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一类 3×3 块鞍点系统解的结构向后误差分析

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摘 要

近年来，一些作者提出了很多求解一类特殊的 3×3 块鞍点系统的有效迭代方法。为了便于评估这些数值算法的强稳定性，本文对这种类型的 3×3 块鞍点系统进行了结构向后误差分析，并给出了结构向后误差的可计算的具体表达式。数值实验表明，该表达式可用于检验实际算法的稳定性。

关键词

3×3 块鞍点问题，向后误差，结构向后误差

Structured Backward Error Analysis for a Class of Block Three-by-Three Saddle Point System

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Abstract

In recent years, a number of authors have proposed efficient iteration methods for

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solving a special class of block 3×3 saddle point systems. In order to evaluate the strong stability of these numerical algorithms, this paper perform the structured backward error analysis for this type of block 3×3 saddle point system and present an explicit and computable formula for the structured backward error. Numerical example show that the expressions are useful for testing the stability of practical algorithms.

Keywords

Block 3×3 Saddle Point Problem, Backward Error, Structured Backward Error

1. 引言

本文研究了以下 3×3 块鞍点问题的结构向后误差

$$\begin{bmatrix} A & B^T & 0 \\ B & 0 & C^T \\ 0 & C & D \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} f \\ g \\ h \end{bmatrix}, \tag{1.1}$$

其中 $A \in \mathbb{R}^{n\times n}$ 是对称正定矩阵, $B \in \mathbb{R}^{m\times n}$ 是行满秩矩阵, 并且 $m \le n$, $D \in \mathbb{R}^{l\times l}$ 是对称半正定矩阵, $C \in \mathbb{R}^{l\times m}$, 且当 D 不是对称正定矩阵时, C 是行满秩矩阵. B^T 代表 B 的转置, C^T 代表 C 的转置. 对 A, B, C, D 的假设保证了线性系统 (1.1) 有唯一解. 向后误差分析可以回答实际解决的问题与我们想要解决的问题有多接近, 并揭示数值算法的稳定性 [1,2]. 线性系统 (1.1) 出现于众多科学与工程领域中, 比如线性等式约束的最小二乘问题 [3,4], 求解非连续系数时变 Maxwell 方程的全离散有限元法 [5,6], 求解发散形式线性二阶椭圆方程的双对偶混合有限元法 [7,8]等.

3×3 块鞍点系统 (1.1) 可以恰当地分为广义鞍点问题, 即:

$$\left[\begin{array}{cc|c} A & B^T & 0 \\ B & 0 & C^T \\ \hline 0 & C & D \end{array}\right] \left[\begin{array}{c} x \\ y \\ \hline z \end{array}\right] = \left[\begin{array}{c} f \\ g \\ \hline h \end{array}\right] \text{ 或 } \left[\begin{array}{c|cc} A & B^T & 0 \\ \hline B & 0 & C^T \\ 0 & C & D \end{array}\right] \left[\begin{array}{c} x \\ \hline y \\ z \end{array}\right] = \left[\begin{array}{c} f \\ \hline g \\ h \end{array}\right].$$

近年来, 很多学者 [9–13]对广义鞍点系统进行了结构向后误差分析, 证明了相应的数值算法具有较强的稳定性. 尽管 3×3 块鞍点系统 (1.1) 可以被认为是广义的 (2×2 块) 鞍点问题, 但因为它具有特

殊的 3×3 块结构, 上述结构向后误差分析并不能准确地表示系统 (1.1). 在本文中, 我们将对 3×3 块鞍点系统 (1.1) 进行结构向后误差分析.

为了简便, 把 (1.1) 记作

$$\mathcal{A}t = d.$$

对任意计算解 $\tilde{t} = \left[\tilde{x}^T, \tilde{y}^T, \tilde{z}^T\right]^T$, 定义其无结构的向后误差 $\eta(\tilde{t})$ 为

$$\eta(\tilde{t}) = \min_{\Delta\mathcal{A},\Delta d}\left\{\left\|\left(\frac{\|\Delta\mathcal{A}\|_F}{\|\mathcal{A}\|_F}, \frac{\|\Delta d\|_2}{\|d\|_2}\right)\right\|_2 : (\mathcal{A}+\Delta\mathcal{A})\tilde{t} = d+\Delta d\right\},$$

上面的式子可以进一步表示为 [14]

$$\eta(\tilde{t}) = \frac{\|d-\mathcal{A}\tilde{t}\|_2}{\sqrt{\|\mathcal{A}\|_F^2\|\tilde{t}\|_2^2+\|d\|_2^2}}, \tag{1.2}$$

其中 $\|\cdot\|_F$ 和 $\|\cdot\|_2$ 分别表示矩阵的 Frobenius 范数和 2-范数. 若 $\eta(\tilde{t})$ 是机器精度的同量级, 则计算解 $\tilde{t}$ 的计算过程是向后稳定的，即 $\tilde{t}$ 是一个向后稳定解. 需要注意的是, 如果系数矩阵 $\mathcal{A}$ 具有某种特殊的结构, 自然要求 $\mathcal{A}+\Delta\mathcal{A}$ 也具有与 $\mathcal{A}$ 相同的结构. 在这种情况下, 自然要考虑结构后向误差.

令 $\tilde{t} = \left(\tilde{x}^T, \tilde{y}^T, \tilde{z}^T\right)^T$ 是系统 (1.1) 的计算解, 定义结构向后误差 $\eta_S(\tilde{x},\tilde{y},\tilde{z})$ 为

$$\eta_S(\tilde{x},\tilde{y},\tilde{z}) = \min_{\begin{pmatrix}\Delta A,\Delta B,\Delta C,\\ \Delta D,\Delta f,\Delta g,\\ \Delta h\end{pmatrix}\in\mathcal{F}}\left\|\begin{bmatrix}\frac{\|\Delta A\|_F}{\|A\|_F} & \frac{\|\Delta B\|_F}{\|B\|_F} & \frac{\|\Delta C\|_F}{\|C\|_F}\\ \frac{\|\Delta D\|_F}{\|D\|_F} & \frac{\|\Delta f\|_2}{\|f\|_2} & \frac{\|\Delta g\|_2}{\|g\|_2}\\ \frac{\|\Delta h\|_2}{\|h\|_2} & 0 & 0\end{bmatrix}\right\|_F,$$

其中

$$\mathcal{F} = \left\{\begin{pmatrix}\Delta A,\Delta B,\Delta C,\\ \Delta D,\Delta f,\Delta g,\\ \Delta h\end{pmatrix} : \begin{bmatrix}A+\Delta A & (B+\Delta B)^T & 0\\ B+\Delta B & 0 & (C+\Delta C)^T\\ 0 & C+\Delta C & D+\Delta D\end{bmatrix}\begin{bmatrix}\tilde{x}\\ \tilde{y}\\ \tilde{z}\end{bmatrix} = \begin{bmatrix}f+\Delta f\\ g+\Delta g\\ h+\Delta h\end{bmatrix}, \begin{array}{c}\Delta A = \Delta A^T,\\ \Delta D = \Delta D^T\end{array}\right\}. \tag{1.3}$$

若计算解 $\tilde{t}$ 的结构向后误差是机器精度的同量级, 则计算解 $\tilde{t}$ 是一个结构向后稳定解, 相应的数值算法是结构向后稳定的 (或强稳定 [15]). 因此, 给出结构向后误差 $\eta_S(\tilde{x},\tilde{y},\tilde{z})$ 的可计算的具体表达式将有助于测试实际数值算法的稳定性. 为此, 进一步定义 $\eta^{(\theta_1,\theta_2,\theta_3,\theta_4,\lambda_1,\lambda_2,\lambda_3)}(\tilde{x},\tilde{y},\tilde{z})$ 为

$$\begin{aligned}&\eta^{(\theta_1,\theta_2,\theta_3,\theta_4,\lambda_1,\lambda_2,\lambda_3)}(\tilde{x},\tilde{y},\tilde{z})\\ &= \min_{\begin{pmatrix}\Delta A,\Delta B,\Delta C,\\ \Delta D,\Delta f,\Delta g,\\ \Delta h\end{pmatrix}\in\mathcal{F}}\left\|\begin{bmatrix}\theta_1\|\Delta A\|_F & \theta_2\|\Delta B\|_F & \theta_3\|\Delta C\|_F\\ \theta_4\|\Delta D\|_F & \lambda_1\|\Delta f\|_2 & \lambda_2\|\Delta g\|_2\\ \lambda_3\|\Delta h\|_2 & 0 & 0\end{bmatrix}\right\|_F,\end{aligned} \tag{1.4}$$

其中 $\theta_1,\theta_2,\theta_3,\theta_4,\lambda_1,\lambda_2$ 和 λ_3 为正参数. 若 $A\neq 0,B\neq 0,C\neq 0,D\neq 0,f\neq 0,g\neq 0$ 和 $h\neq 0$, 则取

$$\tilde{\theta}_1=\frac{1}{\|A\|_F},\tilde{\theta}_2=\frac{1}{\|B\|_F},\tilde{\theta}_3=\frac{1}{\|C\|_F},\tilde{\theta}_4=\frac{1}{\|D\|_F},\tilde{\lambda}_1=\frac{1}{\|f\|_2},\tilde{\lambda}_2=\frac{1}{\|g\|_2},\ \text{and}\ \tilde{\lambda}_3=\frac{1}{\|h\|_2},$$

从而有

$$\eta_S(\tilde{x},\tilde{y},\tilde{z})=\eta^{\left(\tilde{\theta}_1,\tilde{\theta}_2,\tilde{\theta}_3,\tilde{\theta}_4,\tilde{\lambda}_1,\tilde{\lambda}_2,\tilde{\lambda}_3\right)}(\tilde{x},\tilde{y},\tilde{z}). \tag{1.5}$$

本文结构安排如下：

在第 2 节中, 主要介绍了 Kronecker 积，和后面证明中需要用到的几个关键的引理.

在第 3 节中, 推导了 3×3 块鞍点系统 (1.1) 结构向后误差的具体的可计算的表达式.

在第 4 节中, 通过数值实验来证明推导出的表达式对于测试求解 3×3 块鞍点系统的实际数值算法稳定性是有用的.

在第 5 节中, 对全文工作进行了总结.

2. 预备知识

本文用 $\mathbb{R}^{m\times n}$ 和 $\mathbb{SR}^{n\times n}$ 分别表示 $m\times n$ 阶实矩阵的集合和 $n\times n$ 阶实对称矩阵的集合, 用 $A^\dagger$ 表示矩阵 A 的 Moore-Penrose 逆, 设 $X=(x_{ij})\in\mathbb{R}^{m\times n}$, $Z\in\mathbb{R}^{p\times q}$. X 和 Z 之间的 Kronecker 积定义为(见文献 [16], 第4 章])

$$X\otimes Z=(x_{ij}Z)\in\mathbb{R}^{mp\times nq}.$$

由文献 [16]知

$$\operatorname{vec}(XYZ)=\left(Z^T\otimes X\right)\operatorname{vec}(Y)\ , \tag{2.1}$$

$$(X\otimes Z)^T=X^T\otimes Z^T, \tag{2.2}$$

$$(X\otimes Z)(C\otimes G)=(XC\otimes ZG)\ , \tag{2.3}$$

其中 $Y\in\mathbb{R}^{n\times p}$, C 和 G 为合适阶数的矩阵, vec (Y) 定义为将矩阵 Y 的所有列依次堆叠成一个向量后得到的向量.

在本节中, 我们将介绍一些引理, 这些引理将在接下来的章节中使用.

引理2.1 ([11,17]) 设 $u\in\mathbb{R}^m$ 和 $p\in\mathbb{R}^n$ 已知. 定义

$$\mathcal{X}=\left\{X\in\mathbb{R}^{n\times m}:Xu=p\right\}.$$

则, $\mathcal{X}\neq\emptyset$ 当且仅当 $pu^\dagger u=p$, 且在此情况下, 任意的 $X\in\mathcal{X}$ 可表示为

$$X=pu^\dagger+Z\left(I_m-uu^\dagger\right),Z\in\mathbb{R}^{n\times m}.$$

引理2.2 ([11,17]) 设 $b, c \in \mathbb{R}^n$ 已知. 定义

$$\mathcal{H} = \left\{ H \in \mathbb{SR}^{n\times n} : Hb = c \right\}.$$

则, $\mathcal{H} \neq \emptyset$ 当且仅当 b 和 c 满足 $cb^\dagger b = c$, 且在此条件下, 任意的 $H \in \mathcal{H}$ 可表示为

$$H = cb^\dagger + \left(b^\dagger\right)^T c^T \left(I_n - bb^\dagger\right) + \left(I_n - bb^\dagger\right) T \left(I_n - bb^\dagger\right),$$

其中 $T \in \mathbb{SR}^{n\times n}$.

