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高阶分数阶微分方程边值问题解的存在唯一性

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摘 要

研究了具有混合单调非线性项的**Riemann-Liouville** 型高阶分数阶微分方程边值问题。利用**Green** 函数的性质以及混合单调算子的不动点定理证明了该边值问题解的存在唯一性，并给出一个实例验证了结论的正确性。

关键词

分数阶微分方程，边值问题，混合单调算子，存在唯一性

Existence and Uniqueness of Solutions to Boundary Value Problems for Higher-Order Fractional Differential Equations

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Abstract

The boundary value problem of Riemann-Liouville higher order fractional differential

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equations with mixed monotone nonlinear terms is studied. By using the properties of Green's function and the fixed point theorem of mixed monotone operators, the existence and uniqueness of the solution of the boundary value problem are proved, and an example is given to verify the correctness of the conclusion.

Keywords

Fractional Differential Equation, Boundary Value Problem, Mixed Monotone Operator, Existence and Uniqueness

1. 引言

在过去的几十年中,分数阶微分方程因其在电化学、粘弹性、电磁学、多孔介质、控制等各个领域的应用而变得越来越重要. 有关详细信息，请参阅 [1–12] 及其中的参考文献. 近年来, 关于分数阶微分方程的研究受到了许多学者的关注, 如: 不同边值条件下解的存在唯一性, 正解的存在唯一性, 其主要研究方法包括锥上不动点定理, 混合单调算子上的不动点定理等.

文献 [13] 应用混合单调算子的不动点定理研究了分数阶微分方程边值问题

$$\begin{cases} D_{0+}^{\alpha}u(t)+q(t)f(u,u',\cdots,u^{(n-2)})=0,\ 0<t<1,\ n-1<\alpha\le n, \\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=u^{(n-2)}(1)=0 \end{cases} \tag{1}$$

正解的存在唯一性, 其中D_{0+}^{α} 是标准的Riemann-Liouville 型分数阶微分导数, $n\ge 2$, $n\in\mathbb{N}$, 函数f 和q 满足如下特性:

(C_1) $f(x_1,x_2,\cdots,x_{n-1})=g(x_1,x_2,\cdots,x_{n-1})+h(x_1,x_2,\cdots,x_{n-1})$, 其中$g:[0,+\infty)\times R^{n-2}\to[0,+\infty)$ 是连续的, $h:(0,+\infty)\times(R\setminus\{0\})^{n-2}\to(0,+\infty)$ 是连续的.

(C_2) g 在$x_i>0$, $i=1,2,\cdots,n-1$ 时是非减的; 此外, h 在$x_i>0$, $i=1,2,\cdots,n-1$ 是非增的.

(C_3) 存在$\beta\in(0,1)$ 使得对于$x_i>0$, $i=1,2,\cdots,n-1$ 有

$$g(tx_1,\cdots,tx_{n-1})\ge t^{\beta}g(x_1,\cdots,x_{n-1}),\ \ t\in(0,1),$$
$$h(t^{-1}x_1,\cdots,t^{-1}x_{n-1})\ge t^{\beta}h(x_1,\cdots,x_{n-1}),\ \ t\in(0,1).$$

(C_4) $t^rq:[0,1]\to[0,+\infty)$ 是连续的且$\int_0^1 q(s)s^{-\beta(\alpha-1)}ds<+\infty$, $0\le r<1$.

文献 [14] 研究了以下分数阶微分方程边值问题

$$\begin{cases} -D_{0+}^{\alpha}u(t)=f(t,u(t),u(t))+g(t,u(t),u(t))-b,\ \ t\in(0,1),\ n-1<\alpha\le n,\\ u^{(i)}(0)=0,\ \ i=0,1,\cdots,n-2,\\ D_{0+}^{\beta}u(1)=0,\quad 1\le\beta\le n-2 \end{cases} \tag{2}$$

解的存在唯一性, 其中D_{0+}^{α}, D_{0+}^{β} 是标准的Riemann-Liouville 型分数阶微分导数, $n\ge3$, $b>0$ 是常数, f, $g:[0,1]\times(-\infty,+\infty)\times(-\infty,+\infty)\to(-\infty,+\infty)$ 是连续函数. 问题(2) 包括著名的弹性梁方程和文献 [15,16] 中考虑的分数阶问题.

