

由 (2.7) 的 (i) 得 $\frac{\partial F_1^2}{\partial y^h}=2g_{hk}y^k$. 因此,

$$\frac{\partial g^{ih}}{\partial y^j}\frac{\partial F_1^2}{\partial y^h}=2\frac{\partial g^{ih}}{\partial y^j}g_{hk}y^k=-2g^{ih}\frac{\partial g_{hk}}{\partial y^j}y^k=0, \tag{3.15}$$

并注意到 $g^{ih}g_{hj}=\delta^i_j$, 所以

$$\frac{1}{8f_2^2}g^{ih}(\frac{\partial f_2^2}{\partial x^k}y^k+\frac{\partial f_2^2}{\partial u^\alpha}v^\alpha)\frac{\partial^2 F_1^2}{\partial y^h\partial y^j}=\frac{1}{4f_2^2}(\frac{\partial f_2^2}{\partial x^k}y^k+\frac{\partial f_2^2}{\partial u^\alpha}v^\alpha)\delta^i_j, \tag{3.16}$$

把 (3.15) 和 (3.16) 代入 (3.14), 可得

$$\begin{aligned}\frac{\partial \mathbb{G}^i}{\partial y^j}=&\frac{\partial \overset{1}{\mathbb{G}^i}}{\partial y^j}+\frac{1}{4f_2^2}g^{ih}(\frac{\partial f_2^2}{\partial x^j}\frac{\partial F_1^2}{\partial y^h}-\frac{\partial f_2^2}{\partial x^h}\frac{\partial F_1^2}{\partial y^j})+\frac{1}{2f_2^2}(\frac{\partial f_2^2}{\partial x^k}y^k+\frac{\partial f_2^2}{\partial u^\alpha}v^\alpha)\delta^i_j\\&-\frac{1}{4f_2^2}\frac{\partial g^{ih}}{\partial y^j}(\frac{\partial f_2^2}{\partial x^h}F_1^2+\frac{\partial f_1^2}{\partial x^h}F_2^2),\end{aligned} \tag{3.17}$$

(3.17) 两边同时关于 y^l 微分可得

$$\begin{aligned}\frac{\partial^2 \mathbb{G}^i}{\partial y^j\partial y^l}=&\frac{\partial^2 \overset{1}{\mathbb{G}^i}}{\partial y^j\partial y^l}-\frac{1}{4f_2^2}(\frac{\partial g^{ih}}{\partial y^j}\frac{\partial F_1^2}{\partial y^l}+\frac{\partial g^{ih}}{\partial y^l}\frac{\partial F_1^2}{\partial y^j}+2g^{ih}g_{lj})\frac{\partial f_2^2}{\partial x^h}\\&+\frac{1}{2f_2^2}(\frac{\partial f_2^2}{\partial x^l}\delta^i_j+\frac{\partial f_2^2}{\partial x^j}\delta^i_l)-\frac{1}{4f_2^2}\frac{\partial^2 g^{ih}}{\partial y^l\partial y^j}(\frac{\partial f_2^2}{\partial x^h}F_1^2+\frac{\partial f_1^2}{\partial x^h}F_2^2),\end{aligned} \tag{3.18}$$

(3.18) 两边同时关于 y^k 微分, 即可证得(3.1), 同理可得 (3.2)－(3.4) 成立.

类似地, 若令 (3.9) 中的 $A=\alpha$, 同理可得 (3.5)－(3.8) 成立. □

命题3.2. *设 F 是 Finsler 度量 F_1 和 F_2 的双挠积. 那么, F 诱导的平均 Berwald 曲率的系数 $\mathbb{E}_{AB}$ 为:*

$$\begin{aligned}\mathbb{E}_{ij}=&\overset{1}{\mathbb{E}}_{ij}-\frac{1}{8f_2^2}(\frac{\partial g^{kh}}{\partial y^k}\frac{\partial^2 F_1^2}{\partial y^i\partial y^j}+\frac{\partial^2 g^{kh}}{\partial y^j\partial y^k}\frac{\partial F_1^2}{\partial y^i}+\frac{\partial^2 g^{kh}}{\partial y^i\partial y^k}\frac{\partial F_1^2}{\partial y^j}+\frac{\partial^3 g^{kh}}{\partial y^i\partial y^j\partial y^k}F_1^2)\frac{\partial f_2^2}{\partial x^h}\\&-\frac{1}{8f_2^2}\frac{\partial^3 g^{kh}}{\partial y^i\partial y^j\partial y^k}\frac{\partial f_1^2}{\partial x^h}F_2^2-\frac{1}{8f_1^2}\frac{\partial g^{\gamma\mu}}{\partial v^\gamma}\frac{\partial f_2^2}{\partial u^\mu}\frac{\partial^2 F_1^2}{\partial y^i\partial y^j},\end{aligned} \tag{3.19}$$

