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具有离散时滞的Caputo分数阶微分方程的稳定性

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摘 要

本文利用不动点定理，建立了无限时间离散分布时滞分数阶随机微分方程，主要研究无限时间区间内具有布朗运动和离散分布时滞的Caputo分数阶微分方程解的存在性、唯一性和渐近稳定性。其中，运用压缩映射原理和Mittag-Lefler函数的精准估计。最终，运用反证法证明出了渐近稳定性。

关键词

分数阶随机微分方程，渐近稳定性，不动点定理

Stability of Caputo Fractional Differential Equations with Discrete Delay

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Abstract

In this paper, we establish infinite time discrete fractional stochastic differential equa-

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tions with distributed delays by using the fixed point theorem. We mainly study the existence, uniqueness and asymptotic stability of solutions of Caputo fractional differential equations with Brownian motion and discrete distributed delays in infinite time intervals. Among them, the compression mapping principle and accurate estimation of Mittag Leffler function are used. Finally, the asymptotic stability is proved by the method of contradiction.

Keywords

Fractional Stochastic Differential Equations, Asymptotic Stability, Fixed Point Theorems

1. 引言

在过去的几十年中，分数阶微分方程的研究广泛开展，并且相应地产生了许多分数阶类型。分数阶微分方程解的存在唯一性是重要的研究方向。在文献 [1]中，通过应用不动点定理证明了有限时间区间内Caputo分数阶导数系统解的存在性和唯一性。对于Hilfer分数阶微分方程，通过应用 [2]中的逐次逼近讨论了Cauchy型问题解在有限时间区间内的存在性和唯一性。此外，还可以考虑分数阶微分方程的性质，如稳定性。 [3] 通过巧妙地应用Mittag-Lefler函数的渐近展开，得到了线性Riemann-Liouville分数阶微分系统和Caputo分数阶微分方程组的稳定性和渐近稳定性条件，并且讨论了具有Riemann – Liouville分数阶导数的线性微分系统的稳定性和渐近稳定性。

本文主要研究无限时间区间内具有布朗运动和离散分布时滞的Caputo分数阶微分方程解的存在性、唯一性和渐近稳定性。由于我们考虑无限区间，因此我们需要仔细计算估计值。与上述文章的研究相反，因此我们运用了Mittag-Lefler函数的性质。我们考虑无限区域中Mittag-Lefler函数的求值。对于可微函数$f:[0,\infty)\to\mathbb{R}$，p阶$f$的Caputo [4]导数由下式给出

$$\partial_t^p f(t)=\frac{1}{\Gamma(1-p)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-s)^{-p}\left(f(s)-f(0)\right)\mathrm{d}s,$$

其中$0<p\leqslant 1$和$\Gamma(s):=\int_0^\infty t^{s-1}e^{-t}\mathrm{d}t$。

我们将研究以下分数阶微分方程

$$\begin{cases} \partial_t^\beta\left[x_i(t)-\sum\limits_{j=1}^n d_{ij}x_j(t-\tau_j(t))\right]=\sum\limits_{j=1}^n c_{ij}x_j(t)+\sum\limits_{j=1}^n a_{ij}f_j(x_j(t)) \\ \quad+\sum\limits_{j=1}^n b_{ij}g_j(x_j(t-\tau_j(t)))+\sum\limits_{j=1}^n l_{ij}\int_{t-r(t)}^t h_j(x_j(s))ds \\ +\sum\limits_{j=1}^n \partial_t^\gamma\int_0^t \sigma_{ij}(s,x_j(s),x_j(s-\tau(s)))\mathrm{d}W_j(s),\quad t\geqslant 0, \\ x_i(t)=\phi_i(t),\quad t\in[\vartheta,0], \end{cases} \tag{1}$$

其中，∂_t^β和∂_t^γ表示分数阶导数，$\beta\in(\frac{1}{2},1)$，$\gamma\in(\frac{1}{2},\beta+\frac{1}{2})$，以及$x(t)=\phi(t),t\in[\vartheta,0]$是方程(1)的初始条件，其中$t\longmapsto\phi=(\phi_1(t),\cdots,\phi_n(t))^T$和$x(t)=(x_1(t),\cdots,x_n(t))^T\in\mathbb{R}^n$中是与神经元相关的向量；$C=diag(c_1,c_2,\cdots,c_n)$；我们使用$A=(a_{ij})_{n\times n}$、$B=(b_{ij})_{n\times n}$、$C=(c_{ij})_{n\times n}$、$D=(d_{ij})_{n\times n}$和$L=(l_{ij})_{n\times n}$表示不同的矩阵；我们设置了$f_j$、$g_j$和$h_j$的激活函数，其中$f(x(t))=(f_1(x(t)),\cdots,f_n(x(t)))^T\in\mathbb{R}^n$，$g(x(t))=(g_1(x(t)),\cdots,g_n(x(t)))^T\in\mathbb{R}^n$，$h(x(t))=(h_1(x(t)),\cdots,h_n(x(t)))^T\in\mathbb{R}^n$；离散时变延迟和分布式时变延迟的界被表示为$\tau(t)$和$r(t)$。让我们标记$\vartheta=\inf_{t\geqslant 0}\{t-\tau(t),t-r(t)\}$。此外$\tau(t)$和$r(t)$是非负连续函数。此外，$w(t)=(w_1(t),w_2(t),\cdots,w_n(t))^T\in\mathbb{R}^n$是完全概率空间$(\Omega,\mathcal{F},P)$中的n维布朗运动。

论文的其余部分组织如下。在第2节中，我们给出了关于Mittag-Lefler函数的假设并准备了一些结果。第三节利用不动点定理讨论了具有离散和分布时滞的Caputo分数阶随机微分方程解的存在性、唯一性和渐近稳定性。

2. 预备知识

为了实现我们的主要目标，我们可以假设满足以下条件：

(A1) 延迟$\tau(t)$，$r(t)$是连续函数，满足$t-\tau(t)\to\infty$和$t-r(t)\to\infty$ as $t\to\infty$。

(A2) 映射$f_j(\cdot)$、$g_j(\cdot)$和$h_j(\cdot)$，满足$f(0)\equiv 0$，$g(0)\equiv 0$，$h(0)\equiv 0$，$\sigma_{ij}(t,0,0)\equiv 0$ 和具有Lipschitz常数的全局Lipschiz函数α_j，β_j以及γ_j其中$j=1,2\cdots n$,

(A3) 对于每个$i,j=1,2,\cdots,n$，存在常数μ_j and υ_j ，使得

$$(\sigma_{ij}(t,x,y)-\sigma_{ij}(t,u,v))^2\leqslant\mu_j(x_j-u_j)^2+\upsilon_j(y_j-v_j)^2.$$

$f_j(\cdot)$, $g_j(\cdot)$, $h_j(\cdot)$ and $\sigma_{ij}(t,\cdot,\cdot)$的函数具有局部Lipschitz和线性增长可以确保方程(1)解的存在性和唯一性的条件。

定义2.1. [5] *Mittag-Lefler函数类型的双参数函数由级数展开定义*

$$E_{\alpha,\beta}(z)=\sum_{k=0}^\infty\frac{z^k}{\Gamma(k\alpha+\beta)},\quad \alpha,\beta>0,\quad z\in\mathbb{C}.$$

引理2.1. [6] 假设$E_{\alpha,\beta}$是*Mittag-Leffler*函数，我们可以得到

$$\int_0^\infty e^{-st}t^{\beta-1}E_{\alpha,\beta}(\pm at^\alpha)\mathrm{d}t=\frac{s^{\alpha-\beta}}{(s^\alpha\mp a)},\quad \Re(\alpha)>0,\quad \Re(\beta)>0,\quad \Re(s)>0,$$

另外

$$\mathcal{L}\left\{t^{\beta-1}E_{\alpha,\beta}(\pm at^\alpha)\right\}(s)=\frac{s^{\alpha-\beta}}{(s^\alpha\mp a)},\quad \Re(\alpha)>0,\quad \Re(\beta)>0,\quad \Re(s)>0,$$

其中$\Re$ 表示实部，$\mathcal{L}$表示拉普拉斯变换。

引理2.2. [7] 如果$0<\alpha<2$, β是任意一个数, $\pi\alpha/2<\mu<\min\{\pi,\pi\alpha\}$, 然后存在$M>0$ 满足

$$E_{\alpha,\beta}(z)\leqslant\frac{M}{1+|z|},\quad \mu<|arg(z)|<\pi,\quad |z|\geqslant 0.$$

引理2.3. [7] 假设$E_{\alpha,\beta}$ 是*Mittag-Leffler*函数, 我们可以得到

$$\int_0^t E_{\alpha,\beta}(\lambda s^\alpha)s^{\beta-1}\mathrm{d}s=t^\beta E_{\alpha,\beta+1}(\lambda t^\alpha),\quad \alpha>0,\quad \beta>0.$$