引理2.3 ([11,17]) 假设 $F \in \mathbb{R}^{p\times m}, G \in \mathbb{R}^{n\times q}$ 和 $K \in \mathbb{R}^{p\times q}$ 已知, 并令 $X^* = F^\dagger K G^\dagger$. 则

$$\min_{X\in\mathbb{R}^{m\times n}} \|FXG - K\|_F = \|FX^*G - K\|_F.$$

3. 求解系统 (1.1) 的结构向后误差问题

为了给出 $\eta^{(\theta_1,\theta_2,\theta_3,\theta_4,\lambda_1,\lambda_2,\lambda_3)}(\tilde{x},\tilde{y},\tilde{z})$ 的结构向后误差的明确表达式. 我们首先研究了部分结构向后误差 $\eta^{(\theta_1,\theta_2,\theta_3,\theta_4)}(\tilde{x},\tilde{y},\tilde{z})$, 其定义为

$$\eta^{(\theta_1,\theta_2,\theta_3,\theta_4)}(\tilde{x},\tilde{y},\tilde{z}) = \min_{(\Delta A,\Delta B,\Delta C,\Delta D)\in\mathcal{F}_0} \left\| \begin{bmatrix} \theta_1\|\Delta A\|_F & \theta_2\|\Delta B\|_F \\ \theta_3\|\Delta C\|_F & \theta_4\|\Delta D\|_F \end{bmatrix} \right\|_F, \tag{3.1}$$

其中

$$\mathcal{F}_0 = \left\{ \begin{pmatrix} \Delta A, \Delta B, \\ \Delta C, \Delta D \end{pmatrix} : \begin{bmatrix} A+\Delta A & (B+\Delta B)^T & 0 \\ B+\Delta B & 0 & (C+\Delta C)^T \\ 0 & C+\Delta C & D+\Delta D \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix} = \begin{bmatrix} f \\ g \\ h \end{bmatrix},\ \begin{matrix} \Delta A = \Delta A^T, \\ \Delta D = \Delta D^T \end{matrix} \right\}. \tag{3.2}$$

下面给出 $\eta^{(\theta_1,\theta_2,\theta_3,\theta_4)}(\tilde{x},\tilde{y},\tilde{z})$ 的明确表达式.

定理 3.1 假设 $(\tilde{x}^T,\tilde{y}^T,\tilde{z}^T)^T$ 满足 $\tilde{x}\neq 0$ 和 $\tilde{z}\neq 0$ 为系统 (1.1) 的一个计算解. 则

$$\begin{aligned}\left[\eta^{(\theta_1,\theta_2,\theta_3,\theta_4)}(\tilde{x},\tilde{y},\tilde{z})\right]^2 =& \frac{2\theta_1^2\theta_2^2}{\gamma_1}\|r_f\|_2^2 + \frac{\theta_2^2\theta_3^2}{\gamma_2}\|r_g\|_2^2 + \frac{2\theta_3^2\theta_4^2}{\gamma_3}\|r_h\|_2^2 + \frac{\theta_1^2\gamma_5}{\gamma_1\gamma_4}\left(r_f^T\tilde{x}\right)^2 \\ &+ \frac{\gamma_6}{\gamma_2\gamma_4}\left(r_g^T\tilde{y}\right)^2 + \frac{\theta_4^2\gamma_7}{\gamma_3\gamma_4}\left(r_h^T\tilde{z}\right)^2 - \frac{2\theta_1^2\gamma_8}{\gamma_4}\left(r_f^T\tilde{x}\right)\left(r_g^T\tilde{y}\right) \\ &+ \frac{2\theta_1^2\theta_4^2\|\tilde{y}\|_2^2}{\gamma_4}\left(r_f^T\tilde{x}\right)\left(r_h^T\tilde{z}\right) - \frac{2\theta_4^2\gamma_9}{\gamma_4}\left(r_g^T\tilde{y}\right)\left(r_h^T\tilde{z}\right),\end{aligned} \tag{3.3}$$

其中

$$r_f = f - A\tilde{x} - B^T\tilde{y},\ \ r_g = g - B\tilde{x} - C^T\tilde{z},\ \ r_h = h - C\tilde{y} - D\tilde{z},$$

$$\gamma_1=\theta_2^2\|\tilde{x}\|_2^2+2\theta_1^2\|\tilde{y}\|_2^2,\ \ \gamma_2=\theta_3^2\|\tilde{x}\|_2^2+\theta_2^2\|\tilde{z}\|_2^2,\ \ \gamma_3=\theta_3^2\|\tilde{z}\|_2^2+2\theta_4^2\|\tilde{y}\|_2^2,$$

$$\gamma_4=\theta_1^2\|\tilde{y}\|_2^2\|\tilde{z}\|_2^4+\theta_3^2\|\tilde{x}\|_2^4\|\tilde{z}\|_2^2+\theta_2^2\|\tilde{x}\|_2^2\|\tilde{z}\|_2^4+\theta_4^2\|\tilde{x}\|_2^4\|\tilde{y}\|_2^2,$$

$$\gamma_5=2\theta_1^2\theta_3^2\|\tilde{y}\|_2^2\|\tilde{z}\|_2^2+2\theta_1^2\theta_4^2\|\tilde{y}\|_2^4-\theta_2^2\theta_3^2\|\tilde{x}\|_2^2\|\tilde{z}\|_2^2-\theta_2^4\|\tilde{z}\|_2^4-\theta_2^2\theta_4^2\|\tilde{x}\|_2^2\|\tilde{y}\|_2^2,$$

$$\gamma_6=\theta_1^2\theta_3^4\|\tilde{x}\|_2^2\|\tilde{z}\|_2^2+\theta_1^2\theta_3^2\theta_4^2\|\tilde{x}\|_2^2\|\tilde{y}\|_2^2+\theta_1^2\theta_2^2\theta_4^2\|\tilde{y}\|_2^2\|\tilde{z}\|_2^2+\theta_2^4\theta_4^2\|\tilde{x}\|_2^2\|\tilde{z}\|_2^2,$$

$$\gamma_7=2\theta_1^2\theta_4^2\|\tilde{y}\|_2^4+2\theta_2^2\theta_4^2\|\tilde{x}\|_2^2\|\tilde{y}\|_2^2-\theta_1^2\theta_3^2\|\tilde{y}\|_2^2\|\tilde{z}\|_2^2-\theta_3^4\|\tilde{x}\|_2^4-\theta_2^2\theta_3^2\|\tilde{x}\|_2^2\|\tilde{z}\|_2^2,$$

$$\gamma_8=\theta_3^2\|\tilde{z}\|_2^2+\theta_4^2\|\tilde{y}\|_2^2,\ \ \gamma_9=\theta_2^2\|\tilde{x}\|_2^2+\theta_1^2\|\tilde{y}\|_2^2.$$

证明 由 (3.2) 知, $(\Delta A,\Delta B,\Delta C,\Delta D)\in\mathcal{F}_0$ 当且仅当 $\Delta A,\Delta B,\Delta C$ 和 ΔD 满足

$$\Delta A\tilde{x}=r_f-\Delta B^T\tilde{y},\ \ \Delta B\tilde{x}=r_g-\Delta C^T\tilde{z},\ \ \Delta D\tilde{z}=r_h-\Delta C\tilde{y},\ \ \Delta A=\Delta A^T,\ \ \Delta D=\Delta D^T.\quad(3.4)$$

将引理 2.1 应用于 (3.4) 的第二个等式, 可得

$$\Delta B^T=\left(\tilde{x}^\dagger\right)^T\left(r_g-\Delta C^T\tilde{z}\right)^T+\left(I_n-\tilde{x}\tilde{x}^\dagger\right)Z,\quad Z\in\mathbb{R}^{n\times m}.\quad(3.5)$$

考虑到 $\Delta D=\Delta D^T$, 应用引理 2.2 于 (3.4) 的第三个等式, 对于任意 $\Delta C\in\mathbb{R}^{l\times m}$, 可得

$$\Delta D=\left(r_h-\Delta C\tilde{y}\right)\tilde{z}^\dagger+\left(\tilde{z}^\dagger\right)^T\left(r_h-\Delta C\tilde{y}\right)^T\left(I_l-\tilde{z}\tilde{z}^\dagger\right)+\left(I_l-\tilde{z}\tilde{z}^\dagger\right)T_2\left(I_l-\tilde{z}\tilde{z}^\dagger\right),\quad(3.6)$$

其中 $T_2\in\mathbb{SR}^{l\times l}$.

考虑到 $\Delta A=\Delta A^T$, 把 (3.5) 带入到 (3.4) 的第一个等式, 再次应用引理 2.2, 对于任意 $\Delta C\in\mathbb{R}^{l\times m}$, 可推出

$$\Delta A=r\tilde{x}^\dagger+\left(\tilde{x}^\dagger\right)^Tr^T\left(I_n-\tilde{x}\tilde{x}^\dagger\right)+\left(I_n-\tilde{x}\tilde{x}^\dagger\right)T_1\left(I_n-\tilde{x}\tilde{x}^\dagger\right),\ \ T_1\in\mathbb{SR}^{n\times n},\quad(3.7)$$

其中

$$r=r_f-\left(\tilde{x}^\dagger\right)^T\left(r_g^T\tilde{y}-\tilde{z}^T\Delta C\tilde{y}\right)-\left(I_n-\tilde{x}\tilde{x}^\dagger\right)Z\tilde{y}$$

考虑到 $\tilde{x}^\dagger\left(I_n-\tilde{x}\tilde{x}^\dagger\right)=0$ 这一事实, 对 (3.5), (3.6) 和 (3.7) 的等号两边同时取 Frobenius 范数, 可得

$$\|\Delta B\|_F^2=\frac{\left\|r_g-\Delta C^T\tilde{z}\right\|_2^2}{\|\tilde{x}\|_2^2}+\left\|\left(I_n-\tilde{x}\tilde{x}^\dagger\right)Z\right\|_F^2,\quad(3.8)$$

$$\begin{aligned}\|\Delta D\|_F^2&=\frac{\|r_h-\Delta C\tilde{y}\|_2^2}{\|\tilde{z}\|_2^2}+\left\|\left(I_l-\tilde{z}\tilde{z}^\dagger\right)T_2\left(I_l-\tilde{z}\tilde{z}^\dagger\right)\right\|_F^2+\frac{\left\|\left(I_l-\tilde{z}\tilde{z}^\dagger\right)\left(r_h-\Delta C\tilde{y}\right)\right\|_2^2}{\|\tilde{z}\|_2^2}\\&=\frac{2\left\|r_h-\Delta C\tilde{y}\right\|_2^2}{\|\tilde{z}\|_2^2}-\frac{\left(r_h^T\tilde{z}-\tilde{z}^T\Delta C\tilde{y}\right)^2}{\|\tilde{z}\|_2^4}+\left\|\left(I_l-\tilde{z}\tilde{z}^\dagger\right)T_2\left(I_l-\tilde{z}\tilde{z}^\dagger\right)\right\|_F^2,\end{aligned}\quad(3.9)$$

和

$$\begin{aligned}\|\Delta A\|_F^2 &= \frac{\|r\|_2^2}{\|\tilde{x}\|_2^2} + \left\|\left(I_n - \tilde{x}\tilde{x}^\dagger\right) T_1 \left(I_n - \tilde{x}\tilde{x}^\dagger\right)\right\|_F^2 + \frac{\left\|\left(I_n - \tilde{x}\tilde{x}^\dagger\right) r\right\|_2^2}{\|\tilde{x}\|_2^2} \\ &= \frac{2\left\|r_f - \left(\tilde{x}^\dagger\right)^T \left(r_g^T\tilde{y} - \tilde{z}^T\Delta C\tilde{y}\right) - \left(I_n - \tilde{x}\tilde{x}^\dagger\right) Z\tilde{y}\right\|_2^2}{\|\tilde{x}\|_2^2} \\ &\quad - \frac{\left(r_f^T\tilde{x} - r_g^T\tilde{y} + \tilde{z}^T\Delta C\tilde{y}\right)^2}{\|\tilde{x}\|_2^4} + \left\|\left(I_n - \tilde{x}\tilde{x}^\dagger\right) T_1 \left(I_n - \tilde{x}\tilde{x}^\dagger\right)\right\|_F^2,\end{aligned} \tag{3.10}$$

根据 $\eta^{(\theta_1,\theta_2,\theta_3,\theta_4)}(\tilde{x},\tilde{y},\tilde{z})$ 的定义 (3.1), 以及表达式 (3.8), (3.9) 和 (3.10) 可以得出

$$\begin{aligned}&\left[\eta^{(\theta_1,\theta_2,\theta_3,\theta_4)}(\tilde{x},\tilde{y},\tilde{z})\right]^2 \\ =& \min_{\substack{\Delta C\in\mathbb{R}^{l\times m}, Z\in\mathbb{R}^{n\times m} \\ T_1\in S\mathbb{R}^{n\times n}, T_2\in S\mathbb{R}^{l\times l}}} \left\{\theta_1^2\|\Delta A\|_F^2 + \theta_2^2\|\Delta B\|_F^2 + \theta_3^2\|\Delta C\|_F^2 + \theta_4^2\|\Delta D\|_F^2\right\} \\ =& \min_{\Delta C\in\mathbb{R}^{l\times m}, Z\in\mathbb{R}^{n\times m}} p(\Delta C),\end{aligned} \tag{3.11}$$

其中

$$p(\Delta C) = p_1(Z,\Delta C) + p_2(\Delta C), \tag{3.12}$$

并且

$$p_1(Z,\Delta C) = \frac{2\theta_1^2}{\|\tilde{x}\|_2^2}\|r_f - \left(\tilde{x}^\dagger\right)^T \left(r_g^T\tilde{y} - \tilde{z}^T\Delta C\tilde{y}\right) - \left(I_n - \tilde{x}\tilde{x}^\dagger\right) Z\tilde{y}\|_2^2 + \theta_2^2\|\left(I_n - \tilde{x}\tilde{x}^\dagger\right) Z\|_F^2,$$