$$\mathbb{E}_{i\beta}=\mathbb{E}_{\beta i}=-\frac{1}{8f_2^2}\frac{\partial^2 g^{kh}}{\partial y^i\partial y^k}\frac{\partial f_1^2}{\partial x^h}\frac{\partial F_2^2}{\partial v^\beta}-\frac{1}{8f_1^2}\frac{\partial^2 g^{\gamma\mu}}{\partial v^\beta\partial v^\gamma}\frac{\partial f_2^2}{\partial u^\mu}\frac{\partial F_1^2}{\partial y^i}, \tag{3.20}$$

$$\begin{aligned}\mathbb{E}_{\alpha\beta}=&\overset{2}{\mathbb{E}}_{\alpha\beta}-\frac{1}{8f_1^2}(\frac{\partial g^{\gamma\mu}}{\partial v^\gamma}\frac{\partial^2 F_2^2}{\partial v^\alpha\partial v^\beta}+\frac{\partial^2 g^{\gamma\mu}}{\partial v^\beta\partial v^\gamma}\frac{\partial F_2^2}{\partial v^\alpha}+\frac{\partial^2 g^{\gamma\mu}}{\partial v^\alpha\partial v^\gamma}\frac{\partial F_2^2}{\partial v^\beta}+\frac{\partial^3 g^{\gamma\mu}}{\partial v^\alpha\partial v^\beta\partial v^\gamma}F_2^2)\frac{\partial f_1^2}{\partial u^\mu}\\&-\frac{1}{8f_1^2}\frac{\partial^3 g^{\gamma\mu}}{\partial v^\alpha\partial v^\beta\partial v^\gamma}\frac{\partial f_2^2}{\partial u^\mu}F_1^2-\frac{1}{8f_2^2}\frac{\partial g^{kh}}{\partial y^k}\frac{\partial f_1^2}{\partial x^h}\frac{\partial^2 F_2^2}{\partial v^\alpha\partial v^\beta}.\end{aligned} \tag{3.21}$$

证明. 根据 (2.4) 知 $\mathbb{E}_{AB}=\frac{1}{2}\mathbb{B}^k_{ABk}+\frac{1}{2}\mathbb{B}^\gamma_{AB\gamma}$, 从而

$$\mathbb{E}_{ij}=\frac{1}{2}\mathbb{B}^k_{ijk}+\frac{1}{2}\mathbb{B}^\gamma_{ij\gamma}, \tag{3.22}$$

把 (3.1) 和 (3.6) 代入 (3.22) 式, 并注意到 $\overset{1}{\mathbb{E}}_{ij} = \frac{1}{2}\overset{1}{\mathbb{B}}^{k}_{ijk}$, 则

$$\begin{aligned}\mathbb{E}_{ij} =&\overset{1}{\mathbb{E}}_{ij} - \frac{1}{8f_2^2}(\frac{\partial g^{kh}}{\partial y^j}\frac{\partial^2 F_1^2}{\partial y^i\partial y^k} + \frac{\partial g^{kh}}{\partial y^k}\frac{\partial^2 F_1^2}{\partial y^i\partial y^j} + 2\frac{\partial g^{kh}}{\partial y^i}g_{jk} + 2g^{kh}\frac{\partial g_{ij}}{\partial y^k}\\ &+ \frac{\partial^2 g^{kh}}{\partial y^j\partial y^k}\frac{\partial F_1^2}{\partial y^i} + \frac{\partial^2 g^{kh}}{\partial y^i\partial y^k}\frac{\partial F_1^2}{\partial y^j} + \frac{\partial^2 g^{kh}}{\partial y^i\partial y^j}\frac{\partial F_1^2}{\partial y^k} + \frac{\partial^3 g^{kh}}{\partial y^i\partial y^j\partial y^k}F_1^2)\frac{\partial f_2^2}{\partial x^h}\\ &- \frac{1}{8f_2^2}\frac{\partial^3 g^{kh}}{\partial y^i\partial y^j\partial y^k}\frac{\partial f_1^2}{\partial x^h}F_2^2 - \frac{1}{8f_1^2}\frac{\partial g^{\gamma\mu}}{\partial v^\gamma}\frac{\partial f_2^2}{\partial u^\mu}\frac{\partial^2 F_1^2}{\partial y^i\partial y^j},\end{aligned} \tag{3.23}$$