最近, 自Guo 和Lakshmikantham [17] 引入混合单调算子以来, 许多学者都研究了Banach 空间中各种类型的混合单调算子, 并构造了许多相关的定理. 在文献 [18] 中, Bhaskar 和Lakshikantham 在半序度量空间中研究了混合单调算子的一些耦合不动点定理. 在文献 [19] 中, Li 和Zhao 考虑了一类$\tau-\phi$ 的混合单调算子. 此外, 具有扰动的混合单调算子已被广泛研究. 在 [20] 中, Liu 等人考虑了算子方程:

$$A(x,x)+B(x,x)=x$$

在半序Banach 空间上正解的存在唯一性, 其中A 和B 是两个混合单调算子, 此外, 作者还给出了该算子在非线性分数阶微分方程中应用.

受上述工作的启发, 本文将运用文献 [14] 中混合单调算子的不动点定理研究如下具有混合单调非线性项的高阶分数阶微分方程边值问题

$$\begin{cases} D_{0+}^{\alpha}u(t)+q(t)f(u(t),u'(t),\cdots,u^{(n-2)}(t))-b=0,\ t\in[0,1],\\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=u^{(n-2)}(1)=0 \end{cases} \tag{3}$$

解的存在唯一性, 其中D_{0+}^{α} 是标准的Riemann-Liouville 型分数阶微分导数, $b>0$ 是一个常数, $n-1<\alpha\le n$, $n\ge2$, f 是一个非线性函数, $q\in C[0,1]$ 满足$q(t)\ge0$, $q(t)\not\equiv0$.

2. 预备知识

方便起见, 首先给出一些必要的定义和引理, 为后续的研究工作提供数学工具.

定义 2.1 [2,4] 函数$y\in C[0,1]$ 的$\alpha>0$ 阶Riemann-Liouville 分数积分定义为

$$I_{0+}^{\alpha}y(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{y(s)}{(t-s)^{1-\alpha}}ds.$$

定义 2.2 [2,4] 函数$y\in C[0,1]$ 的$\alpha>0$ 阶Riemann-Liouville 分数阶导数定义为

$$D_{0+}^{\alpha}y(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_0^t\frac{y(s)}{(t-s)^{\alpha-n-1}}ds,$$

其中$n = [\alpha] + 1$.

引理 2.1 [13] 令$y \in C[0,1]$, 那么边值问题

$$\begin{cases} D_{0^+}^{\alpha-n+2}x(t) + y(t) = 0, 0 \le t \le 1, n-1 < \alpha \le n, n \ge 2, \\ x(0) = x(1) = 0 \end{cases} \tag{4}$$

有唯一解

$$x(t) = \int_0^1 G(t,s)y(s)ds,$$

其中

$$G(t,s) = \begin{cases} \frac{(t(1-s))^{\alpha-n+1} - (t-s)^{\alpha-n+1}}{\Gamma(\alpha-n+2)}, & 0 \le s \le t \le 1, \\ \frac{(t(1-s))^{\alpha-n+1}}{\Gamma(\alpha-n+2)}, & 0 \le t \le s \le 1. \end{cases} \tag{5}$$

引理 2.2 [13]定义(5) 中的函数$G(t,s)$ 满足如下条件:
(1) $G(t,s) \ge 0$, $G(t,s) \le t^{\alpha-n+1}/\Gamma(\alpha-n+2)$, $G(t,s) \le G(s,s)$，$0 \le t, s \le 1$;
(2) 存在一个正函数$\rho \in C(0,1)$ 使得$min_{\gamma \le t \le \delta} G(t,s) \ge \rho(s)G(s,s)$, $s \in (0,1)$, 其中$0 < \gamma < \delta < 1$.