和

$$\begin{aligned}p_2(\Delta C) =& \frac{2\theta_4^2}{\|\tilde{z}\|_2^2}\|r_h - \Delta C\tilde{y}\|_2^2 + \frac{\theta_2^2}{\|\tilde{x}\|_2^2}\left\|r_g - \Delta C^T\tilde{z}\right\|_2^2 - \frac{\theta_4^2}{\|\tilde{z}\|_2^4}\left(r_h^T\tilde{z} - \tilde{z}^T\Delta C\tilde{y}\right)^2 \\ &- \frac{\theta_1^2}{\|\tilde{x}\|_2^4}\left(r_f^T\tilde{x} - r_g^T\tilde{y} + \tilde{z}^T\Delta C\tilde{y}\right)^2 + \theta_3^2\|\Delta C\|_F^2.\end{aligned} \tag{3.13}$$

为了便于书写, 记

$$\hat{g} = r_f - \left(\tilde{x}^\dagger\right)^T \left(r_g^T\tilde{y} - \tilde{z}^T\Delta C\tilde{y}\right), \ \ Y = \left(I_n - \tilde{x}\tilde{x}^\dagger\right) Z$$

则

$$\begin{aligned}p_1(Z,\Delta C) &= \frac{2\theta_1^2\|\hat{g} - Y\tilde{y}\|_2^2}{\|\tilde{x}\|_2^2} + \theta_2^2\|Y\|_F^2 \\ &= \frac{2\theta_1^2\|\hat{g}\|_2^2}{\|\tilde{x}\|_2^2} + \operatorname{tr}\left[Y\left(\theta_2^2 I_m + \frac{2\theta_1^2\tilde{y}\tilde{y}^T}{\|\tilde{x}\|_2^2}\right)Y^T - \frac{2\theta_1^2 Y\tilde{y}\hat{g}^T}{\|\tilde{x}\|_2^2} - \frac{2\theta_1^2\hat{g}\tilde{y}^T Y^T}{\|\tilde{x}\|_2^2}\right].\end{aligned}$$

另外, 记

$$M = Y\left(\theta_2^2 I_m + \frac{2\theta_1^2\tilde{y}\tilde{y}^T}{\|\tilde{x}\|_2^2}\right)^{\frac{1}{2}}, \quad N^T = \frac{2\theta_1^2}{\|\tilde{x}\|_2^2}\left(\theta_2^2 I_m + \frac{2\theta_1^2\tilde{y}\tilde{y}^T}{\|\tilde{x}\|_2^2}\right)^{-\frac{1}{2}}\tilde{y}\hat{g}^T.$$

利用引理 2.3, 并注意到

$$\|N\|_F^2 - \left\|\tilde{x}\tilde{x}^\dagger N\right\|_F^2 = \left\|\left(I_n - \tilde{x}\tilde{x}^\dagger\right) N\right\|_F^2,$$

则

$$\begin{aligned}&\min_{Z\in\mathbb{R}^{n\times m}} p_1\left(Z,\Delta C\right)\\=&\min_{Z\in\mathbb{R}^{n\times m}}\left\{\frac{2\theta_1^2\|\hat{g}\|_2^2}{\|\tilde{x}\|_2^2}+\|M-N\|_F^2-\|N\|_F^2\right\}\\=&\frac{2\theta_1^2\|\hat{g}\|_2^2}{\|\tilde{x}\|_2^2}-\left\|\left(I_n-\tilde{x}\tilde{x}^{\dagger}\right)N\right\|_F^2\\=&\frac{2\theta_1^2\|\hat{g}\|_2^2}{\|\tilde{x}\|_2^2}-\frac{4\theta_1^4}{\|\tilde{x}\|_2^4}\operatorname{tr}\left[\left(I_n-\tilde{x}\tilde{x}^{\dagger}\right)\hat{g}\tilde{y}^T\left(\theta_2^2I_m+\frac{2\theta_1^2\tilde{y}\tilde{y}^T}{\|\tilde{x}\|_2^2}\right)^{-1}\tilde{y}\hat{g}^T\left(I_n-\tilde{x}\tilde{x}^{\dagger}\right)\right].\end{aligned}$$

此外, 使用 Sherman-Morrison-Woodbury 公式(见文献 [18])可以将上面的式子改写为下面的形式

$$\begin{aligned}&\min_{Z\in\mathbb{R}^{n\times m}} p_1\left(Z,\Delta C\right)\\=&\frac{2\theta_1^2\|\hat{g}\|_2^2}{\|\tilde{x}\|_2^2}-\frac{4\theta_1^4}{\theta_2^2\|\tilde{x}\|_2^4}\operatorname{tr}\left[\left(I_n-\tilde{x}\tilde{x}^{\dagger}\right)\hat{g}\tilde{y}^T\left(I_m-\frac{2\theta_1^2\tilde{y}\tilde{y}^T}{\theta_2^2\|\tilde{x}\|_2^2+2\theta_1^2\|\tilde{y}\|_2^2}\right)\tilde{y}\hat{g}^T\left(I_n-\tilde{x}\tilde{x}^{\dagger}\right)\right]\\=&\frac{2\theta_1^2\|\hat{g}\|_2^2}{\|\tilde{x}\|_2^2}-\frac{4\theta_1^4\|\tilde{y}\|_2^2\left\|\left(I_n-\tilde{x}\tilde{x}^{\dagger}\right)\hat{g}\right\|_2^2}{\|\tilde{x}\|_2^2\left(\theta_2^2\|\tilde{x}\|_2^2+2\theta_1^2\|\tilde{y}\|_2^2\right)}\\=&\frac{2\theta_1^2\theta_2^2\left\|r_f\right\|_2^2}{\theta_2^2\|\tilde{x}\|_2^2+2\theta_1^2\|\tilde{y}\|_2^2}+\frac{4\theta_1^4\|\tilde{y}\|_2^2\left(r_f^T\tilde{x}\right)^2}{\|\tilde{x}\|_2^4\left(\theta_2^2\|\tilde{x}\|_2^2+2\theta_1^2\|\tilde{y}\|_2^2\right)}-\frac{4\theta_1^2\left(r_f^T\tilde{x}\right)\left(r_g^T\tilde{y}\right)}{\|\tilde{x}\|_2^4}+\frac{2\theta_1^2\left(r_g^T\tilde{y}\right)^2}{\|\tilde{x}\|_2^4}\\&+\frac{4\theta_1^2\left(r_f^T\tilde{x}\right)\left(\tilde{z}^T\Delta C\tilde{y}\right)}{\|\tilde{x}\|_2^4}-\frac{4\theta_1^2\left(r_g^T\tilde{y}\right)\left(\tilde{z}^T\Delta C\tilde{y}\right)}{\|\tilde{x}\|_2^4}+\frac{2\theta_1^2\left(\tilde{z}^T\Delta C\tilde{y}\right)^2}{\|\tilde{x}\|_2^4}.\end{aligned}\tag{3.14}$$

把 (3.13) 和 (3.14) 带入 (3.12) 中, 可以得到

$$\begin{aligned}&p(\Delta C)\\=&\frac{2\theta_1^2\theta_2^2\left\|r_f\right\|_2^2}{\theta_2^2\|\tilde{x}\|_2^2+2\theta_1^2\|\tilde{y}\|_2^2}+\frac{4\theta_1^4\|\tilde{y}\|_2^2\left(r_f^T\tilde{x}\right)^2}{\|\tilde{x}\|_2^4\left(\theta_2^2\|\tilde{x}\|_2^2+2\theta_1^2\|\tilde{y}\|_2^2\right)}-\frac{4\theta_1^2\left(r_f^T\tilde{x}\right)\left(r_g^T\tilde{y}\right)}{\|\tilde{x}\|_2^4}+\frac{2\theta_1^2\left(r_g^T\tilde{y}\right)^2}{\|\tilde{x}\|_2^4}\\&+\frac{4\theta_1^2\left(r_f^T\tilde{x}\right)\left(\tilde{z}^T\Delta C\tilde{y}\right)}{\|\tilde{x}\|_2^4}-\frac{4\theta_1^2\left(r_g^T\tilde{y}\right)\left(\tilde{z}^T\Delta C\tilde{y}\right)}{\|\tilde{x}\|_2^4}+\frac{2\theta_1^2\left(\tilde{z}^T\Delta C\tilde{y}\right)^2}{\|\tilde{x}\|_2^4}+\frac{2\theta_4^2\left\|r_h-\Delta C\tilde{y}\right\|_2^2}{\|\tilde{z}\|_2^2}\\&+\frac{\theta_2^2\left\|r_g-\Delta C^T\tilde{z}\right\|_2^2}{\|\tilde{x}\|_2^2}-\frac{\theta_4^2\left(r_h^T\tilde{z}-\tilde{z}^T\Delta C\tilde{y}\right)^2}{\|\tilde{z}\|_2^4}-\frac{\theta_1^2\left(r_f^T\tilde{x}-r_g^T\tilde{y}+\tilde{z}^T\Delta C\tilde{y}\right)^2}{\|\tilde{x}\|_2^4}+\theta_3^2\|\Delta C\|_F^2\\=&\frac{2\theta_1^2\theta_2^2\left\|r_f\right\|_2^2}{\theta_2^2\|\tilde{x}\|_2^2+2\theta_1^2\|\tilde{y}\|_2^2}+\frac{\theta_2^2\left\|r_g\right\|_2^2}{\|\tilde{x}\|_2^2}+\frac{2\theta_4^2\left\|r_h\right\|_2^2}{\|\tilde{z}\|_2^2}+\frac{\theta_1^2\left(2\theta_1^2\|\tilde{y}\|_2^2-\theta_2^2\|\tilde{x}\|_2^2\right)\left(r_f^Tx\right)^2}{\|\tilde{x}\|_2^4\left(\theta_2^2\|\tilde{x}\|_2^2+2\theta_1^2\left\|\tilde{y}\right\|_2^2\right)}\\&-\frac{2\theta_1^2\left(r_f^T\tilde{x}\right)\left(r_g^T\tilde{y}\right)}{\|\tilde{x}\|_2^4}+\frac{\theta_1^2\left(r_g^T\tilde{y}\right)^2}{\|\tilde{x}\|_2^4}-\frac{\theta_4^2\left(r_h^T\tilde{z}\right)^2}{\|\tilde{z}\|_2^4}+\frac{2\theta_1^2\left(r_f^T\tilde{x}\right)\left(\tilde{z}^T\Delta C\tilde{y}\right)}{\|\tilde{x}\|_2^4}-\frac{2\theta_1^2\left(r_g^T\tilde{y}\right)\left(\tilde{z}^T\Delta C\tilde{y}\right)}{\|\tilde{x}\|_2^4}\\&+\frac{2\theta_4^2\left(r_h^T\tilde{z}\right)\left(\tilde{z}^T\Delta C\tilde{y}\right)}{\|\tilde{z}\|_2^4}+\frac{\left(\theta_1^2\|\tilde{z}\|_2^4-\theta_4^2\|\tilde{x}\|_2^4\right)\left(\tilde{z}^T\Delta C\tilde{y}\right)^2}{\|\tilde{x}\|_2^4\|\tilde{z}\|_2^4}-\frac{2\theta_2^2\left(\tilde{z}^T\Delta Cr_g\right)}{\|\tilde{x}\|_2^2}+\frac{\theta_2^2\left\|\tilde{z}^T\Delta C\right\|_2^2}{\|\tilde{x}\|_2^2}\\&-\frac{4\theta_4^2\left(r_h^T\Delta C\tilde{y}\right)}{\|\tilde{z}\|_2^2}+\frac{2\theta_4^2\|\Delta C\tilde{y}\|_2^2}{\|\tilde{z}\|_2^2}+\theta_3^2\|\Delta C\|_F^2.\end{aligned}$$