引理2.4. [7] 假设$E_{\alpha,\beta}$ 是*Mittag-Leffler*函数, 我们可以得到

$$\frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}E_{\alpha,\beta}(\lambda s^\alpha)s^{\beta-1}\mathrm{d}s=t^{\beta+\gamma-1}E_{\alpha,\beta+\gamma}(\lambda t^\alpha),\alpha>0,\beta>0,\gamma>0.$$

引理2.5. [8] 假设(X,Σ,μ) 是σ有限测度空间, M是连续局部鞅, $[M]:[0,T]\times\Omega\to\mathbb{R}^+$ 是它的二次变差。若$\psi:X\times[0,T]\times\Omega\to\mathbb{R}$, $T\in(0,\infty)$ 可逐步测量，对于$\omega\in\Omega$, 得到

$$\int_X\left(\int_0^T|\psi(x,t,\omega)|^2d[M](t,\omega)\right)^{\frac{1}{2}}d\mu(x)<\infty,$$

那么对于几乎所有$\omega\in\Omega$ 对于所有$t\in[0,T]$, 得到

$$\int_X\int_0^t\psi(x,r,\omega)dM(r)d\mu(x)=\int_0^t\int_X\psi(x,t,\omega)d\mu(x)dM(r).$$

在本文中，我们将考虑以下空间上方程(1)的解。定义$\mathcal{S}_\phi$ 是$\mathcal{F}_0$适应过程的空间：$\varphi(t,\omega)$: $[\vartheta,\infty)\times\Omega\to\mathbb{R}^n$ 满足$\varphi\in C\left([\vartheta,\infty),L^P_{\mathcal{F}_0}(\Omega;\mathbb{R}^n)\right)$ 范数定义如下：

$$\|\varphi\|^p:=\sup_{t\geqslant\vartheta}\left(\mathbb{E}\sum_{i=1}^n|\varphi_i(t)|^p\right).$$

另外, 我们令$\varphi(t,\cdot)=\phi(t)$ 在$t\in[\vartheta,0]$上且$\sum_{i=1}^n\mathbb{E}|\varphi_i(t)|^p\to 0$ 当$t\to\infty$, 得到$\mathcal{S}_\phi$ 是一个完备空间。

3. 主要结果

让$\tilde{c}_{i,i}(\cdot)=c_{i,i}(\cdot)+\lambda_i$ 其中λ_i 是正数且$\tilde{c}_{i,j}(\cdot)=c_{i,j}(\cdot)$ 其中$i\neq j$, $i,j=1,2,\cdots,n$. 系统(1) 能

被写做

$$\begin{cases} \partial_t^\beta\big[x_i(t)-\sum\limits_{j=1}^n d_{ij}x_j(t-\tau_j(t))\big]=-\lambda_i\big[x_i(t)-\sum\limits_{j=1}^n d_{ij}x_j(t-\tau_j(t))\big] \\ \quad -\lambda_i\sum\limits_{j=1}^n d_{ij}x_j(t-\tau_j(t))+\sum\limits_{j=1}^n \tilde{c}_{ij}x_j(t)+\sum\limits_{j=1}^n a_{ij}f_j(x_j(t)) \\ \quad +\sum\limits_{j=1}^n b_{ij}g_j(x_j(t-\tau_j(t)))+\sum\limits_{j=1}^n l_{ij}\int_{t-r(t)}^t h_j(x_j(s))ds \\ \quad +\sum\limits_{j=1}^n \partial_t^\gamma\int_0^t \sigma_{ij}(s,x_j(s),x_j(s-\tau(s)))\mathrm{d}W_j(s),\quad t\geqslant 0, \\ x_i(t)=\phi_i(t),\quad t\in[\vartheta,0]. \end{cases}$$

引理3.1. 用以下形式表示的*Caputo*分数阶随机微分系统

$$\begin{cases} \partial_t^\beta\varphi(t)=\lambda\varphi(s)+af(\varphi(t))+\partial_t^\alpha\int_0^t B(s)\mathrm{d}W_s, \\ \varphi(0)=\varphi_0, \end{cases}$$

其中a 是常数且$B(s):[0,\infty)\to\mathbb{R}$, 那么，这个系统等价于积分方程

$$\begin{aligned}\varphi(s)=&\varphi_0E_{\beta,1}(\lambda t^\beta)+a\int_0^t(t-s)^{\beta-1}E_{\beta,\beta}(\lambda(t-s)^\beta)f(\varphi(s))\mathrm{d}s \\ &+\int_0^t(t-s)^{\beta-\alpha}E_{\beta,\beta-\alpha+1}(\lambda(t-s)^\beta)B(s)\mathrm{d}W_s.\end{aligned}$$

Proof. 通过定义的Caputo分数导数，我们可以得到

$$\begin{aligned}\frac{1}{\Gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t(t-s)^{-\beta}\left(\varphi(s)-\varphi(0)\right)\mathrm{d}s=&\lambda\varphi(s)+af(\varphi(t)) \\ &+\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t(t-s)^{-\alpha}\int_0^s B(u)\mathrm{d}W_u\mathrm{d}s.\end{aligned}$$

对等式两边积分并且使用定理2.5, 得到

$$\begin{aligned}\frac{1}{\Gamma(1-\beta)}\int_0^t(t-s)^{-\beta}\left(\varphi(s)-\varphi(0)\right)\mathrm{d}s=*\lambda\int_0^t\varphi(s)\mathrm{d}s+a\int_0^t f(\varphi(s))\mathrm{d}s \\ +\frac{1}{\Gamma(2-\alpha)}\int_0^t(t-s)^{1-\alpha}B(s)\mathrm{d}W_s.\end{aligned}$$

然后

$$\begin{aligned}\frac{1}{\Gamma(1-\beta)}\int_0^t(t-s)^{-\beta}\varphi(s)\mathrm{d}s=&\frac{\varphi(0)}{(1-\beta)\Gamma(1-\beta)}t^{1-\beta} \\ &+\lambda\int_0^t\varphi(s)\mathrm{d}s+a\int_0^t f(\varphi(s))\mathrm{d}s \\ &+\frac{1}{\Gamma(2-\alpha)}\int_0^t(t-s)^{1-\alpha}B(s)\mathrm{d}W_s.\end{aligned}$$

在两边使用拉普拉斯变换，我们可以得到

$$\frac{1}{\Gamma(1-\beta)}\frac{\Gamma(1-\beta)}{s^{1-\beta}}\hat{\varphi}(s)=\frac{\varphi(0)}{(1-\beta)\Gamma(1-\beta)}\frac{\Gamma(2-\beta)}{s^{2-\beta}}+\lambda\frac{1}{s}\hat{\varphi}(s)+a\frac{1}{s}\hat{f}(\varphi(s))$$
$$+\frac{1}{\Gamma(2-\alpha)}\frac{\Gamma(2-\alpha)}{s^{2-\alpha}}\int_0^\infty e^{-st}B(t)\mathrm{d}W_t,$$

关于$\int_0^t(t-s)^{1-\alpha}B(s)\mathrm{d}W_s$的拉普拉斯变换, 我们使用引理2.5，得到

$$\begin{aligned}
&\int_0^\infty e^{-st}\int_0^t(t-v)^{1-\alpha}B(v)\mathrm{d}W_v\mathrm{d}t\\
=&\int_0^\infty B(v)\mathrm{d}W_v\int_v^\infty e^{-st}(t-v)^{1-\alpha}\mathrm{d}t\\
=&\int_0^\infty B(v)\mathrm{d}W_v\int_0^\infty e^{-s(u+v)}u^{1-\alpha}\mathrm{d}u\\
=&\int_0^\infty e^{-sv}B(v)\mathrm{d}W_v\int_0^\infty e^{-su}u^{1-\alpha}\mathrm{d}u,
\end{aligned}$$

因此,

$$\hat{\varphi}(s)=\frac{s^{\beta-1}}{s^\beta-\lambda}\varphi(0)+\frac{1}{s^\beta-\lambda}\hat{f}(\varphi(s))+\frac{s^{\alpha-1}}{s^\beta-\lambda}\int_0^\infty e^{-st}B(t)\mathrm{d}W_t.$$

两边逆拉普拉斯变换以及根据引理2.1, 我们得到

$$\varphi(s)=\varphi_0E_{\beta,1}(\lambda t^\beta)+a\int_0^t(t-s)^{\beta-1}E_{\beta,\beta}(\lambda(t-s)^\beta)f(\varphi(s))\mathrm{d}s$$
$$+\int_0^t(t-s)^{\beta-\alpha}E_{\beta,\beta-\alpha+1}(\lambda(t-s)^\beta)B(s)\mathrm{d}W_s.$$