引理 2.3 *令$u(t) = I_{0^+}^{n-2}x(t)$, $x(t) \in C[0,1]$, 那么$D_{0^+}^{n-2}u(t) = x(t)$. 问题(3) 可以转变成如下问题(6):*

$$\begin{cases} D_{0^+}^{\alpha-n+2}x(t) + q(t)f(I_{0^+}^{n-2}x(t), I_{0^+}^{n-3}x(t), ..., x(t)) - b = 0, 0 \le t \le 1, n-1 < \alpha \le n, n \ge 2, \\ x(0) = x(1) = 0. \end{cases} \tag{6}$$

如果$x \in C[0,1]$ 是问题(6) 的解, 那么$u(t) = I_{0^+}^{n-2}x(t)$ 是问题(3) 的解.

证. 该证明与文献 [13] 中引理2.7 类似, 此处省略. □

本文中, E 是赋范的实Banach 空间, θ 是E 中的零元. 一个非空闭凸集$P \in E$ 如果满足(1) $x \in P$, $\lambda \ge 0 \Rightarrow \lambda x \in P$; (2) $x \in P$, $-x \in P \Rightarrow x = \theta$, 则称$P$ 是E 上的一个锥. E 中的半序关系为$x \lesssim y$ 当且仅当$y - x \in P$. 此外, 如果存在一个常数$N > 0$ 使得对所有的$x, y \in E$, $\theta \lesssim x \lesssim y$ 有$\|x\| \le N\|y\|$, 则称P 是一个正规锥, 其中最小的N 被称为是P 的正规常数. 给定$h > \theta$($i.e.$, $\theta \lesssim h$ 且 $h \ne \theta$), 我们定义集合C_h 为

$$C_h = \{x \in E \mid 存在\ \lambda > 0\ 和\ \mu > 0\ 使得\ \lambda h \lesssim x \lesssim \mu h\}.$$

设$e \in P$, 且$\theta \lesssim e \lesssim h$. 定义

$$C_{h,e} = \{x \in E \mid x + e \in C_h\}.$$

定义 2.3 [21] $A : C_{h,e} \times C_{h,e} \to E$, 如果$A(x,y)$ 关于x 非减关于y 非增, 则称A 是混合单调的, 即如果$x_i, y_i \in C_{h,e}(i = 1,2)$, $x_1 \le x_2$, $y_1 \ge y_2$, 则$A(x_1,y_1) \le A(x_2,y_2)$.

引理 2.4 [14] 设$A, B : C_{h,e} \times C_{h,e} \to E$ 是两个混合单调算子且满足如下条件:

(i) 对于所有的$t \in (0,1)$, 存在$\psi(t) \in (t,1)$ 使得

$$A(tx+(t-1)e, t^{-1}y+(t^{-1}-1)e) \geq \psi(t)A(x,y)+(\psi(t)-1)e,\ \forall x,y \in C_{h,e};$$

(ii) 对于所有的$t \in (0,1)$ 和$x,y \in C_{h,e}$,

$$B(tx+(t-1)e, t^{-1}y+(t^{-1}-1)e) \geq tB(x,y)+(t-1)e;$$

(iii) $A(h,h) \in C_{h,e}$ 且$B(h,h) \in C_{h,e}$;
(iv) 存在一个常数$\delta > 0$, 使得对于所有的$x,y \in C_{h,e}$, 有

$$A(x,y) \geq \delta B(x,y)+(\delta-1)e.$$

那么算子方程$A(x,x)+B(x,x)+e=x$ 在$C_{h,e}$ 上有唯一解x^*, 并且对于任何初值$x_0, y_0 \in C_{h,e}$, 构造序列

$$\begin{aligned} x_n =& A(x_{n-1},y_{n-1})+B(x_{n-1},y_{n-1})+e, \\ y_n =& A(y_{n-1},x_{n-1})+B(y_{n-1},x_{n-1})+e, \quad n=1,2,\cdots, \end{aligned}$$

当$n \to \infty$ 时, 在E 中有$x_n \to x^*$, $y_n \to x^*$.