记 $s = \mathrm{vec}(\Delta C) \in \mathbb{R}^{ml}$, 利用 Kronecker 积的性质 (2.1) 和 (2.2), 上面的式子可以进一步化为

$$\begin{aligned}
&p(\Delta C)\\
=\ &\frac{2\theta_1^2\theta_2^2\left\|r_f\right\|_2^2}{\theta_2^2\|\tilde{x}\|_2^2+2\theta_1^2\|\tilde{y}\|_2^2}+\frac{\theta_2^2\left\|r_g\right\|_2^2}{\|\tilde{x}\|_2^2}+\frac{2\theta_4^2\left\|r_h\right\|_2^2}{\|\tilde{z}\|_2^2}+\frac{\theta_1^2\left(2\theta_1^2\|\tilde{y}\|_2^2-\theta_2^2\|\tilde{x}\|_2^2\right)\left(r_f^T x\right)^2}{\|\tilde{x}\|_2^4\left(\theta_2^2\|\tilde{x}\|_2^2+2\theta_1^2\left\|\tilde{y}\right\|_2^2\right)}\\
&-\frac{2\theta_1^2\left(r_f^T\tilde{x}\right)\left(r_g^T\tilde{y}\right)}{\|\tilde{x}\|_2^4}+\frac{\theta_1^2\left(r_g^T\tilde{y}\right)^2}{\|\tilde{x}\|_2^4}-\frac{\theta_4^2\left(r_h^T\tilde{z}\right)^2}{\|\tilde{z}\|_2^4}+\theta_3^2 s^T I_{ml} s+\frac{2\theta_1^2\left(r_f^T\tilde{x}\right)\left(\tilde{y}^T\otimes\tilde{z}^T\right)s}{\|\tilde{x}\|_2^4}\\
&-\frac{2\theta_1^2\left(r_g^T\tilde{y}\right)\left(\tilde{y}^T\otimes\tilde{z}^T\right)s}{\|\tilde{x}\|_2^4}+\frac{2\theta_4^2\left(r_h^T\tilde{z}\right)\left(\tilde{y}^T\otimes\tilde{z}^T\right)s}{\|\tilde{z}\|_2^4}\\
&+\frac{\left(\theta_1^2\|\tilde{z}\|_2^4-\theta_4^2\|\tilde{x}\|_2^4\right)s^T(\tilde{y}\otimes\tilde{z})\left(\tilde{y}^T\otimes\tilde{z}^T\right)s}{\|\tilde{x}\|_2^4\|\tilde{z}\|_2^4}-\frac{2\theta_2^2\left(r_g^T\otimes\tilde{z}^T\right)s}{\|\tilde{x}\|_2^2}\\
&+\frac{\theta_2^2 s^T\left(I_m\otimes\tilde{z}\right)\left(I_m\otimes\tilde{z}^T\right)s}{\|\tilde{x}\|_2^2}-\frac{4\theta_4^2\left(\tilde{y}^T\otimes r_h^T\right)s}{\|\tilde{z}\|_2^2}+\frac{2\theta_4^2 s^T\left(\tilde{y}\otimes I_l\right)\left(\tilde{y}^T\otimes I_l\right)s}{\|\tilde{z}\|_2^2}
\end{aligned}$$

将上面的等式带入到(3.11)中, 并考虑(2.2), 可以得到

$$\begin{aligned}
\left[\eta^{(\theta_1,\theta_2,\theta_3,\theta_4)}(\tilde{z},\tilde{x},\tilde{y})\right]^2 =&\frac{2\theta_1^2\theta_2^2\left\|r_f\right\|_2^2}{\theta_2^2\|\tilde{x}\|_2^2+2\theta_1^2\|\tilde{y}\|_2^2}+\frac{\theta_2^2\left\|r_g\right\|_2^2}{\|\tilde{x}\|_2^2}+\frac{2\theta_4^2\left\|r_h\right\|_2^2}{\|\tilde{z}\|_2^2}-\frac{2\theta_1^2\left(r_f^T\tilde{x}\right)\left(r_g^T\tilde{y}\right)}{\|\tilde{x}\|_2^4}\\
&+\frac{\theta_1^2\left(2\theta_1^2\|\tilde{y}\|_2^2-\theta_2^2\|\tilde{x}\|_2^2\right)\left(r_f^T x\right)^2}{\|\tilde{x}\|_2^4\left(\theta_2^2\|\tilde{x}\|_2^2+2\theta_1^2\left\|\tilde{y}\right\|_2^2\right)}+\frac{\theta_1^2\left(r_g^T\tilde{y}\right)^2}{\|\tilde{x}\|_2^4}-\frac{\theta_4^2\left(r_h^T\tilde{z}\right)^2}{\|\tilde{z}\|_2^4}\\
&+\min_{s\in\mathbb{R}^{ml}} H(s),
\end{aligned}$$

这里 $H(s) = s^T E s - 2e^T s$, 其中

$$\begin{aligned}
E =&\theta_3^2 I_{ml}+\frac{\theta_2^2\left(I_m\otimes\tilde{z}\right)\left(I_m\otimes\tilde{z}^T\right)}{\|\tilde{x}\|_2^2}+\frac{2\theta_4^2\left(\tilde{y}\otimes I_l\right)\left(\tilde{y}^T\otimes I_l\right)}{\|\tilde{z}\|_2^2}\\
&+\frac{\left(\theta_1^2\|\tilde{z}\|_2^4-\theta_4^2\|\tilde{x}\|_2^4\right)\left(\tilde{y}\otimes\tilde{z}\right)\left(\tilde{y}^T\otimes\tilde{z}^T\right)}{\|\tilde{x}\|_2^4\|\tilde{z}\|_2^4},
\end{aligned}$$

和

$$e =\frac{\theta_2^2\left(r_g\otimes\tilde{z}\right)}{\|\tilde{x}\|_2^2}+\frac{2\theta_4^2\left(\tilde{y}\otimes r_h\right)}{\|\tilde{z}\|_2^2}-\left(\frac{\theta_1^2\left(r_f^T\tilde{x}\right)}{\|\tilde{x}\|_2^4}-\frac{\theta_1^2\left(r_g^T\tilde{y}\right)}{\|\tilde{x}\|_2^4}+\frac{\theta_4^2\left(r_h^T\tilde{z}\right)}{\|\tilde{z}\|_2^4}\right)(\tilde{y}\otimes\tilde{z}),$$

利用(2.3), 并注意到 $\left(I_l-\tilde{z}\tilde{z}^{\dagger}\right)^2=\left(I_l-\tilde{z}\tilde{z}^{\dagger}\right)$ 和 $\tilde{z}^{\dagger}=\tilde{z}^T/\|\tilde{z}\|_2^2$, 可以得出

$$\begin{aligned}
E =&\theta_3^2 I_{ml}+\frac{\theta_2^2\left(I_m\otimes\tilde{z}\right)\left(I_m\otimes\tilde{z}^T\right)}{\|\tilde{x}\|_2^2}+\frac{\theta_4^2\left(\tilde{y}\otimes I_l\right)\left(\tilde{y}^T\otimes I_l\right)}{\|\tilde{z}\|_2^2}+\frac{\theta_1^2(\tilde{y}\otimes\tilde{z})\left(\tilde{y}^T\otimes\tilde{z}^T\right)}{\|\tilde{x}\|_2^4}\\
&+\frac{\theta_4^2\left[\tilde{y}\otimes\left(I_l-\tilde{z}\tilde{z}^{\dagger}\right)\right]\left[\tilde{y}^T\otimes\left(I_l-\tilde{z}\tilde{z}^{\dagger}\right)\right]}{\|\tilde{z}\|_2^2}.
\end{aligned}$$

明显地, E 是一个对称正定矩阵. 从而当 $s=E^{-1}e$ 时, $H(s)$ 能取得最小值. 相应的,

$$\begin{aligned}\left[\eta^{(\theta_1,\theta_2,\theta_3,\theta_4)}(\tilde{x},\tilde{y},\tilde{z})\right]^2 =&\frac{2\theta_1^2\theta_2^2\left\|r_f\right\|_2^2}{\theta_2^2\|\tilde{x}\|_2^2+2\theta_1^2\|\tilde{y}\|_2^2}+\frac{\theta_2^2\left\|r_g\right\|_2^2}{\|\tilde{x}\|_2^2}+\frac{2\theta_4^2\left\|r_h\right\|_2^2}{\|\tilde{z}\|_2^2}-\frac{2\theta_1^2\left(r_f^T\tilde{x}\right)\left(r_g^T\tilde{y}\right)}{\|\tilde{x}\|_2^4}\\&+\frac{\theta_1^2\left(2\theta_1^2\|\tilde{y}\|_2^2-\theta_2^2\|\tilde{x}\|_2^2\right)\left(r_f^Tx\right)^2}{\|\tilde{x}\|_2^4\left(\theta_2^2\|\tilde{x}\|_2^2+2\theta_1^2\left\|\tilde{y}\right\|_2^2\right)}+\frac{\theta_1^2\left(r_g^T\tilde{y}\right)^2}{\|\tilde{x}\|_2^4}-\frac{\theta_4^2\left(r_h^T\tilde{z}\right)^2}{\|\tilde{z}\|_2^4}\\&-e^TE^{-1}e.\end{aligned}\tag{3.15}$$

多次利用 Sherman-Morrison-Woodbury 公式(见文献 [18]), 经过一些初等计算可得

$$\begin{aligned}E^{-1}=&\frac{1}{\theta_3^2}I_{ml}-\frac{\theta_2^2\left(I_m\otimes\tilde{z}\tilde{z}^T\right)}{\theta_3^2\left(\theta_3^2\|\tilde{x}\|_2^2+\theta_2^2\|\tilde{z}\|_2^2\right)}-\frac{2\theta_4^2\left(\tilde{y}\tilde{y}^T\otimes I_l\right)}{\theta_3^2\left(\theta_3^2\|\tilde{z}\|_2^2+2\theta_4^2\|\tilde{y}\|_2^2\right)}\\&+\frac{\omega\left(\tilde{y}\tilde{y}^T\otimes\tilde{z}\tilde{z}^T\right)}{\theta_3^2\left(\theta_3^2\|\tilde{z}\|_2^2+2\theta_4^2\|\tilde{y}\|_2^2\right)}\end{aligned}$$

其中

$$\omega=\frac{\|\tilde{x}\|_2^2\left(2\theta_2^2\theta_4^2\|\tilde{x}\|_2^2\|\tilde{z}\|_2^2-\theta_1^2\theta_3^2\|\tilde{z}\|_2^4+\theta_3^2\theta_4^2\|\tilde{x}\|_2^4\right)}{\gamma_2\gamma_3\gamma_4}+\frac{2\theta_2^2\theta_4^2}{\theta_3^2\gamma_2\gamma_3}.$$

经过一些繁琐的计算, 可以得出

$$\begin{aligned}e^TE^{-1}e=&\frac{\theta_2^4\|\tilde{z}\|_2^2}{\|\tilde{x}\|_2^2\gamma_2}\left\|r_g\right\|_2^2+\frac{4\theta_4^4\|\tilde{y}\|_2^2}{\|\tilde{z}\|_2^2\gamma_3}\left\|r_h\right\|_2^2+\frac{\theta_1^4\|\tilde{y}\|_2^2\|\tilde{z}\|_2^4}{\|\tilde{x}\|_2^4\gamma_4}\left(r_f^T\tilde{x}\right)^2+\frac{\|\tilde{z}\|_2^2\gamma_{10}}{\|\tilde{x}\|_2^4\gamma_2\gamma_4}\left(r_g^T\tilde{y}\right)^2\\&-\frac{\theta_4^4\|\tilde{y}\|_2^2\gamma_{11}}{\|\tilde{z}\|_2^4\gamma_3\gamma_4}\left(r_h^T\tilde{z}\right)^2-\frac{2\theta_1^2\|\tilde{z}\|_2^4\gamma_9}{\|\tilde{x}\|_2^4\gamma_4}\left(r_f^T\tilde{x}\right)\left(r_g^T\tilde{y}\right)-\frac{2\theta_1^2\theta_4^2\|\tilde{y}\|_2^2}{\gamma_4}\left(r_f^T\tilde{x}\right)\left(r_h^T\tilde{z}\right)\\&+\frac{2\theta_4^2\gamma_9}{\gamma_4}\left(r_g^T\tilde{y}\right)\left(r_h^T\tilde{z}\right),\end{aligned}$$

其中

$$\gamma_{10}=\theta_1^2\theta_2^4\|\tilde{x}\|_2^2\|\tilde{z}\|_2^4+\theta_1^4\theta_2^2\|\tilde{y}\|_2^2\|\tilde{z}\|_2^4+\theta_1^4\theta_3^2\|\tilde{x}\|_2^2\|\tilde{y}\|_2^2\|\tilde{z}\|_2^2+2\theta_1^2\theta_2^2\theta_3^2\|\tilde{x}\|_2^4\|\tilde{z}\|_2^2-\theta_2^4\theta_4^2\|\tilde{x}\|_2^6,$$

$$\gamma_{11}=3\theta_3^2\|\tilde{x}\|_2^4\|\tilde{z}\|_2^2+2\theta_4^2\|\tilde{x}\|_2^4\|\tilde{y}\|_2^2+4\theta_1^2\|\tilde{y}\|_2^2\|\tilde{z}\|_2^4+4\theta_2^2\|\tilde{x}\|_2^2\|\tilde{z}\|_2^4.$$

其中 $\gamma_2,\gamma_3,\gamma_4$ 和 γ_9 在定理 3.1 的前面部分定义过, 将上式带入到 (3.15) 中, 推出 (3.3). 证明完毕.

下面, 利用定理 3.1 中部分结构向后误差 $\eta^{(\theta_1,\theta_2,\theta_3,\theta_4)}(\tilde{x},\tilde{y},\tilde{z})$ 的表达式推出结构向后误差 $\eta^{(\theta_1,\theta_2,\theta_3,\theta_4,\lambda_1,\lambda_2,\lambda_3)}(\tilde{x},\tilde{y},\tilde{z})$ 的具体表达式.