根据 (3.15) 知 $\frac{\partial g^{kh}}{\partial y^j}\frac{\partial F_1^2}{\partial y^k} = 0$, 该式两边同时关于 y^i 微分可得

$$\frac{\partial g^{kh}}{\partial y^j}\frac{\partial^2 F_1^2}{\partial y^k\partial y^i} + \frac{\partial^2 g^{kh}}{\partial y^j\partial y^i}\frac{\partial F_1^2}{\partial y^k} = 0, \tag{3.24}$$

注意到

$$2\frac{\partial g^{kh}}{\partial y^i}g_{jk} + 2g^{kh}\frac{\partial g_{ij}}{\partial y^k} = 2(\frac{\partial g^{kh}}{\partial y^i}g_{jk} + g^{kh}\frac{\partial g_{jk}}{\partial y^i}) = 2\frac{\partial g^{kh}g_{jk}}{\partial y^i} = 2\frac{\partial \delta^h_j}{\partial y^i} = 0, \tag{3.25}$$

把 (3.24) 和 (3.25) 代入(3.23) 可得 (3.19) 成立.

同理, 可证明 (3.20) 和 (3.21) 成立. □

4. 弱 Berwald 双挠积 Finsler 度量

本节研究弱 Berwald 双挠积 Finsler 度量, 探索具有迷向平均 Berwald 曲率的双挠积 Finsler 度量与弱 Berwald 度量之间的关系.

定理4.1. *设 F 是 Finsler 度量 F_1 和 F_2 的双挠积. F 是弱 Berwald 度量当且仅当下列方程组成立:*

$$\begin{cases}\overset{1}{\mathbb{E}}_{ij} = \frac{1}{8f_2^2}(\frac{\partial g^{kh}}{\partial y^k}\frac{\partial^2 F_1^2}{\partial y^i\partial y^j} + \frac{\partial^2 g^{kh}}{\partial y^j\partial y^k}\frac{\partial F_1^2}{\partial y^i} + \frac{\partial^2 g^{kh}}{\partial y^i\partial y^k}\frac{\partial F_1^2}{\partial y^j} + \frac{\partial^3 g^{kh}}{\partial y^i\partial y^j\partial y^k}F_1^2)\frac{\partial f_2^2}{\partial x^h}, & (4.1)\\ \overset{2}{\mathbb{E}}_{\alpha\beta} = \frac{1}{8f_1^2}(\frac{\partial g^{\gamma\mu}}{\partial v^\gamma}\frac{\partial^2 F_2^2}{\partial v^\alpha\partial v^\beta} + \frac{\partial^2 g^{\gamma\mu}}{\partial v^\beta\partial v^\gamma}\frac{\partial F_2^2}{\partial v^\alpha} + \frac{\partial^2 g^{\gamma\mu}}{\partial v^\alpha\partial v^\gamma}\frac{\partial F_2^2}{\partial v^\beta} + \frac{\partial^3 g^{\gamma\mu}}{\partial v^\alpha\partial v^\beta\partial v^\gamma}F_2^2)\frac{\partial f_1^2}{\partial u^\mu}, & (4.2)\\ \frac{\partial g^{kh}}{\partial y^k}\frac{\partial f_1^2}{\partial x^h} = 0, & (4.3)\\ \frac{\partial g^{\gamma\mu}}{\partial v^\gamma}\frac{\partial f_2^2}{\partial u^\mu} = 0. & (4.4)\end{cases}$$

证明. 充分性. (4.3) 两边同时关于 y^i 微分得

$$\frac{\partial^2 g^{kh}}{\partial y^i\partial y^k}\frac{\partial f_1^2}{\partial x^h} = 0, \tag{4.5}$$