3. 主要结果

方便起见, 我们定义如下符号.
对于$t \in [0,1]$,

$$\begin{aligned} e(t) &= \frac{b}{\Gamma(\alpha-n+3)}\left(t^{\alpha-n+1}-t^{\alpha-n+2}\right), \\ E^* &= max\{I_{0^+}^{n-2}e(t), I_{0^+}^{n-3}e(t), \ldots, e(t)\}. \end{aligned}$$

定理 3.1 如果以下条件成立:
(H_1) 对于任意的$x_i \in [-E^*,+\infty)(i=0,1,2,\cdots,n-2)$, 有$f(x_0,x_1,\cdots,x_{n-2}) = g(x_0,x_1,\cdots,x_{n-2},$

$x_0,$

$x_1,\ldots,x_{n-2}) + \phi(x_0,x_1,\cdots,x_{n-2},x_0,x_1,\cdots,x_{n-2})$ 其中$g \in C([-E^*,+\infty)^{2(n-1)},(-\infty,+\infty))$,

$\phi \in C([-E^*,+\infty)^{2(n-1)},(-\infty,+\infty))$.
(H_2) 对于任意固定的$y_i \in [-E^*,+\infty)(i=0,1,2,\cdots,n-2)$, $g(x_0,x_1,\ldots,x_{n-2},y_0,y_1,\cdots,y_{n-2})$ 和 $\phi(x_0,x_1,\cdots,x_{n-2},y_0,y_1,\cdots,y_{n-2})$ 在$x_i \in [-E^*,+\infty)$ 非减; 对于任意固定的$x_i \in [-E^*,+\infty)$

$(i=0,1,2,\ldots,n-2)$, $g(x_0,x_1,\cdots,x_{n-2},y_0,y_1,\cdots,y_{n-2})$和$\phi(x_0,x_1,\cdots,x_{n-2},y_0,y_1,\cdots,y_{n-2})$ 在y_i $\in [-E^*,+\infty)$ 非增.
(H_3) 对于任意的$\tau \in (0,1)$, 存在$\varphi(\tau) \in (\tau,1)$ 使得对于所有的$x_i, y_i \in [-E^*,+\infty)(i=0,1,\cdots,n-$

2), 有

$$
\begin{aligned}
& g(\tau x_0+(\tau-1)\rho_0, \tau x_1+(\tau-1)\rho_0, \cdots, \tau x_{n-2}+(\tau-1)\rho_0, \tau^{-1}y_0+(\tau^{-1}-1)\rho_0, \\
& \quad \tau^{-1}y_1+(\tau^{-1}-1)\rho_0, \cdots, \tau^{-1}y_{n-2}+(\tau^{-1}-1)\rho_0) \geq \varphi(\tau)g(x_0, x_1, \cdots, x_{n-2}, y_0, y_1, \cdots, y_{n-2}),\ \rho_0 \in [0, E^*], \\
& \phi(\tau x_0+(\tau-1)\rho_0, \tau x_1+(\tau-1)\rho_0, \ldots, \tau x_{n-2}+(\tau-1)\rho_0, \tau^{-1}y_0+(\tau^{-1}-1)\rho_0, \\
& \quad \tau^{-1}y_1+(\tau^{-1}-1)e, \ldots, \tau^{-1}y_{n-2}+(\tau^{-1}-1)e) \geq \tau h(x_0, x_1, \ldots, x_{n-2}, y_0, y_1, \ldots, y_{n-2}),\ \rho_0 \in [0, E^*].
\end{aligned}
$$

(H_4) 对于任意固定的$x_i, y_i \in [-E^*, +\infty)(i=0,1,2,\ldots,n-2)$, 存在一个常数$\delta_0 > 0$ 使得

$$
g(x_0, x_1, \ldots, x_{n-2}, y_0, y_1, \ldots, y_{n-2}) \geq \delta_0 \phi(x_0, x_1, \ldots, x_{n-2}, y_0, y_1, \ldots, y_{n-2}).
$$