定理 3.2 假设 $(\tilde{x}^T,\tilde{y}^T,\tilde{z}^T)^T$ 满足 $\tilde{x}\neq 0$ 和 $\tilde{z}\neq 0$ 为系统 (1.1) 的一个计算解. 且

$\eta^{(\theta_1,\theta_2,\theta_3,\theta_4,\lambda_1,\lambda_2,\lambda_3)}(\tilde{x},\tilde{y},\tilde{z})$ 的定义由 (1.3) 和 (1.4) 给出. 则

$$\begin{aligned}&\left[\eta^{(\theta_1,\theta_2,\theta_3,\theta_4,\lambda_1,\lambda_2,\lambda_3)}(\tilde{x},\tilde{y},\tilde{z})\right]^2\\=&\frac{2\theta_1^2\theta_2^2\left(\mu_1-2\theta_1^2\theta_2^2\right)}{\gamma_1\mu_1}\left\|r_f\right\|_2^2+\frac{\theta_2^2\theta_3^2\left(\mu_2-\theta_2^2\theta_3^2\right)}{\gamma_2\mu_2}\left\|r_g\right\|_2^2+\frac{2\theta_3^2\theta_4^2\left(\mu_3-2\theta_4^2\right)}{\gamma_3\mu_3}\left\|r_h\right\|_2^2\\&+\left(r_f^T\tilde{x}\right)^2k_1+\left(r_g^T\tilde{y}\right)^2k_2+\left(r_h^T\tilde{z}\right)^2k_3+\left(r_f^T\tilde{x}\right)\left(r_g^T\tilde{y}\right)k_4\\&+\left(r_f^T\tilde{x}\right)\left(r_h^T\tilde{z}\right)k_5+\left(r_g^T\tilde{y}\right)\left(r_h^T\tilde{z}\right)k_6,\end{aligned}\tag{3.16}$$

其中

$$\begin{aligned}\mu_1&=\gamma_1\lambda_1^2+2\theta_1^2\theta_2^2,\\\mu_2&=\gamma_2\lambda_2^2+\theta_2^2\theta_3^2,\\\mu_3&=\gamma_3\lambda_3^2+2\theta_3^2\theta_4^2,\\\mu_4&=\theta_1^2\|\tilde{x}\|_2^2\gamma_5+\gamma_1\gamma_4\lambda_1^2+2\theta_1^2\theta_2^2\gamma_4,\\\mu_5&=\theta_1^2\|\tilde{x}\|_2^2\gamma_5\gamma_6+\gamma_1\gamma_4\gamma_6\lambda_1^2+2\theta_1^2\theta_2^2\gamma_4\gamma_6-\theta_1^4\|\tilde{x}\|_2^2\gamma_1\gamma_2\gamma_8^2,\\\mu_6&=\theta_1^4\|\tilde{x}\|_2^2\|\tilde{y}\|_2^2\gamma_1\gamma_8-\theta_1^2\|\tilde{x}\|_2^2\gamma_5\gamma_9-\gamma_1\gamma_4\gamma_9\lambda_1^2-2\theta_1^2\theta_2^2\gamma_4\gamma_9,\\\mu_7&=2\theta_1^4\theta_4^4\|\tilde{x}\|_2^2\|\tilde{y}\|_2^4\gamma_1\gamma_2\gamma_3\gamma_8\gamma_9-\theta_1^2\theta_4^4\|\tilde{x}\|_2^2\|\tilde{y}\|_2^2\gamma_2\gamma_3\gamma_5\gamma_9^2-\theta_4^4\|\tilde{y}\|_2^2\gamma_1\gamma_2\gamma_3\gamma_4\gamma_9^2\lambda_1^2,\\&\quad-2\theta_1^2\theta_2^2\theta_4^4\|\tilde{y}\|_2^2\gamma_2\gamma_3\gamma_4\gamma_9^2-\theta_1^4\theta_4^4\|\tilde{x}\|_2^2\|\tilde{y}\|_2^4\gamma_1\gamma_2\gamma_3\gamma_4\lambda_2^2-\theta_1^4\theta_2^2\theta_3^2\theta_4^4\|\tilde{x}\|_2^2\|\tilde{y}\|_2^4\gamma_1\gamma_3\gamma_4,\\&\quad-\theta_1^4\theta_4^4\|\tilde{x}\|_2^2\|\tilde{y}\|_2^6\gamma_1\gamma_3\gamma_6,\end{aligned}$$

和

$$\begin{aligned}\xi_1&=\gamma_4\mu_3^2\left(\gamma_4\mu_2\mu_4+\|\tilde{y}\|_2^2\mu_5\right)+\|\tilde{z}\|_2^2\mu_3\left[\mu_7+\theta_4^2\gamma_7\left(\gamma_4\mu_2\mu_4+\|\tilde{y}\|_2^2\mu_5\right)\right],\\\xi_2&=\gamma_2\gamma_8\mu_6+\gamma_4\mu_2\mu_4+\|\tilde{y}\|_2^2\mu_5,\\\xi_3&=\theta_1^2\|\tilde{y}\|_2^2\gamma_1\gamma_2\gamma_4\gamma_8^2\mu_1-\gamma_4\gamma_5\mu_2\mu_4-\|\tilde{y}\|_2^2\gamma_5\mu_5,\end{aligned}$$

和

$$\Omega_1=\frac{\theta_1^2\gamma_1\mu_4\xi_1\xi_3+\theta_1^4\theta_4^4\|\tilde{y}\|_2^4\|\tilde{z}\|_2^2\gamma_1^2\gamma_3\gamma_4\mu_1\mu_3\xi_2^2}{\mu_1\mu_4^2\xi_1\left(\gamma_4\mu_2\mu_4+\|\tilde{y}\|_2^2\mu_5\right)},$$

$$\Omega_4=\frac{\theta_4^4\|\tilde{z}\|_2^2\gamma_2^2\gamma_3\gamma_4\mu_2\mu_3\mu_6^2-\gamma_2\mu_5\xi_1}{\mu_2\xi_1\left(\gamma_4\mu_2\mu_4+\|\tilde{y}\|_2^2\mu_5\right)},$$

$$\Omega_2=\frac{\theta_1^2\gamma_1\gamma_2\gamma_4\gamma_8\mu_4\xi_1+\theta_1^2\theta_4^4\|\tilde{y}\|_2^2\|\tilde{z}\|_2^2\gamma_1\gamma_2\gamma_3\gamma_4\mu_3\mu_6\xi_2}{\mu_4\xi_1\left(\gamma_4\mu_2\mu_4+\|\tilde{y}\|_2^2\mu_5\right)},$$

$$\Omega_3=\frac{\theta_1^2\theta_4^2\|\tilde{y}\|_2^2\gamma_1\gamma_3\gamma_4\mu_3\xi_2}{\mu_4\xi_1},\ \Omega_5=\frac{\theta_4^2\gamma_2\gamma_3\gamma_4\mu_3\mu_6}{\xi_1},$$

$$\Omega_6=\frac{\gamma_3\left[\mu_7+\theta_4^2\gamma_7\left(\gamma_4\mu_2\mu_4+\|\tilde{y}\|_2^2\mu_5\right)\right]}{\xi_1},$$

和

$$k_1 = -\frac{\theta_1^4\|\tilde{y}\|_2^2\gamma_8\left(2\theta_4^2\|\tilde{y}\|_2^2\|\tilde{z}\|_2^2\gamma_1\Omega_5 + \|\tilde{y}\|_2^2\gamma_1\gamma_8\Omega_4 - 4\theta_2^2\gamma_4\Omega_2 - 2\|\tilde{x}\|_2^2\gamma_5\Omega_2\right)}{\gamma_1{\gamma_4}^2}$$
$$+\frac{\theta_1^4\theta_4^2\|\tilde{y}\|_2^2\|\tilde{z}\|_2^2\left(4\theta_2^2\gamma_4\Omega_3 + 2\|\tilde{x}\|_2^2\gamma_5\Omega_3 + \theta_4^2\|\tilde{y}\|_2^2\|\tilde{z}\|_2^2\gamma_1\Omega_6\right)}{\gamma_1\gamma_4^2}$$
$$+\frac{\theta_1^2\left(\gamma_1\gamma_4\gamma_5 - 4\theta_1^2\theta_2^4\gamma_4^2\Omega_1 - 4\theta_1^2\theta_2^2\|\tilde{x}\|_2^2\gamma_4\gamma_5\Omega_1 - \theta_1^2\|\tilde{x}\|_2^4\gamma_5^2\Omega_1\right)}{\gamma_1^2\gamma_4^2}$$
$$-\frac{\theta_1^4\left(4\theta_2^2\gamma_4\gamma_5\mu_2\mu_3 + \|\tilde{x}\|_2^2\gamma_5^2\mu_2\mu_3 + \|\tilde{y}\|_2^2\gamma_1\gamma_2\gamma_8^2\mu_1\mu_3 + \theta_4^4\|\tilde{y}\|_2^4\|\tilde{z}\|_2^2\gamma_1\gamma_3\mu_1\mu_2\right)}{\gamma_1\gamma_4^2\mu_1\mu_2\mu_3},$$

$$k_2 = -\frac{\theta_1^2\|\tilde{x}\|_2^2\gamma_8\left(\theta_1^2\|\tilde{x}\|_2^2\gamma_2\gamma_8\Omega_1 - 2\theta_4^2\|\tilde{z}\|_2^2\gamma_2\gamma_9\Omega_3 - 2\theta_2^2\theta_3^2\gamma_4\Omega_2 - 2\|\tilde{y}\|_2^2\gamma_6\Omega_2\right)}{\gamma_2{\gamma_4}^2}$$
$$-\frac{\theta_4^2\|\tilde{z}\|_2^2\gamma_9\left(2\theta_2^2\theta_3^2\gamma_4\Omega_5 + 2\|\tilde{y}\|_2^2\gamma_6\Omega_5 - \theta_4^2\|\tilde{z}\|_2^2\gamma_2\gamma_9\Omega_6\right)}{\gamma_2\gamma_4^2}$$
$$+\frac{\gamma_6\left(\gamma_2\gamma_4\mu_2 - 2\theta_2^2\theta_3^2\|\tilde{y}\|_2^2\gamma_4\mu_2\Omega_4 - \|\tilde{y}\|_2^2\gamma_2\gamma_6 - \|\tilde{y}\|_2^4\gamma_6\mu_2\Omega_4\right)}{\gamma_2^2\gamma_4^2\mu_2}$$
$$-\frac{2\theta_2^2\theta_3^2\gamma_2\gamma_4\gamma_6\mu_1\mu_3 + \theta_1^4\|\tilde{x}\|_2^2\gamma_1\gamma_2^2\gamma_8^2\mu_2\mu_3 + \theta_2^4\theta_3^4\gamma_4^2\mu_1\mu_2\mu_3\Omega_4 + \theta_4^4\|\tilde{z}\|_2^2\gamma_2^2\gamma_3\gamma_9^2\mu_1\mu_2}{\gamma_2^2\gamma_4^2\mu_1\mu_2\mu_3},$$

$$k_3 = -\frac{\theta_4^4\|\tilde{y}\|_2^2\gamma_9\left(\|\tilde{y}\|_2^2\gamma_3\gamma_9\Omega_4 - 2\theta_1^2\|\tilde{x}\|_2^2\|\tilde{y}\|_2^2\gamma_3\Omega_2 + 4\theta_3^2\gamma_4\Omega_5 + 2\|\tilde{z}\|_2^2\gamma_7\Omega_5\right)}{\gamma_3\gamma_4^2}$$
$$-\frac{\theta_1^2\theta_4^4\|\tilde{x}\|_2^2\|\tilde{y}\|_2^2\left(\theta_1^2\|\tilde{x}\|_2^2\|\tilde{y}\|_2^2\gamma_3\Omega_1 - 4\theta_3^2\gamma_4\Omega_3 - 2\|\tilde{z}\|_2^2\gamma_7\Omega_3\right)}{\gamma_3\gamma_4^2}$$
$$+\frac{\theta_4^2\gamma_7\left(\gamma_3\gamma_4\mu_3 - \theta_4^2\|\tilde{z}\|_2^2\gamma_3\gamma_7 + 4\theta_3^2\theta_4^2\|\tilde{z}\|_2^2\gamma_4\mu_3\Omega_6 + \theta_4^2\|\tilde{z}\|_2^4\gamma_7\mu_3\Omega_6\right)}{\gamma_3^2\gamma_4^2\mu_3}$$
$$-\frac{\theta_4^4\left(4\theta_3^2\gamma_3\gamma_4\gamma_7\mu_1\mu_2 + \theta_1^4\|\tilde{x}\|_2^2\|\tilde{y}\|_2^4\gamma_1\gamma_3^2\mu_2\mu_3 + \|\tilde{y}\|_2^2\gamma_2\gamma_3^2\gamma_9^2\mu_1\mu_3 - 4\theta_3^4\gamma_4^2\mu_1\mu_2\mu_3\Omega_6\right)}{\gamma_3^2\gamma_4^2\mu_1\mu_2\mu_3},$$