□

为了简便, 记$\tilde{E}_{\alpha,\beta}(t)=t^{\beta-1}E_{\alpha,\beta}(-\lambda_i t^\alpha)$。因此，我们可以通过与引理*3.1*的证明类似的方法重写方程(1)。

$$\begin{aligned}
x_i(t)\quad=\quad&\Big[\phi_i(0)-\sum_{j=1}^n d_{ij}\phi_j(0-\tau_j(0))\Big]E_{\beta,1}(-\lambda_i t^\beta)+\sum_{j=1}^n d_{ij}x_j(t-\tau_j(t))\\
&-\lambda_i\int_0^t\tilde{E}_{\beta,\beta}(t-s)\sum_{j=1}^n d_{ij}x_j(s-\tau_j(s))ds+\int_0^t\tilde{E}_{\beta,\beta}(t-s)\sum_{j=1}^n\tilde{c}_{ij}x_j(s)ds\\
&+\int_0^t\tilde{E}_{\beta,\beta}(t-s)\sum_{j=1}^n a_{ij}f_j(x_j(s))ds+\int_0^t\tilde{E}_{\beta,\beta}(t-s)\sum_{j=1}^n b_{ij}g_j(x_j(s-\tau_j(s)))ds\\
&+\int_0^t\tilde{E}_{\beta,\beta}(t-s)\sum_{j=1}^n l_{ij}\int_{s-r(s)}^s h_j(x_j(u))duds\\
&+\int_0^t\tilde{E}_{\beta,\beta-\gamma+1}(t-s)\sum_{j=1}^n\sigma_{ij}(s,x_j(s),x_j(s-\tau(s)))\mathrm{d}W_j(s).
\end{aligned}$$

通过$(Q\varphi)(t)=\phi(t)$ 定义算子，其中$t\in[-\tau,0]$, $t>0$, $i=1,2,3,\cdots,n$,

$$
\begin{aligned}
(Q\varphi)_i(t) \quad = \quad & \Big[\phi_i(0)-\sum_{j=1}^{n} d_{ij}\phi_j(0-\tau_j(0))\Big] E_{\beta,1}(-\lambda_i t^{\beta})+\sum_{j=1}^{n} d_{ij}\varphi_j(t-\tau_j(t)) \\
& -\lambda_i \int_0^t \tilde{E}_{\beta,\beta}(t-s)\sum_{j=1}^{n} d_{ij}\varphi_j(s-\tau_j(s))ds \qquad (2)\\
& +\int_0^t \tilde{E}_{\beta,\beta}(t-s)\sum_{j=1}^{n} \tilde{c}_{ij}\varphi_j(s)ds \qquad (3)\\
& +\int_0^t \tilde{E}_{\beta,\beta}(t-s)\sum_{j=1}^{n} a_{ij}f_j(\varphi_j(s))ds \qquad (4)\\
& +\int_0^t \tilde{E}_{\beta,\beta}(t-s)\sum_{j=1}^{n} b_{ij}g_j(\varphi_j(s-\tau_j(s)))ds \\
& +\int_0^t \tilde{E}_{\beta,\beta}(t-s)\sum_{j=1}^{n} l_{ij}\int_{s-r(s)}^{s} h_j(\varphi_j(u))duds \\
& +\int_0^t \tilde{E}_{\beta,\beta-\gamma+1}(t-s)\sum_{j=1}^{n} \sigma_{ij}(s,\varphi_j(s),\varphi_j(s-\tau(s)))\mathrm{d}W_j(s) \\
=: \quad & \sum_{k=1}^{8} J_{ki}.
\end{aligned}
$$

定理3.1. 假设（*A1*）-（*A3*）成立并且满足以下条件，

(i) 函数*r(t)* 以常数r为界，其中>0;

*(ii)*存在$T_1>0$, 满足

$$
\begin{aligned}
&8^{p-1}\sum_{i=1}^{n}\Bigg\{\left(\sum_{j=1}^{n}|d_{ij}|^q\right)^{\frac{p}{q}}+\left(\frac{M}{\lambda_i}\right)^{p}\Bigg[\lambda_i^p\left(\sum_{j=1}^{n}|d_{ij}|^q\right)^{\frac{p}{q}}+\left(\sum_{j=1}^{n}|\tilde{c_{ij}}|^q\right)^{\frac{p}{q}} \qquad (5)\\
&+\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{\frac{p}{q}}+\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{\frac{p}{q}}+r^p\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{\frac{p}{q}}\Bigg]\\
&+n^{p-1}2^{\frac{p}{2}-1}\left(\mu^{\frac{p}{2}}+\upsilon^{\frac{p}{2}}\right)\left(\frac{M^2}{2\beta-2\gamma+1}T_1^{2\beta-2\gamma+1}+\frac{M^2}{(2\gamma-1)\lambda_i^2T_1^{2\gamma-1}}\right)^{\frac{p}{2}}\Bigg\}<1,
\end{aligned}
$$

其中M 是定理2.2中的, $\mu=max\{\mu_1,\mu_2,\cdots,\mu_n\},\nu=max\{\nu_1,\nu_2,\cdots,\nu_n\}$, 而$Q:\mathcal{S}_\phi\to\mathcal{S}_\phi$ 是压缩映射且方程*(1)*的解是唯一的并且在p阶矩中渐近稳定的.

Proof. 我们将使用不动点定理来从以下步骤证明。

步骤1. 我们证明连续性。让$x \in \mathcal{S}_\phi$, $t_1 \geqslant 0$, $r \in \mathbb{R}$ 且$|r|$ 足够小并且$r>0$当$t_1 = 0$. 由引理2.2, 使用Hölder 不等式, 我们有

$$\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^{n}|J_{3i}(t_1+r)-J_{3i}(t_1)|^p\right]\\
=\ &\mathbb{E}\Bigg[\sum_{i=1}^{n}\Bigg|\lambda_i\int_0^{t_1}\left[\tilde{E}_{\beta,\beta}(t_1-s)-\tilde{E}_{\beta,\beta}(t_1+r-s))\right]\sum_{j=1}^{n}d_{ij}\varphi_j(s-\tau(s))\mathrm{d}s\\
&-\lambda_i\int_{t_1}^{t_1+r}\tilde{E}_{\beta,\beta}(t_1+r-s)\sum_{j=1}^{n}d_{ij}\varphi_j(s-\tau(s))\mathrm{d}s\Bigg|^p\Bigg]\\
\leqslant\ &(2n)^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i^p\mathbb{E}\left[\left|\int_0^{t_1}\left[\tilde{E}_{\beta,\beta}(t_1-s)-\tilde{E}_{\beta,\beta}(t_1+r-s))\right]d_{ij}\varphi_j(s-\tau(s))\mathrm{d}s\right|^p\right]\\
&+(2n)^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i^p\mathbb{E}\left[\left|\int_{t_1}^{t_1+r}\tilde{E}_{\beta,\beta-\gamma+1}(t_1+r-s)d_{ij}\varphi_j(s-\tau(s))\mathrm{d}s\right|^p\right]\\
=:\ &(2n)^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i^pI_1+(2n)^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i^pI_2.
\end{aligned}$$

现在我们讨论

$$\begin{aligned}
I_1\ \leqslant\ &2^{p-1}\Bigg\{\mathbb{E}\Bigg[\Bigg|\int_0^{t_1}\Big[(t_1-s)^{\beta-1}E_{\beta,\beta}(-\lambda_i(t_1-s)^\beta)\\
&-(t_1-s)^{\beta-1}E_{\beta,\beta}(-\lambda_i(t_1+r-s)^\beta)\Big]\times d_{ij}\varphi_j(s-\tau(s))\mathrm{d}s\Bigg|^p\Bigg]\\
&+\mathbb{E}\Bigg[\Bigg|\int_0^{t_1}\Big[(t_1-s)^{\beta-1}E_{\beta,\beta}(-\lambda_i(t_1+r-s)^\beta)\\
&-(t_1+r-s)^{\beta-1}E_{\beta,\beta}(-\lambda_i(t_1+r-s)^\beta)\Big]\times d_{ij}\varphi_j(s-\tau(s))\mathrm{d}s\Bigg|^p\Bigg]\Bigg\}\\
=:\ &2^{p-1}I_{11}+2^{p-1}I_{12}.
\end{aligned}$$