(4.4) 两边同时关于 v^β 微分得

$$\frac{\partial^2 g^{\gamma\mu}}{\partial v^\beta \partial v^\gamma}\frac{\partial f_2^2}{\partial u^\mu}=0, \tag{4.6}$$

把 (4.5) 和 (4.6) 代入 (3.20) 可得

$$\mathbb{E}_{i\beta}=\mathbb{E}_{\beta i}=0, \tag{4.7}$$

(4.5) 两边同时关于 y^j 微分得

$$\frac{\partial^3 g^{kh}}{\partial y^i \partial y^j \partial y^k}\frac{\partial f_1^2}{\partial x^h}=0, \tag{4.8}$$

把 (4.1), (4.4) 和 (4.8) 代入到 (3.19) 得

$$\mathbb{E}_{ij}=0, \tag{4.9}$$

同理, 可证

$$\mathbb{E}_{\alpha\beta}=0, \tag{4.10}$$

由 (4.7), (4.9) 和 (4.10)可得 $\mathbb{E}_{AB}=0$, 即 F 是弱 Berwald 度量.

必要性, 设 F 是弱 Berwald 度量, 那么 $\mathbb{E}_{AB}=0$, 即 $\mathbb{E}_{ij}=\mathbb{E}_{i\beta}=\mathbb{E}_{\beta i}=\mathbb{E}_{\alpha\beta}=0$.

根据 (3.20) 式等于零可得

$$\frac{1}{f_2^2}\frac{\partial^2 g^{kh}}{\partial y^i \partial y^k}\frac{\partial f_1^2}{\partial x^h}\frac{\partial F_2^2}{\partial v^\beta}=-\frac{1}{f_1^2}\frac{\partial^2 g^{\gamma\mu}}{\partial v^\beta \partial v^\gamma}\frac{\partial f_2^2}{\partial u^\mu}\frac{\partial F_1^2}{\partial y^i}, \tag{4.11}$$

(4.11) 两边同时关于 y^j 微分, 再与 v^β 缩并可得

$$\frac{1}{f_2^2}\frac{\partial^3 g^{kh}}{\partial y^i \partial y^j \partial y^k}\frac{\partial f_1^2}{\partial x^h}F_2^2=\frac{1}{2f_1^2}\frac{\partial g^{\gamma\mu}}{\partial v^\gamma}\frac{\partial f_2^2}{\partial u^\mu}\frac{\partial^2 F_1^2}{\partial y^i \partial y^j}, \tag{4.12}$$

把 (4.12) 代入 (3.19) 得

$$\begin{aligned}\overset{1}{\mathbb{E}}_{ij}=&\frac{1}{8f_2^2}\Big(\frac{\partial g^{kh}}{\partial y^k}\frac{\partial^2 F_1^2}{\partial y^i \partial y^j}+\frac{\partial^2 g^{kh}}{\partial y^j \partial y^k}\frac{\partial F_1^2}{\partial y^i}+\frac{\partial^2 g^{kh}}{\partial y^i \partial y^k}\frac{\partial F_1^2}{\partial y^j}+\frac{\partial^3 g^{kh}}{\partial y^i \partial y^j \partial y^k}F_1^2\Big)\frac{\partial f_2^2}{\partial x^h}\\&+\frac{3}{16f_1^2}\frac{\partial g^{\gamma\mu}}{\partial v^\gamma}\frac{\partial f_2^2}{\partial u^\mu}\frac{\partial^2 F_1^2}{\partial y^i \partial y^j},\end{aligned} \tag{4.13}$$

(4.13) 等式两边同时关于 v^λ 微分, 再与 v^λ 缩并得 (4.4), 把 (4.4) 代回 (4.13), 即得 (4.1).

同理可证 (4.2) 和 (4.3) 成立. □

推论4.1. *设 F 是 Finsler 度量 F_1 和 F_2 的双挠积, 且 $f_2(x,u)=f_2(u)$, $f_1(x,u)=f_1(x)$. 那么 F 是*

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弱 Berwald 度量当且仅当 F_1 和 F_2 是弱 Berwald 度量, 并且 $\frac{\partial g^{kh}}{\partial y^k}\frac{\partial f_1^2}{\partial x^h}=\frac{\partial g^{\gamma\mu}}{\partial v^\gamma}\frac{\partial f_2^2}{\partial u^\mu}=0$ 成立.