$$k_4 = -\frac{2\theta_1^4\|\tilde{x}\|_2^2\gamma_8\left(\|\tilde{y}\|_2^2\gamma_1\gamma_8\Omega_2 + \theta_4^2\|\tilde{y}\|_2^2\|\tilde{z}\|_2^2\gamma_1\Omega_3 - 2\theta_2^2\gamma_4\Omega_1 - \|\tilde{x}\|_2^2\gamma_5\Omega_1\right)}{\gamma_1\gamma_4^2}$$
$$-\frac{2\theta_1^2\theta_4^2\|\tilde{z}\|_2^2\gamma_9\left(2\theta_2^2\gamma_4\Omega_3 + \|\tilde{x}\|_2^2\gamma_5\Omega_3 - \|\tilde{y}\|_2^2\gamma_1\gamma_8\Omega_5 + \theta_4^2\|\tilde{y}\|_2^2\|\tilde{z}\|_2^2\gamma_1\Omega_6\right)}{\gamma_1\gamma_4^2}$$
$$+\frac{2\theta_1^2\gamma_8\left(2\theta_1^2\theta_2^2\gamma_4\mu_2 + \theta_2^2\theta_3^2\gamma_4\mu_1 + \theta_1^2\|\tilde{x}\|_2^2\gamma_5\mu_2 + \|\tilde{y}\|_2^2\gamma_6\mu_1\right)}{\gamma_4^2\mu_1\mu_2}$$
$$+\frac{2\theta_1^2\|\tilde{y}\|_2^2\left(\theta_2^2\theta_3^2\theta_4^2\|\tilde{z}\|_2^2\gamma_4\Omega_5 + \theta_4^2\|\tilde{y}\|_2^2\|\tilde{z}\|_2^2\gamma_6\Omega_5 + \theta_2^2\theta_3^2\gamma_4\gamma_8\Omega_4 + \|\tilde{y}\|_2^2\gamma_6\gamma_8\Omega_4\right)}{\gamma_2\gamma_4^2}$$
$$-\frac{2\theta_1^2\theta_2^2\Omega_2\left(2\theta_2^2\theta_3^2\gamma_4 + \theta_3^2\|\tilde{x}\|_2^2\gamma_5 + 2\|\tilde{y}\|_2^2\gamma_6\right)}{\gamma_1\gamma_2\gamma_4}$$
$$+\frac{2\theta_1^2\left(\theta_4^4\|\tilde{y}\|_2^2\|\tilde{z}\|_2^2\gamma_1\gamma_2\gamma_3\gamma_9 - \gamma_1\gamma_2\gamma_4\gamma_8\mu_3 - \|\tilde{x}\|_2^2\|\tilde{y}\|_2^2\gamma_5\gamma_6\mu_3\Omega_2\right)}{\gamma_1\gamma_2\gamma_4^2\mu_3},$$

$$
\begin{aligned}
k_5 =& -\frac{2\theta_1^2\theta_4^2\|\tilde{y}\|_2^2\left(\theta_4^2\|\tilde{y}\|_2^2\|\tilde{z}\|_2^2\gamma_9\Omega_5-\theta_1^2\|\tilde{x}\|_2^2\|\tilde{y}\|_2^2\gamma_8\Omega_2-\theta_1^2\theta_4^2\|\tilde{x}\|_2^2\|\tilde{y}\|_2^2\|\tilde{z}\|_2^2\Omega_3+\|\tilde{y}\|_2^2\gamma_8\gamma_9\Omega_4\right)}{\gamma_4^2} \\
& -\frac{4\theta_1^2\theta_4^2\|\tilde{y}\|_2^2\left(\theta_1^2\theta_2^2\gamma_1\mu_3+\theta_3^2\theta_4^2\gamma_1\mu_1-\theta_2^2\gamma_9\mu_1\mu_3\Omega_2+\theta_1^2\theta_2^2\|\tilde{x}\|_2^2\mu_1\mu_3\Omega_1\right)}{\gamma_1\gamma_4\mu_1\mu_3} \\
& -\frac{4\theta_1^2\theta_4^2\left(\theta_3^2\|\tilde{y}\|_2^2\gamma_1\gamma_8\Omega_5-\theta_2^2\|\tilde{z}\|_2^2\gamma_7\Omega_3-\theta_3^2\|\tilde{x}\|_2^2\gamma_5\Omega_3-\theta_3^2\theta_4^2\|\tilde{y}\|_2^2\|\tilde{z}\|_2^2\gamma_1\Omega_6\right)}{\gamma_1\gamma_3\gamma_4} \\
& -\frac{2\theta_1^2\theta_4^2\left(\|\tilde{y}\|_2^2\|\tilde{z}\|_2^2\gamma_1\gamma_7\gamma_8\Omega_5-4\theta_2^2\theta_3^2\gamma_4^2\Omega_3-\|\tilde{x}\|_2^2\|\tilde{z}\|_2^2\gamma_5\gamma_7\Omega_3-\theta_4^2\|\tilde{y}\|_2^2\|\tilde{z}\|_2^4\gamma_1\gamma_7\Omega_6\right)}{\gamma_1\gamma_3\gamma_4^2} \\
& -\frac{2\theta_1^2\theta_4^2\|\tilde{x}\|_2^2\|\tilde{y}\|_2^2\gamma_5\left(\theta_1^2\gamma_1-\gamma_9\mu_1\Omega_2+\theta_1^2\|\tilde{x}\|_2^2\mu_1\Omega_1\right)}{\gamma_1\gamma_4^2\mu_1} \\
& +\frac{2\theta_1^2\theta_4^2\|\tilde{y}\|_2^2\left(\gamma_4\mu_2\mu_3-\gamma_2\gamma_8\gamma_9\mu_3-\theta_4^2\|\tilde{z}\|_2^2\gamma_7\mu_2\right)}{\gamma_4^2\mu_2\mu_3},
\end{aligned}
$$

$$
\begin{aligned}
k_6 =& -\frac{2\theta_4^4\|\tilde{z}\|_2^2\gamma_9\left(\theta_1^2\|\tilde{x}\|_2^2\|\tilde{y}\|_2^2\gamma_3\Omega_3-\|\tilde{y}\|_2^2\gamma_3\gamma_9\Omega_5+2\theta_3^2\gamma_4\Omega_6+\|\tilde{z}\|_2^2\gamma_7\Omega_6\right)}{\gamma_3\gamma_4^2} \\
& -\frac{2\theta_3^2\theta_4^2\left(\theta_1^2\theta_2^2\|\tilde{x}\|_2^2\|\tilde{y}\|_2^2\mu_2\mu_3\Omega_2-\theta_2^2\gamma_2\gamma_9\mu_3-\theta_2^2\|\tilde{y}\|_2^2\gamma_9\mu_2\mu_3\Omega_4-2\theta_4^2\gamma_2\gamma_9\mu_2\right)}{\gamma_2\gamma_4\mu_2\mu_3} \\
& -\frac{2\theta_1^2\theta_4^2\|\tilde{x}\|_2^2\|\tilde{y}\|_2^2\gamma_8\left(\gamma_9\mu_1\Omega_2-\theta_1^2\gamma_1-\theta_1^2\|\tilde{x}\|_2^2\mu_1\Omega_1\right)}{\gamma_4^2\mu_1} \\
& -\frac{2\theta_4^2\left(\gamma_3\gamma_4\gamma_9+2\theta_1^2\theta_3^2\|\tilde{x}\|_2^2\gamma_4\gamma_8\Omega_3+\theta_1^2\|\tilde{x}\|_2^2\|\tilde{z}\|_2^2\gamma_7\gamma_8\Omega_3\right)}{\gamma_3\gamma_4^2} \\
& -\frac{2\theta_4^2\|\tilde{y}\|_2^2\gamma_6\left(\theta_1^2\|\tilde{x}\|_2^2\|\tilde{y}\|_2^2\gamma_3\Omega_2-\|\tilde{y}\|_2^2\gamma_3\gamma_9\Omega_4-2\theta_3^2\gamma_4\Omega_5-\|\tilde{z}\|_2^2\gamma_7\Omega_5\right)}{\gamma_2\gamma_3\gamma_4^2} \\
& +\frac{2\theta_4^2\left(\|\tilde{y}\|_2^2\gamma_2\gamma_3\gamma_6\gamma_9\mu_3+2\theta_2^2\theta_3^4\gamma_4^2\mu_2\mu_3\Omega_5+\theta_2^2\theta_3^2\|\tilde{z}\|_2^2\gamma_4\gamma_7\mu_2\mu_3\Omega_5+\theta_4^2\|\tilde{z}\|_2^2\gamma_2\gamma_3\gamma_7\gamma_9\mu_2\right)}{\gamma_2\gamma_3\gamma_4^2\mu_2\mu_3}.
\end{aligned}
$$

证明 根据 $\eta^{(\theta_1,\theta_2,\theta_3,\theta_4,\lambda_1,\lambda_2,\lambda_3)}(\tilde{x},\tilde{y},\tilde{z})$ 的定义 (1.4) 和部分结构向后误差 $\eta^{(\theta_1,\theta_2,\theta_3,\theta_4)}(\tilde{x},\tilde{y},\tilde{z})$ 的表达式 (3.3), 可以推出

$$
\left[\eta^{(\theta_1,\theta_2,\theta_3,\theta_4,\lambda_1,\lambda_2,\lambda_3)}(\tilde{x},\tilde{y},\tilde{z})\right]^2=\min_{\Delta f\in\mathbb{R}^n,\Delta g\in\mathbb{R}^m,\Delta h\in\mathbb{R}^l}\mathcal{X}(\Delta f,\Delta g,\Delta h),
$$

其中

$$
\begin{aligned}
&\mathcal{X}(\Delta f,\Delta g,\Delta h) \\
=&\lambda_1^2\|\Delta f\|_2^2+\lambda_2^2\|\Delta g\|_2^2+\lambda_3^2\|\Delta h\|_2^2+\frac{2\theta_1^2\theta_2^2}{\gamma_1}\|r_f+\Delta f\|_2^2+\frac{\theta_2^2\theta_3^2}{\gamma_2}\|r_g+\Delta g\|_2^2 \\
&+\frac{2\theta_3^2\theta_4^2}{\gamma_3}\|r_h+\Delta h\|_2^2+\frac{\theta_1^2\gamma_5}{\gamma_1\gamma_4}\left[(r_f+\Delta f)^T\tilde{x}\right]^2+\frac{\gamma_6}{\gamma_2\gamma_4}\left[(r_g+\Delta g)^T\tilde{y}\right]^2 \\
&+\frac{\theta_4^2\gamma_7}{\gamma_3\gamma_4}\left[(r_h+\Delta h)^T\tilde{z}\right]^2-\frac{2\theta_1^2\gamma_8}{\gamma_4}\left[(r_f+\Delta f)^T\tilde{x}\right]\left[(r_g+\Delta g)^T\tilde{y}\right] \\
&+\frac{2\theta_1^2\theta_4^2\|\tilde{y}\|_2^2}{\gamma_4}\left[(r_f+\Delta f)^T\tilde{x}\right]\left[(r_h+\Delta h)^T\tilde{z}\right]-\frac{2\theta_4^2\gamma_9}{\gamma_4}\left[(r_g+\Delta g)^T\tilde{y}\right]\left[(r_h+\Delta h)^T\tilde{z}\right].
\end{aligned}
$$

经过一系列初级运算, 可以推出

$$\mathcal{X}(\Delta f, \Delta g, \Delta h) = \left[\eta^{(\theta_1,\theta_2,\theta_3,\theta_4)}(\tilde{x},\tilde{y},\tilde{z})\right]^2 + \begin{bmatrix} \Delta f \\ \Delta g \\ \Delta h \end{bmatrix}^T \Phi \begin{bmatrix} \Delta f \\ \Delta g \\ \Delta h \end{bmatrix} + 2 \begin{bmatrix} \Delta f \\ \Delta g \\ \Delta h \end{bmatrix}^T q,$$

且

$$\Phi = \begin{bmatrix} \frac{\gamma_1\lambda_1^2+2\theta_1^2\theta_2^2}{\gamma_1} I_n + \frac{\theta_1^2\gamma_5\tilde{x}\tilde{x}^T}{\gamma_1\gamma_4} & -\frac{\theta_1^2\gamma_8\tilde{x}\tilde{y}^T}{\gamma_4} & \frac{\theta_1^2\theta_4^2\|\tilde{y}\|_2^2\tilde{x}\tilde{z}^T}{\gamma_4} \\ -\frac{\theta_1^2\gamma_8\tilde{y}\tilde{x}^T}{\gamma_4} & \frac{\gamma_2\lambda_2^2+\theta_2^2\theta_3^2}{\gamma_2} I_m + \frac{\gamma_6\tilde{y}\tilde{y}^T}{\gamma_2\gamma_4} & -\frac{\theta_4^2\gamma_9\tilde{y}\tilde{z}^T}{\gamma_4} \\ \frac{\theta_1^2\theta_4^2\|\tilde{y}\|_2^2\tilde{z}\tilde{x}^T}{\gamma_4} & -\frac{\theta_4^2\gamma_9\tilde{z}\tilde{y}^T}{\gamma_4} & \frac{\gamma_3\lambda_3^2+2\theta_3^2\theta_4^2}{\gamma_3} I_l + \frac{\theta_4^2\gamma_7\tilde{z}\tilde{z}^T}{\gamma_3\gamma_4} \end{bmatrix},$$