(H_5) 存在一个常数$M = \frac{b}{(\alpha-n+1)\Gamma(\alpha-n+3)}$, 使得函数$g$ 和函数ϕ 满足

$$
\begin{aligned}
& g(M, M, \ldots, M, 0, 0, \ldots, 0) < +\infty, \\
& \phi(M, M, \ldots, M, 0, 0, \ldots, 0) < +\infty, \\
& \phi(0, 0, \ldots, 0, M, M, \ldots, M) > 0.
\end{aligned}
$$

那么问题*(6)* 在$C_{h,e}$ 有唯一的平凡解x^*, 其中$h(t) = Mt^{\alpha-n+1}$, $t \in [0,1]$. 此外, 可以构建以下两个序列:

$$
\begin{aligned}
\omega_n(t) = & \int_0^1 G(t,s) g\big(I_{0^+}^{n-2}\omega_{n-1}(s), I_{0^+}^{n-3}\omega_{n-1}(s), ..., \omega_{n-1}(s), I_{0^+}^{n-2}\tau_{n-1}(s), I_{0^+}^{n-3}\tau_{n-1}(s), \\
& ..., \tau_{n-1}(s)\big) ds + \int_0^1 G(t,s) q(s) \phi\big(I_{0^+}^{n-2}\omega_{n-1}(s), I_{0^+}^{n-3}\omega_{n-1}(s), ..., \omega_{n-1}(s), \\
& I_{0^+}^{n-2}\tau_{n-1}(s), I_{0^+}^{n-3}\tau_{n-1}(s), ..., \tau_{n-1}(s)\big) ds - \frac{b}{\Gamma(\alpha-n+3)} \left(t^{\alpha-n+1} - t^{\alpha-n+2}\right), \\
& n = 1, 2, \cdots, \\
\tau_n(t) = & \int_0^1 G(t,s) g\big(I_{0^+}^{n-2}\tau_{n-1}(s), I_{0^+}^{n-3}\tau_{n-1}(s), ..., \tau_{n-1}(s), I_{0^+}^{n-2}\omega_{n-1}(s), I_{0^+}^{n-3}\omega_{n-1}(s), \\
& ..., \omega_{n-1}(s)\big) ds + \int_0^1 G(t,s) q(s) \phi\big(I_{0^+}^{n-2}\tau_{n-1}(s), I_{0^+}^{n-3}\tau_{n-1}(s), ..., \tau_{n-1}(s), \\
& I_{0^+}^{n-2}\omega_{n-1}(s), I_{0^+}^{n-3}\omega_{n-1}, ..., \omega_{n-1}\big) ds - \frac{b}{\Gamma(\alpha-n+3)} \left(t^{\alpha-n+1} - t^{\alpha-n+2}\right), \\
& n = 1, 2, \cdots,
\end{aligned}
$$

对于任意给定的$\omega_0, \tau_0 \in C_{h,e}$, $t \in [0,1]$, 有$\{\omega_n(t)\}$ 和$\{\tau_n(t)\}$ 都一致收敛于x^*.

证. 对于$t \in [0,1]$,

$$
\begin{aligned}
& e(t) = \frac{b}{\Gamma(\alpha-n+3)} \left(t^{\alpha-n+1} - t^{\alpha-n+2}\right) \geq 0, \\
& E^* = max\{I_{0^+}^{n-2}e(t), I_{0^+}^{n-3}e(t), \ldots, e(t)\}.
\end{aligned}
$$

即e, $E^* \in P$. 此外, $e(t) \leq E^* \leq \frac{b}{(\alpha-n+1)\Gamma(\alpha-n+3)} t^{\alpha-n+1} = h(t)$, $C_{h,e} = \{x \in C[0,1] | x + e \in C_h\}$.