使用Hölder 不等式, 我们得到

$$\begin{aligned}
I_{11}\ =\ &\mathbb{E}\left[\left|\int_0^{t_1}\left[(t_1-s)^{\beta-1}\Big(E_{\beta,\beta}(-\lambda_i(t_1-s)^\beta)-E_{\beta,\beta}(-\lambda_i(t_1+r-s)^\beta)\Big)\right]d_{ij}\varphi_j(s-\tau(s))\mathrm{d}s\right|^p\right]\\
=\ &\mathbb{E}\Bigg[\Bigg|\int_0^{t_1}\Big[(t_1-s)^{\frac{\beta-1}{q}}(t_1-s)^{\frac{\beta-1}{p}}\Big(E_{\beta,\beta}(-\lambda_i(t_1-s)^\beta)\\
&-E_{\beta,\beta}(-\lambda_i(t_1+r-s)^\beta)\Big)\Big]d_{ij}\varphi_j(s-\tau(s))\mathrm{d}s\Bigg|^p\Bigg]\\
\leqslant\ &\mathbb{E}\Bigg[\Bigg|\left(\int_0^{t_1}(t_1-s)^{(\beta-1)}ds\right)^{\frac{p}{q}}\int_0^{t_1}(t_1-s)^{(\beta-1)}\Big[\Big(E_{\beta,\beta}(-\lambda_i(t_1-s)^\beta)\\
&-E_{\beta,\beta}(-\lambda_i(t_1+r-s)^\beta)\Big)d_{ij}\varphi_j(s-\tau(s))\Big]^p\mathrm{d}s\Bigg|\Bigg]\\
\leqslant\ &\left(\frac{t_1^\beta}{\beta}\right)^{p-1}d_{ij}^p\int_0^{t_1}(t_1-s)^{(\beta-1)}\mathbb{E}\left[\left|\varphi_j(s-\tau(s))\right|^p\right]\mathrm{d}s\\
&\times\sup_{0\leqslant s\leqslant t_1}\left|\left[E_{\beta,\beta}(-\lambda_i(t_1-s)^\beta)-E_{\beta,\beta}(-\lambda_i(t_1+r-s)^\beta)\right]^p\right|.
\end{aligned}$$

因为$(E_{\beta,\beta}(-\lambda_i(t_1-s)^\beta)$ 是连续的且$\mathbb{E}\Big|\varphi_j(s-\tau(s))\Big|^p$ 有界, 所以我们容易得到$I_{11}\to 0$, $r\to 0$.

使用Hölder 不等式和引理2.2，

$$\begin{aligned}
I_{12} &= \mathbb{E}\Big[\Big|\int_0^{t_1}\left[(t_1-s)^{\beta-1}-(t_1+r-s)^{\beta-1}\right]E_{\beta,\beta}(-\lambda_i(t_1+r-s)^{\beta})d_{ij}\varphi_j(s-\tau(s))\mathrm{d}s\Big|^p\Big]\\
&\leqslant \mathbb{E}\Big[\Big|\Big(\int_0^{t_1}\left[(t_1-s)^{\beta-1}-(t_1+r-s)^{\beta-1}\right]ds\Big)^{\frac{p}{q}}\\
&\quad\times\int_0^{t_1}\left[(t_1-s)^{\beta-1}-(t_1+r-s)^{\beta-1}\right]\left[E_{\beta,\beta}(-\lambda_i(t_1+r-s)^{\beta})d_{ij}\varphi_j(s-\tau(s))\right]^p\mathrm{d}s\Big|\Big]\\
&= \Big(\frac{1}{\beta}t_1^{\beta}+\frac{1}{\beta}r^{\beta}-\frac{1}{\beta}(t_1+r)^{\beta}\Big)^{\frac{p}{q}}\\
&\quad\times\mathbb{E}\Big[\Big|\int_0^{t_1}\left[(t_1-s)^{\beta-1}-(t_1+r-s)^{\beta-1}\right]\left[E_{\beta,\beta}(-\lambda_i(t_1+r-s)^{\beta})d_{ij}\varphi_j(s-\tau(s))\right]^p\mathrm{d}s\Big|\Big]\\
&\leqslant \Big(\frac{1}{\beta}t_1^{\beta}+\frac{1}{\beta}r^{\beta}-\frac{1}{\beta}(t_1+r)^{\beta}\Big)^{\frac{p}{q}}\\
&\quad\times\mathbb{E}\Big[\Big|\int_0^{t_1}\left[(t_1-s)^{\beta-1}-(t_1+r-s)^{\beta-1}\right]\frac{M}{(1+\lambda_i(t_1+r-s)^{\beta})^p}\left[d_{ij}\varphi_j(s-\tau(s))\right]^p\mathrm{d}s\Big|\Big]\\
&\leqslant \Big(\frac{1}{\beta}t_1^{\beta}+\frac{1}{\beta}r^{\beta}-\frac{1}{\beta}(t_1+r)^{\beta}\Big)^{p}d_{ij}^p\frac{M}{(1+\lambda_i(r)^{\beta})^p}\Big|\mathbb{E}\left[|\varphi_j(s-\tau(s))|^p\right]\Big|.
\end{aligned}$$

$\left(\frac{1}{\beta}t_1^{\beta}+\frac{1}{\beta}r^{\beta}-\frac{1}{\beta}(t_1+r)^{\beta}\right)\to 0$, 当$r\to 0$ 且$\varphi\in\mathcal{S}_{\phi}$, $\mathbb{E}\left[\left|\varphi_j(s-\tau(s))\right|^p\right]$ 有界，可以得到$I_{12}\to 0$, 当$r\to 0$. 所以$I_1\to 0$, 当$r\to 0$.
对于I_2,我们相同的讨论

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\int_{t_1}^{t_1+r}\tilde{E}_{\beta,\beta}(t_1+r-s)d_{ij}\varphi_j(s-\tau(s))\mathrm{d}s\Big|^p\Big]\\
&= \mathbb{E}\Big[\Big|\int_{t_1}^{t_1+r}\left[\tilde{E}_{\beta,\beta}(t_1+r-s)\right]^{\frac{1}{q}}\left[\tilde{E}_{\beta,\beta}(t_1+r-s)\right]^{\frac{1}{p}}d_{ij}\varphi_j(s-\tau(s))\mathrm{d}s\Big|^p\Big]\\
&\leqslant \mathbb{E}\Big[\Big|\Big(\int_{t_1}^{t_1+r}\tilde{E}_{\beta,\beta}(t_1+r-s)ds\Big)^{\frac{p}{q}}\int_{t_1}^{t_1+r}\tilde{E}_{\beta,\beta}(t_1+r-s)d_{ij}^p\varphi_j(s-\tau(s))^p\mathrm{d}s\Big|\Big]\\
&\leqslant \Big(\int_{t_1}^{t_1+r}\frac{(t_1+r-s)^{\beta-1}}{1+\lambda_i(t_1+r-s)^{\beta}}ds\Big)^{\frac{p}{q}}\mathbb{E}\Big[\Big|\int_{t_1}^{t_1+r}\tilde{E}_{\beta,\beta}(t_1+r-s)d_{ij}^p\varphi_j(s-\tau(s))^p\mathrm{d}s\Big|\Big]\\
&\leqslant d_{ij}^p(\frac{1}{\lambda_i\beta})^{(p-1)}(\ln(1+\lambda_i r^{\beta}))^{p-1}\Big|\int_{t_1}^{t_1+r}\tilde{E}_{\beta,\beta}(t_1+r-s)\mathbb{E}\left[|\varphi_j(s-\tau(s))|^p\right]\mathrm{d}s\Big|.
\end{aligned}$$

显然, $\ln(1+\lambda_i r)\to 0$ 当$r\to 0$ 且φ的L^P-范数有界,我们得到$I_2\to 0$ 当$r\to 0$. 综上所得, $\mathbb{E}\left[\sum_{i=1}^n|J_{3i}(t_1+r)-J_{3i}|^p\right]\to 0$, 当$r\to 0$.
类似地，我们可以得到

$$\mathbb{E}\left[\sum_{i=1}^n|J_{4i}(t_1+r)-J_{4i}(t_1)|^p\right]\to 0, r\to 0,\quad \mathbb{E}\left[\sum_{i=1}^n|J_{5i}(t_1+r)-J_{5i}(t_1)|^p\right]\to 0, r\to 0,$$
$$\mathbb{E}\left[\sum_{i=1}^n|J_{6i}(t_1+r)-J_{6i}(t_1)|^p\right]\to 0, r\to 0,\quad \mathbb{E}\left[\sum_{i=1}^n|J_{7i}(t_1+r)-J_{7i}(t_1)|^p\right]\to 0, r\to 0.$$