和

$$q = \begin{pmatrix} \frac{2\theta_1^2\theta_2^2}{\gamma_1} r_f + \frac{\theta_1^2\gamma_5\left(r_f^T\tilde{x}\right)}{\gamma_1\gamma_4}\tilde{x} - \frac{\theta_1^2\gamma_8\left(r_g^T\tilde{y}\right)}{\gamma_4}\tilde{x} + \frac{\theta_1^2\theta_4^2\|\tilde{y}\|_2^2\left(r_h^T\tilde{z}\right)}{\gamma_4}\tilde{x} \\ \frac{\theta_2^2\theta_3^2}{\gamma_2} r_g + \frac{\gamma_6\left(r_g^T\tilde{y}\right)}{\gamma_2\gamma_4}\tilde{y} - \frac{\theta_1^2\gamma_8\left(r_f^T\tilde{x}\right)}{\gamma_4}\tilde{y} - \frac{\theta_4^2\gamma_9\left(r_h^T\tilde{z}\right)}{\gamma_4}\tilde{y} \\ \frac{2\theta_3^2\theta_4^2}{\gamma_3} r_h + \frac{\theta_4^2\gamma_7\left(r_h^T\tilde{z}\right)}{\gamma_3\gamma_4}\tilde{z} + \frac{\theta_1^2\theta_4^2\|\tilde{y}\|_2^2\left(r_f^T\tilde{x}\right)}{\gamma_4}\tilde{z} - \frac{\theta_4^2\gamma_9\left(r_g^T\tilde{y}\right)}{\gamma_4}\tilde{z} \end{pmatrix}.$$

通过繁琐的计算, 对于任意非零向量 $t = \left(a^T, b^T, c^T\right)^T$, 其中 $a \in \mathbb{R}^n, b \in \mathbb{R}^m$, 和 $c \in \mathbb{R}^l$, 可得

$$\begin{aligned} t^T\Phi t =& \frac{\gamma_1\lambda_1^2 + 2\theta_1^2\theta_2^2}{\gamma_1} a^Ta + \frac{\theta_1^2\gamma_5}{\gamma_1\gamma_4} a^T\tilde{x}\tilde{x}^Ta - \frac{2\theta_1^2\gamma_8}{\gamma_4} a^T\tilde{x}\tilde{y}^Tb + \frac{2\theta_1^2\theta_4^2\|\tilde{y}\|_2^2}{\gamma_4} a^T\tilde{x}\tilde{z}^Tc + \frac{\theta_4^2\gamma_7}{\gamma_3\gamma_4} c^T\tilde{z}\tilde{z}^Tc \\ &+ \frac{\gamma_2\lambda_2^2 + \theta_2^2\theta_3^2}{\gamma_2} b^Tb + \frac{\gamma_6}{\gamma_2\gamma_4} b^T\tilde{y}\tilde{y}^Tb - \frac{2\theta_4^2\gamma_9}{\gamma_4} b^T\tilde{y}\tilde{z}^Tc + \frac{\gamma_3\lambda_3^2 + 2\theta_3^2\theta_4^2}{\gamma_3} c^Tc \\ =& \frac{\gamma_4\lambda_1^2 + \theta_1^2\theta_2^2\|\tilde{z}\|_2^4}{\gamma_4} a^Ta + \frac{\gamma_4\lambda_2^2 + \theta_2^2\theta_3^2\|\tilde{x}\|_2^2\|\tilde{z}\|_2^2}{\gamma_4} b^Tb + \frac{\gamma_4\lambda_3^2 + \theta_3^2\theta_4^2\|\tilde{x}\|_2^4}{\gamma_4} c^Tc \\ &+ \frac{\theta_1^2\theta_4^2\|\tilde{y}\|_2^2}{\gamma_4}\left(a^T\tilde{x} - b^T\tilde{y} + c^T\tilde{z}\right)^2 + \frac{\theta_1^2\theta_3^2\|\tilde{z}\|_2^2}{\gamma_4}\left(a^T\tilde{x} - b^T\tilde{y}\right)^2 + \frac{\theta_2^2\theta_4^2\|\tilde{x}\|_2^2}{\gamma_4}\left(b^T\tilde{y} - c^T\tilde{z}\right)^2 \\ &+ \frac{\theta_1^2\left(\gamma_1\gamma_8 - \gamma_5\right)}{\gamma_1\gamma_4} a^T\left(\|\tilde{x}\|_2^2 I_n - \tilde{x}\tilde{x}^T\right)a + \frac{\theta_2^2\theta_3^2\left(\theta_4^2\|\tilde{x}\|_2^4 + \theta_1^2\|\tilde{z}\|_2^4\right)}{\gamma_2\gamma_4} b^T\left(\|\tilde{y}\|_2^2 I_m - \tilde{y}\tilde{y}^T\right)b \\ &+ \frac{\theta_4^2\left(\gamma_3\gamma_9 - \gamma_7\right)}{\gamma_3\gamma_4} c^T\left(\|\tilde{z}\|_2^2 I_l - \tilde{z}\tilde{z}^T\right)c \\ >& 0, \end{aligned}$$

这表明 Φ 是一个对称正定矩阵. 因此, $\mathcal{X}(\Delta f, \Delta g, \Delta h)$ 的最小值点为

$$\left(\Delta f^T, \Delta g^T, \Delta h^T\right)^T = -\Phi^{-1}q.$$

相应的,

$$\left[\eta^{(\theta_1,\theta_2,\theta_3,\theta_4,\lambda_1,\lambda_2,\lambda_3)}(\tilde{x},\tilde{y},\tilde{z})\right]^2 = \left[\eta^{(\theta_1,\theta_2,\theta_3,\theta_4)}(\tilde{x},\tilde{y},\tilde{z})\right]^2 - q^T\Phi^{-1}q. \tag{3.17}$$

令

$$\Phi^{-1}=\begin{bmatrix}\Psi_{11} & \Psi_{12} & \Psi_{13}\\ \Psi_{12}^T & \Psi_{22} & \Psi_{23}\\ \Psi_{13}^T & \Psi_{23}^T & \Psi_{33}\end{bmatrix},\ \Psi_{11}\in\mathbb{R}^{n\times n},\ \Psi_{22}\in\mathbb{R}^{m\times m},\ \Psi_{33}\in\mathbb{R}^{l\times l}.$$

经过一系列初等行变换，有

$$\Psi_{11}=\frac{\gamma_1}{\mu_1}I_n+\frac{\theta_1^2\gamma_1\mu_4\xi_1\xi_3+\theta_1^4\theta_4^4\|\tilde{y}\|_2^4\|\tilde{z}\|_2^2\gamma_1^2\gamma_3\gamma_4\mu_1\mu_3\xi_2^2}{\mu_1\mu_4^2\xi_1\left(\gamma_4\mu_2\mu_4+\|\tilde{y}\|_2^2\mu_5\right)}\tilde{x}\tilde{x}^T,$$

$$\Psi_{12}=\frac{\theta_1^2\gamma_1\gamma_2\gamma_4\gamma_8\mu_4\xi_1+\theta_1^2\theta_4^4\|\tilde{y}\|_2^2\|\tilde{z}\|_2^2\gamma_1\gamma_2\gamma_3\gamma_4\mu_3\mu_6\xi_2}{\mu_4\xi_1\left(\gamma_4\mu_2\mu_4+\|\tilde{y}\|_2^2\mu_5\right)}\tilde{x}\tilde{y}^T,$$

$$\Psi_{13}=-\frac{\theta_1^2\theta_4^2\|\tilde{y}\|_2^2\gamma_1\gamma_3\gamma_4\mu_3\xi_2}{\mu_4\xi_1}\tilde{x}\tilde{z}^T,\ \Psi_{22}=\frac{\gamma_2}{\mu_2}I_m+\frac{\theta_4^4\|\tilde{z}\|_2^2\gamma_2^2\gamma_3\gamma_4\mu_2\mu_3\mu_6^2-\gamma_2\mu_5\xi_1}{\mu_2\xi_1\left(\gamma_4\mu_2\mu_4+\|\tilde{y}\|_2^2\mu_5\right)}\tilde{y}\tilde{y}^T,$$

$$\Psi_{23}=-\frac{\theta_4^2\gamma_2\gamma_3\gamma_4\mu_3\mu_6}{\xi_1}\tilde{y}\tilde{z}^T,\ \Psi_{33}=\frac{\gamma_3}{\mu_3}I_l-\frac{\gamma_3\left[\mu_7+\theta_4^2\gamma_7\left(\gamma_4\mu_2\mu_4+\|\tilde{y}\|_2^2\mu_5\right)\right]}{\xi_1}\tilde{z}\tilde{z}^T.$$

经过一系列繁琐的初级代数计算, 可以得到

$$\begin{aligned}q^T\Phi^{-1}q=&\frac{4\theta_1^4\theta_2^4\left\|r_f\right\|_2^2}{\gamma_1\mu_1}+\frac{\theta_2^4\theta_3^4\left\|r_g\right\|_2^2}{\gamma_2\mu_2}+\frac{4\theta_3^2\theta_4^4\left\|r_h\right\|_2^2}{\gamma_3\mu_3}+\left(r_f^T\tilde{x}\right)^2l_1+\left(r_g^T\tilde{y}\right)^2l_2\\&+\left(r_h^T\tilde{z}\right)^2l_3+\left(r_f^T\tilde{x}\right)\left(r_g^T\tilde{y}\right)l_4+\left(r_f^T\tilde{x}\right)\left(r_h^T\tilde{z}\right)l_5+\left(r_g^T\tilde{y}\right)\left(r_h^T\tilde{z}\right)l_6,\end{aligned}\tag{3.18}$$

这里

$$\begin{aligned}l_1=&\frac{\theta_1^4\|\tilde{y}\|_2^2\gamma_8\left(2\theta_4^2\|\tilde{y}\|_2^2\|\tilde{z}\|_2^2\gamma_1\Omega_5+\|\tilde{y}\|_2^2\gamma_1\gamma_8\Omega_4-4\theta_2^2\gamma_4\Omega_2-2\|\tilde{x}\|_2^2\gamma_5\Omega_2\right)}{\gamma_1{\gamma_4}^2}\\&-\frac{\theta_1^4\theta_4^2\|\tilde{y}\|_2^2\|\tilde{z}\|_2^2\left(4\theta_2^2\gamma_4\Omega_3+2\|\tilde{x}\|_2^2\gamma_5\Omega_3+\theta_4^2\|\tilde{y}\|_2^2\|\tilde{z}\|_2^2\gamma_1\Omega_6\right)}{\gamma_1\gamma_4^2}\\&+\frac{\theta_1^4\Omega_1\left(4\theta_2^4\gamma_4^2+4\theta_2^2\|\tilde{x}\|_2^2\gamma_4\gamma_5+\|\tilde{x}\|_2^4\gamma_5^2\right)}{\gamma_1^2\gamma_4^2}\\&+\frac{\theta_1^4\left(4\theta_2^2\gamma_4\gamma_5\mu_2\mu_3+\|\tilde{x}\|_2^2\gamma_5^2\mu_2\mu_3+\|\tilde{y}\|_2^2\gamma_1\gamma_2\gamma_8^2\mu_1\mu_3+\theta_4^4\|\tilde{y}\|_2^4\|\tilde{z}\|_2^2\gamma_1\gamma_3\mu_1\mu_2\right)}{\gamma_1\gamma_4^2\mu_1\mu_2\mu_3},\end{aligned}$$

$$\begin{aligned}l_2=&\frac{\theta_1^2\|\tilde{x}\|_2^2\gamma_8\left(\theta_1^2\|\tilde{x}\|_2^2\gamma_2\gamma_8\Omega_1-2\theta_4^2\|\tilde{z}\|_2^2\gamma_2\gamma_9\Omega_3-2\theta_2^2\theta_3^2\gamma_4\Omega_2-2\|\tilde{y}\|_2^2\gamma_6\Omega_2\right)}{\gamma_2{\gamma_4}^2}\\&+\frac{\theta_4^2\|\tilde{z}\|_2^2\gamma_9\left(2\theta_2^2\theta_3^2\gamma_4\Omega_5+2\|\tilde{y}\|_2^2\gamma_6\Omega_5-\theta_4^2\|\tilde{z}\|_2^2\gamma_2\gamma_9\Omega_6\right)}{\gamma_2\gamma_4^2}\\&+\frac{\|\tilde{y}\|_2^2\gamma_6\left(2\theta_2^2\theta_3^2\gamma_4\mu_2\Omega_4+\gamma_2\gamma_6+\|\tilde{y}\|_2^2\gamma_6\mu_2\Omega_4\right)}{\gamma_2^2\gamma_4^2\mu_2}\\&+\frac{2\theta_2^2\theta_3^2\gamma_2\gamma_4\gamma_6\mu_1\mu_3+\theta_1^4\|\tilde{x}\|_2^2\gamma_1\gamma_2^2\gamma_8^2\mu_2\mu_3+\theta_2^4\theta_3^4\gamma_4^2\mu_1\mu_2\mu_3\Omega_4+\theta_4^4\|\tilde{z}\|_2^2\gamma_2^2\gamma_3\gamma_9^2\mu_1\mu_2}{\gamma_2^2\gamma_4^2\mu_1\mu_2\mu_3},\end{aligned}$$