注4.1. 若 $f_2(x,u)=f_2(u)$, $f_1(x,u)=f_1(x)$, 双挠积 Finsler 度量退化为双扭曲积 Finsler 度量, 此时, 推论 3.1 与文献[13] 中的定理 2 结论一致.

定理4.2. *设 F 是 Finsler 度量 F_1 和 F_2 的双挠积. 如果 F_1 和 F_2 是弱 Berwald 度量, 那么 F 是弱 Berwald 度量当且仅当*

$$\frac{\partial g^{kh}}{\partial y^k}\frac{\partial f_1^2}{\partial x^h}=\frac{\partial g^{kh}}{\partial y^k}\frac{\partial f_2^2}{\partial x^h}=\frac{\partial g^{\gamma\mu}}{\partial v^\gamma}\frac{\partial f_1^2}{\partial u^\mu}=\frac{\partial g^{\gamma\mu}}{\partial v^\gamma}\frac{\partial f_2^2}{\partial u^\mu}=0. \tag{4.14}$$

证明. 必要性. 设 F 是弱 Berwald 度量, 则根据定理 3.2 可得

$$\frac{\partial g^{kh}}{\partial y^k}\frac{\partial f_1^2}{\partial x^h}=\frac{\partial g^{\gamma\mu}}{\partial v^\gamma}\frac{\partial f_2^2}{\partial u^\mu}=0,$$

若 F_1 是弱 Berwald 度量, 即 $\overset{1}{\mathbb{E}}_{ij}=0$, 由 (4.1) 有

$$\left(\frac{\partial g^{kh}}{\partial y^k}\frac{\partial^2 F_1^2}{\partial y^i\partial y^j}+\frac{\partial^2 g^{kh}}{\partial y^j\partial y^k}\frac{\partial F_1^2}{\partial y^i}+\frac{\partial^2 g^{kh}}{\partial y^i\partial y^k}\frac{\partial F_1^2}{\partial y^j}+\frac{\partial^3 g^{kh}}{\partial y^i\partial y^j\partial y^k}F_1^2\right)\frac{\partial f_2^2}{\partial x^h}=0, \tag{4.15}$$

(4.15) 两边同时与 y^k 缩并, 再关于 y^k 微分, 并应用 (3.24) 有

$$\left(\frac{\partial^2 g^{kh}}{\partial y^j\partial y^k}\frac{\partial F_1^2}{\partial y^i}+\frac{\partial^2 g^{kh}}{\partial y^i\partial y^k}\frac{\partial F_1^2}{\partial y^j}+2\frac{\partial^3 g^{kh}}{\partial y^i\partial y^j\partial y^k}F_1^2\right)\frac{\partial f_2^2}{\partial x^h}=0, \tag{4.16}$$

(4.16) 与 (4.15) 作差, 可得

$$\frac{\partial g^{kh}}{\partial y^k}\frac{\partial f_2^2}{\partial x^h}\frac{\partial^2 F_1^2}{\partial y^i\partial y^j}-\frac{\partial^3 g^{kh}}{\partial y^i\partial y^j\partial y^k}\frac{\partial f_2^2}{\partial x^h}F_1^2=0, \tag{4.17}$$

(4.17) 两边同时与 y^k 缩并得

$$\frac{\partial^2 g^{kh}}{\partial y^i\partial y^j}\frac{\partial f_2^2}{\partial x^h}=0, \tag{4.18}$$

(4.18) 等式两边关于 y^k 微分, 可得

$$\frac{\partial^3 g^{kh}}{\partial y^i\partial y^j\partial y^k}\frac{\partial f_2^2}{\partial x^h}=0, \tag{4.19}$$

把 (4.19) 代回 (4.17) 得

$$\frac{\partial g^{kh}}{\partial y^k}\frac{\partial f_2^2}{\partial x^h}=0,$$

同理, 根据 (4.2) 可得 $\frac{\partial g^{\gamma\mu}}{\partial v^\gamma}\frac{\partial f_1^2}{\partial u^\mu}=0$.