使用Burkhölder不等式，我们可以得到

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^{n}\left|J_{8i}(t_1+r)-J_{8i}(t_1)\right|^p\right]\\
=\ &\mathbb{E}\Bigg[\sum_{i=1}^{n}\Bigg|\int_0^{t_1}\left[\tilde{E}_{\beta,\beta-\gamma+1}(t_1+r-s)-\tilde{E}_{\beta,\beta-\gamma+1}(t_1-s)\right]\sum_{j=1}^{n}\sigma_{ij}(s,\varphi_j(s),\varphi_j(s-\tau(s)))\mathrm{d}w_j(s)\\
&+\int_{t_1}^{t_1+r}\tilde{E}_{\beta,\beta-\gamma+1}(t_1+r-s)\sum_{j=1}^{n}\sigma_{ij}(s,\varphi_j(s),\varphi_j(s-\tau(s)))\mathrm{d}w_j(s)\Bigg|^p\Bigg]\\
\leqslant\ &c_p(2n)^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\left|\int_0^{t_1}\left[\tilde{E}_{\beta,\beta-\gamma+1}(t_1+r-s)-\tilde{E}_{\beta,\beta-\gamma+1}(t_1-s)\right]^2\sigma_{ij}^2(s,\varphi_j(s),\varphi_j(s-\tau(s)))\mathrm{d}s\right|^{\frac{p}{2}}\right]\\
&+(2n)^{p-1}\mathbb{E}\left[\left|\int_{t_1}^{t_1+r}\left[\tilde{E}_{\beta,\beta-\gamma+1}(t_1+r-s)\right]^2\sigma_{ij}^2(s,\varphi_j(s),\varphi_j(s-\tau(s)))\mathrm{d}s\right|^{\frac{p}{2}}\right].
\end{aligned}
$$

根据条件(A3)和Hölder不等式，其中$\frac{1}{p'}+\frac{1}{q'}=1,\ p'=\frac{p}{p-2},\ q'=\frac{p}{2}$，我们可以得到

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^{n}\left|J_{8i}(t_1+r)-J_{8i}(t_1)\right|^p\right]\\
\leqslant\ &c_p(2n)^{p-1}2^{\frac{p}{2}-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\left|\int_0^{t_1}\left[\tilde{E}_{\beta,\beta-\gamma+1}(t_1+r-s)-\tilde{E}_{\beta,\beta-\gamma+1}(t_1-s)\right]^2\mu_j\varphi_j(s)^2\mathrm{d}s\right|^{\frac{p}{2}}\right]\\
&+c_p(2n)^{p-1}2^{\frac{p}{2}-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\Bigg[\Bigg|\int_0^{t_1}\left[\tilde{E}_{\beta,\beta-\gamma+1}(t_1+r-s)-\tilde{E}_{\beta,\beta-\gamma+1}(t_1-s)\right]^2\\
&\times\upsilon_j\varphi_j(s-\tau(s))^2\mathrm{d}s\Bigg|^{\frac{p}{2}}\Bigg]\\
&+(2n)^{p-1}2^{\frac{p}{2}-1}\mathbb{E}\left[\left|\int_{t_1}^{t_1+r}\left[\tilde{E}_{\beta,\beta-\gamma+1}(t_1+r-s)\right]^2\mu_j\varphi_j(s)^2\mathrm{d}s\right|^{\frac{p}{2}}\right]\\
&+(2n)^{p-1}2^{\frac{p}{2}-1}\mathbb{E}\left[\left|\int_{t_1}^{t_1+r}\left[\tilde{E}_{\beta,\beta-\gamma+1}(t_1+r-s)\right]^2\upsilon_j\varphi_j(s-\tau(s))^2\mathrm{d}s\right|^{\frac{p}{2}}\right]\\
\leqslant\ &c_p(2n)^{p-1}2^{\frac{p}{2}-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\left|\int_0^{t_1}\left[\tilde{E}_{\beta,\beta-\gamma+1}(t_1+r-s)-\tilde{E}_{\beta,\beta-\gamma+1}(t_1-s)\right]^2\mu_j\varphi_j(s)^2\mathrm{d}s\right|^{\frac{p}{2}}\right]\\
&+c_p(2n)^{p-1}2^{\frac{p}{2}-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\left|\int_0^{t_1}\left[\tilde{E}_{\beta,\beta-\gamma+1}(t_1+r-s)-\tilde{E}_{\beta,\beta-\gamma+1}(t_1-s)\right]^2\upsilon_j\varphi_j(s-\tau(s))^2\mathrm{d}s\right|^{\frac{p}{2}}\right]\\
&+(2n)^{p-1}2^{\frac{p}{2}-1}\left(\int_{t_1}^{t_1+r}\left[\tilde{E}_{\beta,\beta-\gamma+1}(t_1+r-s)\right]^2ds\right)^{\frac{p-2}{2}}\mathbb{E}\left[\left|\int_{t_1}^{t_1+r}\left[\tilde{E}_{\beta,\beta-\gamma+1}(t_1+r-s)\right]^2\mu_j^{\frac{p}{2}}\varphi_j(s)^p\mathrm{d}s\right|\right]\\
&+(2n)^{p-1}2^{\frac{p}{2}-1}\left(\int_{t_1}^{t_1+r}\left[\tilde{E}_{\beta,\beta-\gamma+1}(t_1+r-s)\right]^2ds\right)^{\frac{p-2}{2}}\\
&\times\mathbb{E}\left[\left|\int_{t_1}^{t_1+r}\left[\tilde{E}_{\beta,\beta-\gamma+1}(t_1+r-s)\right]^2\upsilon_j^{\frac{p}{2}}\varphi_j(s-\tau(s))^p\mathrm{d}s\right|\right].
\end{aligned}
$$

由于

$$\int_{t_1}^{t_1+r}(t-s)^{2(\beta-\gamma)}E_{\beta,\beta-\gamma+1}(-\lambda_i(t_1+r-s)^\beta)^2ds \leqslant \int_{t_1}^{t_1+r}M^2\frac{(t_1+r-s)^{2\beta-2\gamma}}{(1+\lambda_i(t_1+r-s)^\beta)^2}ds$$
$$\leqslant M^2\int_{t_1}^{t_1+r}(t_1+r-s)^{2\beta-2\gamma}ds$$
$$= \frac{M^2}{2\beta-2\gamma+1}r^{2\beta-2\gamma+1}\to 0, as \quad r\to 0,$$

且$(t_1+r-s)^{\beta-\gamma}E_{\beta,\beta-\gamma+1}(-\lambda_i(t_1+r-s)^\beta)$连续, 所以我们得到，

当$r\to 0$, $\mathbb{E}\left[\sum_{i=1}^n|J_{8i}(t_1+r)-J_{8i}(t_1)|^p\right]\to 0.$

因此, Q连续。

步骤2. 我们证明$Q(S_\phi)\subseteq S_\phi$. 我们开始讨论以下不等式的右侧,

$$\sum_{i=1}^n\mathbb{E}\left|\left((Q\varphi)_i(s)\right)\right|^p = \sum_{i=1}^n\mathbb{E}\left[\left|\sum_{k=1}^8 J_{ki}(s)\right|^p\right] \leqslant 8^{p-1}\sum_{k=1}^8\sum_{i=1}^n\mathbb{E}\left[|J_{ki}(s)|^p\right].$$

由Hölder不等式得到

$$\mathbb{E}\left[\sum_{i=1}^n|J_{3i}|^p\right] = \mathbb{E}\left[\sum_{i=1}^n\lambda_i^p\left|\int_0^t\tilde{E}_{\beta,\beta}(t-s)^{\frac{1}{q}}\tilde{E}_{\beta,\beta}(t-s)^{\frac{1}{p}}\left(\sum_{j=1}^n d_{ij}\varphi_j(s-\tau(s))\right)\mathrm{d}s\right|^p\right]$$
$$\leqslant \sum_{i=1}^n\lambda_i^p\left(\int_0^t\tilde{E}_{\beta,\beta}(t-s)\mathrm{d}s\right)^{\frac{p}{q}}\mathbb{E}\left[\left|\int_0^t\tilde{E}_{\beta,\beta}(t-s)\left(\sum_{j=1}^n d_{ij}\varphi_j(s-\tau(s))\right)^p\mathrm{d}s\right|\right]$$
$$\leqslant \sum_{i=1}^n\lambda_i^p\left(\sum_{j=1}^n d_{ij}^q\right)^{\frac{p}{q}}\left(\int_0^t\tilde{E}_{\beta,\beta}(t-s)\mathrm{d}s\right)^{\frac{p}{q}}\sum_{j=1}^n\mathbb{E}\left[\left|\int_0^t\tilde{E}_{\beta,\beta}(t-s)\left(\varphi_j(s-\tau(s))\right)^p\mathrm{d}s\right|\right].$$