$$l_3 = \frac{\theta_4^4\|\tilde{y}\|_2^2\gamma_9\left(\|\tilde{y}\|_2^2\gamma_3\gamma_9\Omega_4 - 2\theta_1^2\|\tilde{x}\|_2^2\|\tilde{y}\|_2^2\gamma_3\Omega_2 + 4\theta_3^2\gamma_4\Omega_5 + 2\|\tilde{z}\|_2^2\gamma_7\Omega_5\right)}{\gamma_3\gamma_4^2}$$
$$+ \frac{\theta_1^2\theta_4^4\|\tilde{x}\|_2^2\|\tilde{y}\|_2^2\left(\theta_1^2\|\tilde{x}\|_2^2\|\tilde{y}\|_2^2\gamma_3\Omega_1 - 4\theta_3^2\gamma_4\Omega_3 - 2\|\tilde{z}\|_2^2\gamma_7\Omega_3\right)}{\gamma_3\gamma_4^2}$$
$$+ \frac{\theta_4^4\|\tilde{z}\|_2^2\gamma_7\left(\gamma_3\gamma_7 - 4\theta_3^2\gamma_4\mu_3\Omega_6 - \|\tilde{z}\|_2^2\gamma_7\mu_3\Omega_6\right)}{\gamma_3^2\gamma_4^2\mu_3}$$
$$+ \frac{\theta_4^4\left(4\theta_3^2\gamma_3\gamma_4\gamma_7\mu_1\mu_2 + \theta_1^4\|\tilde{x}\|_2^2\|\tilde{y}\|_2^4\gamma_1\gamma_3^2\mu_2\mu_3 + \|\tilde{y}\|_2^2\gamma_2\gamma_3^2\gamma_9^2\mu_1\mu_3 - 4\theta_3^4\gamma_4^2\mu_1\mu_2\mu_3\Omega_6\right)}{\gamma_3^2\gamma_4^2\mu_1\mu_2\mu_3},$$

$$l_4 = \frac{2\theta_1^4\|\tilde{x}\|_2^2\gamma_8\left(\|\tilde{y}\|_2^2\gamma_1\gamma_8\Omega_2 + \theta_4^2\|\tilde{y}\|_2^2\|\tilde{z}\|_2^2\gamma_1\Omega_3 - 2\theta_2^2\gamma_4\Omega_1 - \|\tilde{x}\|_2^2\gamma_5\Omega_1\right)}{\gamma_1\gamma_4^2}$$
$$+ \frac{2\theta_1^2\theta_4^2\|\tilde{z}\|_2^2\gamma_9\left(2\theta_2^2\gamma_4\Omega_3 + \|\tilde{x}\|_2^2\gamma_5\Omega_3 - \|\tilde{y}\|_2^2\gamma_1\gamma_8\Omega_5 + \theta_4^2\|\tilde{y}\|_2^2\|\tilde{z}\|_2^2\gamma_1\Omega_6\right)}{\gamma_1\gamma_4^2}$$
$$- \frac{2\theta_1^2\gamma_8\left(2\theta_1^2\theta_2^2\gamma_4\mu_2 + \theta_2^2\theta_3^2\gamma_4\mu_1 + \theta_1^2\|\tilde{x}\|_2^2\gamma_5\mu_2 + \|\tilde{y}\|_2^2\gamma_6\mu_1\right)}{\gamma_4^2\mu_1\mu_2}$$
$$- \frac{2\theta_1^2\|\tilde{y}\|_2^2\left(\theta_2^2\theta_3^2\theta_4^2\|\tilde{z}\|_2^2\gamma_4\Omega_5 + \theta_4^2\|\tilde{y}\|_2^2\|\tilde{z}\|_2^2\gamma_6\Omega_5 + \theta_2^2\theta_3^2\gamma_4\gamma_8\Omega_4 + \|\tilde{y}\|_2^2\gamma_6\gamma_8\Omega_4\right)}{\gamma_2\gamma_4^2}$$
$$+ \frac{2\theta_1^2\theta_2^2\Omega_2\left(2\theta_2^2\theta_3^2\gamma_4 + \theta_3^2\|\tilde{x}\|_2^2\gamma_5 + 2\|\tilde{y}\|_2^2\gamma_6\right)}{\gamma_1\gamma_2\gamma_4}$$
$$+ \frac{2\theta_1^2\|\tilde{y}\|_2^2\left(\|\tilde{x}\|_2^2\gamma_5\gamma_6\mu_3\Omega_2 - \theta_4^4\|\tilde{z}\|_2^2\gamma_1\gamma_2\gamma_3\gamma_9\right)}{\gamma_1\gamma_2\gamma_4^2\mu_3},$$

$$l_5 = \frac{2\theta_1^2\theta_4^2\|\tilde{y}\|_2^2\left(\theta_4^2\|\tilde{y}\|_2^2\|\tilde{z}\|_2^2\gamma_9\Omega_5 - \theta_1^2\|\tilde{x}\|_2^2\|\tilde{y}\|_2^2\gamma_8\Omega_2 - \theta_1^2\theta_4^2\|\tilde{x}\|_2^2\|\tilde{y}\|_2^2\|\tilde{z}\|_2^2\Omega_3 + \|\tilde{y}\|_2^2\gamma_8\gamma_9\Omega_4\right)}{\gamma_4^2}$$
$$+ \frac{4\theta_1^2\theta_4^2\|\tilde{y}\|_2^2\left(\theta_1^2\theta_2^2\gamma_1\mu_3 + \theta_3^2\theta_4^2\gamma_1\mu_1 - \theta_2^2\gamma_9\mu_1\mu_3\Omega_2 + \theta_1^2\theta_2^2\|\tilde{x}\|_2^2\mu_1\mu_3\Omega_1\right)}{\gamma_1\gamma_4\mu_1\mu_3}$$
$$+ \frac{4\theta_1^2\theta_4^2\left(\theta_3^2\|\tilde{y}\|_2^2\gamma_1\gamma_8\Omega_5 - \theta_2^2\|\tilde{z}\|_2^2\gamma_7\Omega_3 - \theta_3^2\|\tilde{x}\|_2^2\gamma_5\Omega_3 - \theta_3^2\theta_4^2\|\tilde{y}\|_2^2\|\tilde{z}\|_2^2\gamma_1\Omega_6\right)}{\gamma_1\gamma_3\gamma_4}$$
$$+ \frac{2\theta_1^2\theta_4^2\left(\|\tilde{y}\|_2^2\|\tilde{z}\|_2^2\gamma_1\gamma_7\gamma_8\Omega_5 - 4\theta_2^2\theta_3^2\gamma_4^2\Omega_3 - \|\tilde{x}\|_2^2\|\tilde{z}\|_2^2\gamma_5\gamma_7\Omega_3 - \theta_4^2\|\tilde{y}\|_2^2\|\tilde{z}\|_2^4\gamma_1\gamma_7\Omega_6\right)}{\gamma_1\gamma_3\gamma_4^2}$$
$$+ \frac{2\theta_1^2\theta_4^2\|\tilde{x}\|_2^2\|\tilde{y}\|_2^2\gamma_5\left(\theta_1^2\gamma_1 - \gamma_9\mu_1\Omega_2 + \theta_1^2\|\tilde{x}\|_2^2\mu_1\Omega_1\right)}{\gamma_1\gamma_4^2\mu_1}$$
$$+ \frac{2\theta_1^2\theta_4^2\|\tilde{y}\|_2^2\left(\gamma_2\gamma_8\gamma_9\mu_3 + \theta_4^2\|\tilde{z}\|_2^2\gamma_7\mu_2\right)}{\gamma_4^2\mu_2\mu_3},$$

$$
\begin{aligned}
l_6 =& \frac{2\theta_4^4\|\tilde{z}\|_2^2\gamma_9\left(\theta_1^2\|\tilde{x}\|_2^2\|\tilde{y}\|_2^2\gamma_3\Omega_3-\|\tilde{y}\|_2^2\gamma_3\gamma_9\Omega_5+2\theta_3^2\gamma_4\Omega_6+\|\tilde{z}\|_2^2\gamma_7\Omega_6\right)}{\gamma_3\gamma_4^2} \\
&+\frac{2\theta_3^2\theta_4^2\left(\theta_1^2\theta_2^2\|\tilde{x}\|_2^2\|\tilde{y}\|_2^2\mu_2\mu_3\Omega_2-\theta_2^2\gamma_2\gamma_9\mu_3-\theta_2^2\|\tilde{y}\|_2^2\gamma_9\mu_2\mu_3\Omega_4-2\theta_4^2\gamma_2\gamma_9\mu_2\right)}{\gamma_2\gamma_4\mu_2\mu_3} \\
&+\frac{2\theta_1^2\theta_4^2\|\tilde{x}\|_2^2\|\tilde{y}\|_2^2\gamma_8\left(\gamma_9\mu_1\Omega_2-\theta_1^2\gamma_1-\theta_1^2\|\tilde{x}\|_2^2\mu_1\Omega_1\right)}{\gamma_4^2\mu_1} \\
&+\frac{2\theta_1^2\theta_4^2\|\tilde{x}\|_2^2\gamma_8\Omega_3\left(2\theta_3^2\gamma_4+\|\tilde{z}\|_2^2\gamma_7\right)}{\gamma_3\gamma_4^2} \\
&+\frac{2\theta_4^2\|\tilde{y}\|_2^2\gamma_6\left(\theta_1^2\|\tilde{x}\|_2^2\|\tilde{y}\|_2^2\gamma_3\Omega_2-\|\tilde{y}\|_2^2\gamma_3\gamma_9\Omega_4-2\theta_3^2\gamma_4\Omega_5-\|\tilde{z}\|_2^2\gamma_7\Omega_5\right)}{\gamma_2\gamma_3\gamma_4^2} \\
&-\frac{2\theta_4^2\left(\|\tilde{y}\|_2^2\gamma_2\gamma_3\gamma_6\gamma_9\mu_3+2\theta_2^2\theta_3^4\gamma_4^2\mu_2\mu_3\Omega_5+\theta_2^2\theta_3^2\|\tilde{z}\|_2^2\gamma_4\gamma_7\mu_2\mu_3\Omega_5+\theta_4^2\|\tilde{z}\|_2^2\gamma_2\gamma_3\gamma_7\gamma_9\mu_2\right)}{\gamma_2\gamma_3\gamma_4^2\mu_2\mu_3}.
\end{aligned}
$$

其中 $\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5,\gamma_6,\gamma_7,\gamma_8$ 和 γ_9 在定理 3.1 的前面部分定义过，$\mu_1,\mu_2,\mu_3,\Omega_1,\Omega_2,\Omega_3,\Omega_4,\Omega_5$ 和 Ω_6 在定理 3.2 的前面部分定义过, 最后将 (3.18) 和 (3.3) 带入 (3.17) 中得到期望的表达式 (3.16).

结构向后误差 $\eta^{(\theta_1,\theta_2,\theta_3,\theta_4,\lambda_1,\lambda_2,\lambda_3)}(\tilde{x},\tilde{y},\tilde{z})$ 的表达式 (3.16) 虽然看起来很复杂, 但是从计算的角度来看, 表达式 (3.16) 是容易计算的, 因为它只涉及加、减、乘、除这样基本的运算.

4. 数值实验

本节将给出一个数值例子来比较第 3 节中推出的结构向后误差 $\eta_S(\tilde{x},\tilde{y},\tilde{z})$ 和相应的无结构向后误差 $\eta(\tilde{t})$. 数值实验在 MATLAB R2015b 中进行, 机器精度为 2.2204×10^{-16}.

考虑线性系统 (1.1) 满足

$$
A=MPM,\quad B=\begin{bmatrix}0&0&1&0&0&0\\0&1&0&0&0&0\\10^{-3}&0&0&0&0&0\end{bmatrix},\quad C=\begin{bmatrix}1&-2&1\\-2&-1&0\\1&0&0\\0&0&0\\0&0&0\\0&0&0\end{bmatrix}
$$

和

$$
D=-I_6,\quad f=\left[10^8,10,0,0,0,0\right]^T,\quad g=\left[10^{-8},0,0\right]^T,\quad h=\left[10^{-8},0,0\right]^T
$$

其中

$$
M=\operatorname{diag}(1,5,10,50,100,10000),\quad P=(p_{ij}),\quad p_{ij}=\frac{(i+j-2)!}{(i-1)!(j-1)!}
$$

这个问题是由文 [11]中的 Example 5.2 修改而来. 很显然系数矩阵是非奇异的. 使用列选主元的高斯消去法可以得到一个计算解 $\tilde{t}=\left(\tilde{x}^T,\tilde{y}^T,\tilde{z}^T\right)^T$, 其中

$$\tilde{x}=\begin{pmatrix}2.0264\times 10^{8}\\ 3.0780\times 10^{6}\\ -4.1382\times 10^{7}\\ 1.0019\times 10^{7}\\ -2.5379\times 10^{6}\\ 4.8888\times 10^{3}\end{pmatrix},\ \tilde{y}=\begin{pmatrix}4.8552\times 10^{7}\\ -1.0059\times 10^{8}\\ -2.4993\times 10^{8}\end{pmatrix}\ \text{和}\ \tilde{z}=\begin{pmatrix}-2.0264\times 10^{5}\\ 3.4833\times 10^{6}\\ 4.8552\times 10^{7}\\ 0.0000\\ 0.0000\\ 0.0000\end{pmatrix}.$$

由 (1.2) 得, 无结构向后误差

$$\eta(\tilde{t})=9.7551\times 10^{-22}.$$

由 (1.5) 和 (3.16) 得, 结构向后误差

$$\eta_S(\tilde{x},\tilde{y},\tilde{z})=3.8498\times 10^{-6}.$$

5. 总结

本文首先利用文 [13,19]中的技巧和 Kronecker 积的性质, 将矩阵优化问题转化为正定二次型的最小值问题，然后利用 Sherman-Morrison-Woodbury 公式, 最终获得结构向后误差的可计算的具体表达式. 最后我们给出了一个数值实验, 以证明我们的结果可以很容易地用来测试实际数值算法的稳定性.

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