充分性, 设 F_1 和 F_2 是弱 Berwald 度量, 则

$$\overset{1}{\mathbb{E}}_{ij}=\overset{2}{\mathbb{E}}_{\alpha\beta}=0, \tag{4.20}$$

明显地, 根据 (4.14) 可得

$$\frac{\partial^2 g^{kh}}{\partial y^i \partial y^k}\frac{\partial f_1^2}{\partial x^h}=\frac{\partial^2 g^{\gamma\mu}}{\partial v^\beta \partial v^\gamma}\frac{\partial f_1^2}{\partial u^\mu}=0, \qquad \frac{\partial^3 g^{kh}}{\partial y^i \partial y^j \partial y^k}\frac{\partial f_1^2}{\partial x^h}=\frac{\partial^3 g^{\gamma\mu}}{\partial v^\alpha \partial v^\beta \partial v^\gamma}\frac{\partial f_1^2}{\partial u^\mu}=0, \tag{4.21}$$

把 (4.14), (4.20) 和 (4.21) 代入到 (3.19) 得 $\mathbb{E}_{ij}=0$.

同理可证 $\mathbb{E}_{i\beta}=\mathbb{E}_{\beta i}=\mathbb{E}_{\alpha\beta}=0$. 综上所述, $\mathbb{E}_{AB}=0$, 即 F 是弱 Berwald 度量. □

定理4.3. *设 F 是 Finsler 度量 F_1 和 F_2 的双挠积. 如果 $\frac{\partial g^{kh}}{\partial y^k}\frac{\partial f_1^2}{\partial x^h}=0$ 或 $\frac{\partial g^{\gamma\mu}}{\partial v^\gamma}\frac{\partial f_2^2}{\partial u^\mu}=0$, 那么具有迷向平均 Berwald 曲率的双挠积 Finsler 度量是弱 Berwald 度量.*

证明. 设 F 是 Finsler 度量 F_1 和 F_2 的双挠积. 根据 (2.5) 有

$$\mathbb{E}_{AB}=\frac{1}{2}(n+1)cF_{Y^AY^B}, \tag{4.22}$$

从而

$$\mathbb{E}_{i\beta}=\mathbb{E}_{\beta i}=\frac{1}{2}(n+1)cF_{y^iv^\beta}, \tag{4.23}$$

其中 $c=c(x,u)$ 是 M 上的标量函数.

又由 (3.20) 知

$$\mathbb{E}_{i\beta}=\mathbb{E}_{\beta i}=-\frac{1}{8f_2^2}\frac{\partial^2 g^{kh}}{\partial y^i\partial y^k}\frac{\partial f_1^2}{\partial x^h}\frac{\partial F_2^2}{\partial v^\beta}-\frac{1}{8f_1^2}\frac{\partial^2 g^{\gamma\mu}}{\partial v^\beta\partial v^\gamma}\frac{\partial f_2^2}{\partial u^\mu}\frac{\partial F_1^2}{\partial y^i}, \tag{4.24}$$

所以

$$\frac{1}{8f_2^2}\frac{\partial^2 g^{kh}}{\partial y^i\partial y^k}\frac{\partial f_1^2}{\partial x^h}\frac{\partial F_2^2}{\partial v^\beta}+\frac{1}{8f_1^2}\frac{\partial^2 g^{\gamma\mu}}{\partial v^\beta\partial v^\gamma}\frac{\partial f_2^2}{\partial u^\mu}\frac{\partial F_1^2}{\partial y^i}=-\frac{1}{2}(n+1)cF_{y^iv^\beta}, \tag{4.25}$$

如果 $\frac{\partial g^{\gamma\mu}}{\partial v^\gamma}\frac{\partial f_2^2}{\partial u^\mu}=0$, 那么 (4.25) 可化为

$$\frac{1}{8f_2^2}\frac{\partial^2 g^{kh}}{\partial y^i\partial y^k}\frac{\partial f_1^2}{\partial x^h}=\frac{1}{8}(n+1)c\frac{f_1^2f_2^2}{F^3}\frac{\partial F_1^2}{\partial y^i}, \tag{4.26}$$

(4.26) 两边同时关于 v^λ 微分得

$$\frac{c(n+1)}{F^5}f_1^4f_2^2\frac{\partial F_1^2}{\partial y^i}\frac{\partial F_2^2}{\partial v^\lambda}=0,$$

因此 $c=0$, 将其代入(4.22) 可得 $\mathbb{E}_{AB}=0$, 即 F 是弱 Berwald 度量. □

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