因$\sum_{i=1}^n\mathbb{E}|\varphi_i(t)|^p\to 0$ 当$t\to\infty$, 对于$\epsilon>0$, 存在t 满足$t>T_1$ 表明$\sum_{i=1}^n\mathbb{E}|\varphi_i(t)|^p<\epsilon$. 通过引理2.2和2.3, 得到

$$\mathbb{E}\left[\sum_{i=1}^n|J_{3i}|^p\right] \leqslant \sum_{i=1}^n\lambda_i^p\left(\sum_{j=1}^n d_{ij}^q\right)^{\frac{p}{q}}\left(\int_0^t\tilde{E}_{\beta,\beta}(t-s)\mathrm{d}s\right)^{\frac{p}{q}}\sum_{j=1}^n\mathbb{E}\left[\left|\int_0^t\tilde{E}_{\beta,\beta}(t-s)\left(\varphi_j(s-\tau(s))\right)^p\mathrm{d}s\right|\right]$$
$$= \sum_{i=1}^n\lambda_i^p\left(\sum_{j=1}^n d_{ij}^q\right)^{\frac{p}{q}}\left(t^\beta E_{\beta,\beta+1}\left(-\lambda_i t^\beta\right)\right)^{p-1}\mathbb{E}\left[\left|\int_0^t\tilde{E}_{\beta,\beta}(t-s)\left(\varphi_j(s-\tau(s))\right)^p\mathrm{d}s\right|\right]$$
$$\leqslant \sum_{i=1}^n\lambda_i^p\left(\sum_{j=1}^n d_{ij}^q\right)^{\frac{p}{q}}\left(\frac{Mt^\beta}{1+\lambda_i t^\beta}\right)^{p-1}\mathbb{E}\left[\left|\int_0^t\tilde{E}_{\beta,\beta}(t-s)\left(\varphi_j(s-\tau(s))\right)^p\mathrm{d}s\right|\right].$$

让我们讨论

$$
\begin{aligned}
&\mathbb{E}\left[\left|\int_0^t \tilde{E}_{\beta,\beta}(t-s)\left(\varphi_j\left(s-\tau(s)\right)\right)^p \mathrm{d}s\right|\right] \\
\leqslant\ & \mathbb{E}\left[\left|\int_0^{T_1} \tilde{E}_{\beta,\beta}(t-s)\left(\varphi_j\left(s-\tau(s)\right)\right)^p \mathrm{d}s\right|\right]+\mathbb{E}\left[\left|\int_{T_1}^t \tilde{E}_{\beta,\beta}(t-s)\left(\varphi_j\left(s-\tau(s)\right)\right)^p \mathrm{d}s\right|\right] \\
\leqslant\ & \sup_{0\leqslant s\leqslant T_1}\left[\mathbb{E}\left[\left(\varphi_j\left(s-\tau(s)\right)\right)^p\right]\right]\int_0^{T_1}\tilde{E}_{\beta,\beta}(t-s)\mathrm{d}s+\epsilon\int_{T_1}^t \tilde{E}_{\beta,\beta}(t-s)\mathrm{d}s \\
\leqslant\ & \sup_{0\leqslant s\leqslant T_1}\left[\mathbb{E}\left[\left(\varphi_j\left(s-\tau(s)\right)\right)^p\right]\right]\int_0^{T_1}\frac{M(t-s)^{(\beta-1)}}{1+\lambda_i(t-s)^\beta}\mathrm{d}s+\left(\frac{Mt^\beta}{1+\lambda_i t^\beta}\right)\epsilon.
\end{aligned}
$$

因此, $\mathbb{E}\left[\sum_{i=1}^n |J_{3i}|^p\right]\to 0$ 当$t\to\infty$.

因$\sum_{i=1}^n \mathbb{E}|\varphi_i(t)|^p\to 0$ 当$t\to\infty$, 对于任何ϵ>0, 存在t 满足t>T_2 表明$\sum_{i=1}^n \mathbb{E}|\varphi_i(t)|^p$<$\epsilon$.使用Itô等距公式, Hölder不等式，其中$\frac{1}{p'}+\frac{1}{q'}=1$, $p'=\frac{p}{p-2}$, $q'=\frac{p}{2}$ 和条件(A3), 得到

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^n |J_{8i}|^p\right] \\
\leqslant\ & n^{p-1}\sum_{i=1}^n\sum_{j=1}^n \mathbb{E}\left[\left|\int_0^t \tilde{E}_{\beta,\beta-\gamma+1}(t-s)\sigma_{ij}(s,\varphi_j(s),\varphi_j(s-\tau(s)))dW_j(s)\right|^p\right] \\
=\ & n^{p-1}\sum_{i=1}^n\sum_{j=1}^n \mathbb{E}\left[\left|\int_0^t \left[\tilde{E}_{\beta,\beta-\gamma+1}(t-s)\right]^2\sigma_{ij}^2(s,\varphi_j(s),\varphi_j(s-\tau(s)))ds\right|\right]^{\frac{p}{2}} \\
\leqslant\ & n^{p-1}\sum_{i=1}^n\sum_{j=1}^n \mathbb{E}\Bigg[\bigg|\int_0^t \left[\tilde{E}_{\beta,\beta-\gamma+1}(t-s)\right]^{2-\frac{4}{p}}\left[\tilde{E}_{\beta,\beta-\gamma+1}(t-s)\right]^{\frac{4}{p}} \\
&\times\left(\mu_j\varphi_j^2(s)+\upsilon_j\varphi_j^2(s-\tau(s))\right)ds\bigg|^{\frac{p}{2}}\Bigg] \\
\leqslant\ & n^{p-1}\sum_{i=1}^n\sum_{j=1}^n 2^{\frac{p}{2}-1}\left(\int_0^t\left[\tilde{E}_{\beta,\beta-\gamma+1}(t-s)\right]^2ds\right)^{\frac{p-2}{2}} \\
&\times\mathbb{E}\left[\left|\int_0^t\left[\tilde{E}_{\beta,\beta-\gamma+1}(t-s)\right]^2\left(\mu_j^{\frac{p}{2}}\varphi_j^p(s)\right)ds\right|\right]+n^{p-1}\sum_{i=1}^n\sum_{j=1}^n 2^{\frac{p}{2}-1}\left(\int_0^t\left[\tilde{E}_{\beta,\beta-\gamma+1}(t-s)\right]^2ds\right)^{\frac{p-2}{2}} \\
&\times\mathbb{E}\left[\left|\int_0^t\left[\tilde{E}_{\beta,\beta-\gamma+1}(t-s)\right]^2\left(\upsilon_j^{\frac{p}{2}}\varphi_j^p(s-\tau(s))\right)ds\right|\right] \\
\leqslant\ & n^{p-1}\sum_{i=1}^n 2^{\frac{p}{2}-1}\left(\mu^{\frac{p}{2}}+\upsilon^{\frac{p}{2}}\right) \\
&\times\left(\int_0^t\left[\tilde{E}_{\beta,\beta-\gamma+1}(t-s)\right]^2ds\right)^{\frac{p-2}{2}}\int_0^t\left[\tilde{E}_{\beta,\beta-\gamma+1}(t-s)\right]^2\sum_{j=1}^n\mathbb{E}|\varphi_j(s)|^p ds \\
\leqslant\ & n^{p-1}\sum_{i=1}^n 2^{\frac{p}{2}-1}\left(\mu^{\frac{p}{2}}+\upsilon^{\frac{p}{2}}\right)\left(\frac{M^2}{2\beta-2\gamma+1}T_1^{2\beta-2\gamma+1}+\frac{M^2}{(2\gamma-1)\lambda_i^2T_1^{2\gamma-1}}\right)^{\frac{p-2}{2}} \\
&\times\left(\int_0^{T_1}\left(\frac{Mt^{\beta-1}}{1+\lambda_i t^\beta}\right)^2\sum_{j=1}^n\sup_{0\leqslant s\leqslant T_1}\left[\mathbb{E}\left(\varphi_j^p(s)\right)\right]ds+\left(\frac{M^2}{2\beta-2\gamma+1}T_1^{2\beta-2\gamma+1}+\frac{M^2}{(2\gamma-1)\lambda_i^2T_1^{2\gamma-1}}\right)\epsilon\right).
\end{aligned}
$$

在上一个不等式中，我们借助引理2.2和$0<T_1<t$使用了以下估计，

$$\begin{aligned}
&\int_0^t (t-s)^{2\beta-2\gamma}\left[E_{\beta,\beta-\gamma+1}(-\lambda(t-s)^\beta)\right]^2\mathrm{d}s \\
=\;& \int_0^t s^{2\beta-2\gamma}\left[E_{\beta,\beta-\gamma+1}(-\lambda s^\beta)\right]^2\mathrm{d}s \\
=\;& \int_0^{T_1} s^{2\beta-2\gamma}\left[E_{\beta,\beta-\gamma+1}(-\lambda s^\beta)\right]^2\mathrm{d}s+\int_{T_1}^t s^{2\beta-2\gamma}\left[E_{\beta,\beta-\gamma+1}(-\lambda s^\beta)\right]^2\mathrm{d}s \\
\leqslant\;& \int_0^{T_1}\frac{M^2 s^{(2\beta-2\gamma)}}{(1+\lambda s^\beta)^2}\mathrm{d}s+\int_{T_1}^t\frac{M^2 s^{(2\beta-2\gamma)}}{(1+\lambda s^\beta)^2}\mathrm{d}s \\
\leqslant\;& M^2\int_0^{T_1} s^{(2\beta-2\gamma)}\mathrm{d}s+M^2\int_{T_1}^t\frac{s^{(2\beta-2\gamma)}}{\lambda^2 s^{2\beta}}\mathrm{d}s \\
=\;& M^2\left(\frac{1}{2\beta-2\gamma+1}T_1^{2\beta-2\gamma+1}+\frac{1}{(2\gamma-1)\lambda^2}\left(\frac{1}{T_1^{2\gamma-1}}-\frac{1}{t^{2\gamma-1}}\right)\right) \\
\leqslant\;& M^2\left(\frac{1}{2\beta-2\gamma+1}T_1^{2\beta-2\gamma+1}+\frac{1}{(2\gamma-1)\lambda^2T_1^{2\gamma-1}}\right).
\end{aligned}$$

因此, $\mathbb{E}\left[\sum_{i=1}^n|J_{8i}|^p\right]\to 0$, 当$t\to\infty$. 使用类似的论点，我们得到$\sum_{i=1}^n\mathbb{E}\left[|(Q\varphi)_i(s)|^p\right]\to 0$ 当$t\to\infty$. 因此, $Q(\mathcal{S}_\phi)\subseteq\mathcal{S}_\phi$.

步骤3. 我们证明Q 是一个压缩映射。对于任意$\varphi,\psi\in\mathcal{S}_\phi$, 得到

$$\begin{aligned}
&\sup_{t\geqslant 0}\left\{\mathbb{E}\left[\sum_{i=1}^n\left|(P\varphi)_i(t)-(P\psi)_i(t)\right|^p\right]\right\} \\
\leqslant\;& 7^{p-1}\sup_{t\geqslant 0}\left\{\mathbb{E}\left[\sum_{i=1}^n\left|\sum_{j=1}^n d_{ij}\left(\varphi_j(t-\tau(t))-\psi_j(t-\tau(t))\right)\right|^p\right]\right\} \\
&+7^{p-1}\sup_{t\geqslant 0}\left\{\mathbb{E}\left[\sum_{i=1}^n\left|\lambda_i\int_0^t(t-s)^{\beta-1}E_{\beta,\beta}(-\lambda_i(t-s)^\beta)\sum_{j=1}^n d_{ij}\left(\varphi_j(s-\tau(s))-\psi_j(s-\tau(s))\right)ds\right|^p\right]\right\} \\
&+7^{p-1}\sup_{t\geqslant 0}\left\{\mathbb{E}\left[\sum_{i=1}^n\left|\int_0^t\tilde{E}_{\beta,\beta}(t-s)\sum_{j=1}^n\tilde{c}_{ij}\left(\varphi_j(s)-\psi_j(s)\right)ds\right|^p\right]\right\} \\
&+7^{p-1}\sup_{t\geqslant 0}\left\{\mathbb{E}\left[\sum_{i=1}^n\left|\int_0^t\tilde{E}_{\beta,\beta}(t-s)\sum_{j=1}^n a_{ij}\left(f_j(\varphi_j(s))-f_j(\psi_j(s))\right)ds\right|^p\right]\right\} \\
&+7^{p-1}\sup_{t\geqslant 0}\left\{\mathbb{E}\left[\sum_{i=1}^n\left|\int_0^t\tilde{E}_{\beta,\beta}(t-s)\sum_{j=1}^n b_{ij}\left(g_j(\varphi_j(s-\tau(s)))-g_j(\psi_j(s-\tau(s)))\right)ds\right|^p\right]\right\} \\
&+7^{p-1}\sup_{t\geqslant 0}\left\{\mathbb{E}\left[\sum_{i=1}^n\left|\int_0^t\tilde{E}_{\beta,\beta}(t-s)\sum_{j=1}^n l_{ij}\int_{s-r(s)}^s\left(h_j(\varphi_j(u))-h_j(\psi_j(u))\right)duds\right|^p\right]\right\} \\
&+7^{p-1}\sup_{t\geqslant 0}\left\{\mathbb{E}\left[\sum_{i=1}^n\left|\int_0^t\tilde{E}_{\beta,\beta-\gamma+1}(t-s)\right.\right.\right. \\
&\left.\left.\left.\times\sum_{j=1}^n\left[\sigma_{ij}(s,\varphi_j(s),\varphi_j(s-\tau(s)))-\sigma_{ij}(s,\psi_j(s),\psi_j(s-\tau(s)))\right]dW_j(s)\right|^p\right]\right\}.
\end{aligned}$$

通过步骤2，我们得到

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=1}^{n}|J_{4i}|^{p}\right] &\leqslant \sum_{i=1}^{n}\left(\sum_{j=1}^{n}|\tilde{c}_{ij}|^{q}\right)^{\frac{p}{q}}\sum_{j=1}^{n}\left(\frac{Mt^{\beta}}{1+\lambda_i t^{\beta}}\right)^{p-1}\mathbb{E}\left[\int_0^t \tilde{E}_{\beta,\beta}(t-s)\,|\varphi_j(s)|^p\,\mathrm{d}s\right],\\
\mathbb{E}\left[\sum_{i=1}^{n}|J_{5i}|^{p}\right] &\leqslant \sum_{i=1}^{n}\left(\sum_{j=1}^{n}|a_{ij}|^{q}|\alpha_j|^q\right)^{\frac{p}{q}}\sum_{j=1}^{n}\left(\frac{Mt^{\beta}}{1+\lambda_i t^{\beta}}\right)^{p-1}\mathbb{E}\left[\int_0^t \tilde{E}_{\beta,\beta}(t-s)\,|\varphi_j(s)|^p\,\mathrm{d}s\right],\\
\mathbb{E}\left[\sum_{i=1}^{n}|J_{6i}|^{p}\right] &\leqslant \sum_{i=1}^{n}\left(\sum_{j=1}^{n}|b_{ij}|^{q}|\beta_j|^q\right)^{\frac{p}{q}}\sum_{j=1}^{n}\left(\frac{Mt^{\beta}}{1+\lambda_i t^{\beta}}\right)^{p-1}\mathbb{E}\left[\int_0^t \tilde{E}_{\beta,\beta}(t-s)\,|\varphi_j(s-\tau(s))|^p\,\mathrm{d}s\right],\\
\mathbb{E}\left[\sum_{i=1}^{n}|J_{7i}|^{p}\right] &\leqslant r^p\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|l_{ij}|^{q}|\gamma_j|^q\right)^{\frac{p}{q}}\sum_{j=1}^{n}\left(\frac{Mt^{\beta}}{1+\lambda_i t^{\beta}}\right)^{p-1}\mathbb{E}\left[\int_0^t \tilde{E}_{\beta,\beta}(t-s)\,|\varphi_j(s)|^p\,\mathrm{d}s\right].
\end{aligned}$$

因此，我们有

$$\begin{aligned}
&\sup_{t\geqslant 0}\left\{\mathbb{E}\left[\sum_{i=1}^{n}\left|(P\varphi)_i(t)-(P\psi)_i(t)\right|^p\right]\right\}\\
\leqslant\ & 7^{p-1}\sum_{i=1}^{n}\Bigg\{\left(\sum_{j=1}^{n}|d_{ij}|^q\right)^{\frac{p}{q}}+\left(\int_0^t\tilde{E}_{\beta,\beta}(t-s)\mathrm{d}s\right)^p\Bigg[\lambda_i^p\left(\sum_{j=1}^{n}|d_{ij}|^q\right)^{\frac{p}{q}}\\
&+\left(\sum_{j=1}^{n}|\tilde{c_{ij}}|^q\right)^{\frac{p}{q}}+\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{\frac{p}{q}}+\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{\frac{p}{q}}+r^p\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{\frac{p}{q}}\Bigg]\\
&+n^{p-1}2^{\frac{p}{2}-1}\left(\mu^{\frac{p}{2}}+\upsilon^{\frac{p}{2}}\right)\left(\int_0^t\left[\tilde{E}_{\beta,\beta-\gamma+1}(t-s)\right]^2\mathrm{d}s\right)^{\frac{p}{2}}\Bigg\}\sup_{t\geqslant 0}\left\{\mathbb{E}\left[\sum_{i=1}^{n}|\varphi_i(t)-\psi_i(t)|^p\right]\right\}\\
\leqslant\ & 7^{p-1}\sum_{i=1}^{n}\Bigg\{\left(\sum_{j=1}^{n}|d_{ij}|^q\right)^{\frac{p}{q}}+\left(\frac{M}{\lambda_i}\right)^p\Bigg[\lambda_i^p\left(\sum_{j=1}^{n}|d_{ij}|^q\right)^{\frac{p}{q}}\\
&+\left(\sum_{j=1}^{n}|\tilde{c_{ij}}|^q\right)^{\frac{p}{q}}+\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{\frac{p}{q}}+\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{\frac{p}{q}}+r^p\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{\frac{p}{q}}\Bigg]\\
&+n^{p-1}2^{\frac{p}{2}-1}\left(\mu^{\frac{p}{2}}+\upsilon^{\frac{p}{2}}\right)\left(\frac{M^2}{2\beta-2\gamma+1}T_1^{2\beta-2\gamma+1}+\frac{M^2}{(2\gamma-1)\lambda_i^2T_1^{2\gamma-1}}\right)^{\frac{p}{2}}\sup_{t\geqslant 0}\left\{\mathbb{E}\left[\sum_{i=1}^{n}|\varphi_i(t)-\psi_i(t)|^p\right]\right\}.
\end{aligned}$$

因此，由(5),我们得到$Q:\mathcal{S}_\phi\to\mathcal{S}_\phi$ 是压缩映射。由引理2.2 和压缩映射原理, 我们发现Q 具有唯一的不动点$x(t)$是方程(1)的解。

步骤4. 我们证明方程解是p阶矩稳定的。由引理2.2, 让我们选择$\delta\in(0,\epsilon)$ 满足$8^{p-1}M^p\delta<(1-\alpha)\epsilon$, 其中$\alpha$ 是方程(5)左边界. 如果$x(t)=(x_1(t),\cdots,x_n(t))^T$ 是方程(1)的解且满足$\|\phi\|^p<\delta$, 所以$x(t)=(Qx)(t)$. 我们定义$\sum_{i=1}^{n}\mathbb{E}|x_i(t)|^p<\epsilon$ 对于所有$t\geqslant 0$. 我们假设存在$t^*>0$ 满足$\sum_{i=1}^{n}\mathbb{E}|x_i(t^*)|^p=\epsilon$

且$\sum_{i=1}^{n}\mathbb{E}|x_i(t)|^p<\epsilon$ 对于$\vartheta\leqslant t<t^*$. 我们现在估计$\sum_{i=1}^{n}\mathbb{E}|x_i(t^*)|^p$

$$\begin{aligned}&\sum_{i=1}^{n}\mathbb{E}|x_i(t^*)|^p\\ \leqslant\ & 8^{p-1}\mathbb{E}\Bigg[\sum_{i=1}^{n}\Big|\phi_i(0)-\sum_{j=1}^{n}d_{ij}\phi_j(0-\tau_j(0))\Big|^p\Big|E_{\beta,1}(-\lambda_i t^{*\beta})\Big|^p\Bigg]+8^{p-1}\mathbb{E}\Bigg[\sum_{i=1}^{n}\Big|\sum_{j=1}^{n}d_{ij}x_j(t^*-\tau(t^*))\Big|^p\Bigg]\\ &+8^{p-1}\mathbb{E}\Bigg[\sum_{i=1}^{n}\Big|\lambda_i\int_0^{t^*}(t^*-s)^{\beta-1}E_{\beta,\beta}(-\lambda_i(t^*-s)^\beta)\sum_{j=1}^{n}d_{ij}x_j(s-\tau(s))ds\Big|^p\Bigg]\\ &+8^{p-1}\mathbb{E}\Bigg[\sum_{i=1}^{n}\Big|\int_0^{t^*}(t^*-s)^{\beta-1}E_{\beta,\beta}(-\lambda_i(t^*-s)^\beta)\sum_{j=1}^{n}\tilde{c}_{ij}x_j(s)ds\Big|^p\Bigg]\\ &+8^{p-1}\mathbb{E}\Bigg[\sum_{i=1}^{n}\Big|\int_0^{t^*}(t^*-s)^{\beta-1}E_{\beta,\beta}(-\lambda_i(t^*-s)^\beta)\sum_{j=1}^{n}a_{ij}f_j(x_j(s))ds\Big|^p\Bigg]\\ &+8^{p-1}\mathbb{E}\Bigg[\sum_{i=1}^{n}\Big|\int_0^{t^*}(t^*-s)^{\beta-1}E_{\beta,\beta}(-\lambda_i(t^*-s)^\beta)\sum_{j=1}^{n}b_{ij}g_j(x_j(s-\tau(s))ds\Big|^p\Bigg]\\ &+8^{p-1}\mathbb{E}\Bigg[\sum_{i=1}^{n}\Big|\int_0^{t^*}(t^*-s)^{\beta-1}E_{\beta,\beta}(-\lambda_i(t^*-s)^\beta)\sum_{j=1}^{n}l_{ij}\int_{s-r(s)}^{s}h_j(x_j(v)dvds\Big|^p\Bigg]\\ &+8^{p-1}\mathbb{E}\Bigg[\sum_{i=1}^{n}\Big|\int_0^{t^*}(t^*-s)^{\beta-\gamma}E_{\beta,\beta-\gamma+1}(-\lambda_i(t^*-s)^\beta)\sum_{j=1}^{n}\sigma_{ij}(s,x_j(s),x_j(s-\tau(s)))dW_j(s)\Big|^p\Bigg].\end{aligned}$$

由(5) 和t^*的猜想, 可以得到

$$\begin{aligned}\sum_{i=1}^{n}\mathbb{E}|x_i(t^*)|^p\leqslant\ & 8^{p-1}\mathbb{E}\left[\sum_{i=1}^{n}\Big|\phi_i(0)-\sum_{j=1}^{n}d_{ij}\phi_j(t^*-\tau_j(0))\Big|^p\ \big|E_{\beta,1}(-\lambda_i t^{*\beta})\big|^p\right]\\ &+8^{p-1}\sum_{i=1}^{n}\Bigg\{\left(\sum_{j=1}^{n}|d_{ij}|^q\right)^{\frac{p}{q}}+\left(\frac{Mt^\beta}{1+\lambda_i t^\beta}\right)^p\Bigg[\lambda_i^p\left(\sum_{j=1}^{n}|d_{ij}|^q\right)^{\frac{p}{q}}\\ &+\left(\sum_{j=1}^{n}|\tilde{c_{ij}}|^q\right)^{\frac{p}{q}}+\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{\frac{p}{q}}+\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{\frac{p}{q}}+r^p\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{\frac{p}{q}}\Bigg]\\ &+n^{p-1}2^{\frac{p}{2}-1}\left(\mu^{\frac{p}{2}}+\upsilon^{\frac{p}{2}}\right)\left(\frac{M^2}{2\beta-2\gamma+1}T_1^{2\beta-2\gamma+1}+\frac{M^2}{(2\gamma-1)\lambda_i^2T_1^{2\gamma-1}}\right)^{\frac{p}{2}}\Bigg\}\epsilon\\ \leqslant\ & 8^{p-1}M^p\delta+8^{p-1}\sum_{i=1}^{n}\Bigg\{\left(\sum_{j=1}^{n}|d_{ij}|^q\right)^{\frac{p}{q}}+\left(\frac{M}{\lambda_i}\right)^p\Bigg[\lambda_i^p\left(\sum_{j=1}^{n}|d_{ij}|^q\right)^{\frac{p}{q}}\\ &+\left(\sum_{j=1}^{n}|\tilde{c_{ij}}|^q\right)^{\frac{p}{q}}+\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{\frac{p}{q}}+\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{\frac{p}{q}}+r^p\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{\frac{p}{q}}\Bigg]\\ &+n^{p-1}2^{\frac{p}{2}-1}\left(\mu^{\frac{p}{2}}+\upsilon^{\frac{p}{2}}\right)\left(\frac{M^2}{2\beta-2\gamma+1}T_1^{2\beta-2\gamma+1}+\frac{M^2}{(2\gamma-1)\lambda_i^2T_1^{2\gamma-1}}\right)^{\frac{p}{2}}\Bigg\}\epsilon\\ <\ & (1-\alpha)\epsilon+\alpha\epsilon=\epsilon,\end{aligned}$$

是矛盾的。因此，方程(1)平凡解在p阶矩中是渐近稳定的。

证明结束。

4. 结论

在这项工作中，我们利用不动点定理研究了无限时间中具有离散和分布时滞的分数阶随机微分方程解的存在性、唯一性和渐近稳定性。在不同的空间和规范下，这项工作可能会得到不同的结果，我们将在未来继续进一步研究。

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