证明之前, 首先转换$x(t)$. 由引理2.1 和(H_1), 问题(6) 的积分公式为

$$
\begin{aligned}
x(t) &= \int_0^1 G(t,s)[q(s)f(I_{0+}^{n-2}x(s), I_{0+}^{n-3}x(s), ..., x(s)) - b]ds \\
&= \int_0^1 G(t,s)q(s)f\big(I_{0+}^{n-2}x(s), I_{0+}^{n-3}x(s), ..., x(s)\big)ds - b\int_0^1 G(t,s)ds \\
&= \int_0^1 G(t,s)q(s)[g\big(I_{0+}^{n-2}x(s), I_{0+}^{n-3}x(s), ..., x(s), I_{0+}^{n-2}x(s), I_{0+}^{n-3}x(s), ..., x(s)\big) + \\
&\quad \phi\big(I_{0+}^{n-2}x(s), I_{0+}^{n-3}x(s), ..., x(s), I_{0+}^{n-2}x(s), I_{0+}^{n-3}x(s), ..., x(s)\big)]ds - \frac{b\,(t^{\alpha-n+1} - t^{\alpha-n+2})}{\Gamma(\alpha-n+3)} \\
&= \int_0^1 G(t,s)q(s)g\big(I_{0+}^{n-2}x(s), I_{0+}^{n-3}x(s), ..., x(s), I_{0+}^{n-2}x(s), I_{0+}^{n-3}x(s), ..., x(s)\big)ds + \\
&\quad \int_0^1 G(t,s)q(s)\phi\big(I_{0+}^{n-2}x(s), I_{0+}^{n-3}x(s), ..., x(s), I_{0+}^{n-2}x(s), I_{0+}^{n-3}x(s), ..., x(s)\big)ds - e(t) \\
&= \int_0^1 G(t,s)q(s)g\big(I_{0+}^{n-2}x(s), I_{0+}^{n-3}x(s), ..., x(s), I_{0+}^{n-2}x(s), I_{0+}^{n-3}x(s), ..., x(s)\big)ds - e(t) + \\
&\quad \int_0^1 G(t,s)q(s)\phi\big(I_{0+}^{n-2}x(s), I_{0+}^{n-3}x(s), ..., x(s), I_{0+}^{n-2}x(s), I_{0+}^{n-3}x(s), ..., x(s)\big)ds - e(t) + e(t),
\end{aligned}
$$

其中$G(t,s)$ 在(5) 给出. 对于每个$t \in [0,1]$, $x, y \in C_{h,e}$, 定义如下算子:

$$
\begin{aligned}
A(x,y)(t) &= \int_0^1 G(t,s)q(s)g\big(I_{0+}^{n-2}x(s), I_{0+}^{n-3}x(s), ..., x(s), I_{0+}^{n-2}y(s), I_{0+}^{n-3}y(s), ..., y(s)\big)ds - e(t), \\
B(x,y)(t) &= \int_0^1 G(t,s)q(s)\phi\big(I_{0+}^{n-2}x(s), I_{0+}^{n-3}x(s), ..., x(s), I_{0+}^{n-2}y(s), I_{0+}^{n-3}y(s), ..., y(s)\big)ds - e(t).
\end{aligned}
$$

很容易证明x 是问题(6) 的解当且仅当$x = A(x,x) + B(x,x) + e$. 接下来, 分四步证明A 和B 满足引理2.4中的条件.

(1) 首先, 证明$A, B : C_{h,e} \times C_{h,e} \to E$ 是混合单调算子. 事实上, 对于所有的$x_i, y_i \in C_{h,e}(i = 1,2)$ 且$x_1 \geq x_2, y_1 \leq y_2$, 由$(H_2)$, 可得

$$
\begin{aligned}
A(x_1,y_1)(t) &= \int_0^1 G(t,s)q(s)g\big(I_{0+}^{n-2}x_1(s), I_{0+}^{n-3}x_1(s), ..., x_1(s), I_{0+}^{n-2}y_1(s), I_{0+}^{n-3}y_1(s), ..., y_1(s)\big)ds - e(t) \\
&\geq \int_0^1 G(t,s)q(s)g\big(I_{0+}^{n-2}x_2(s), I_{0+}^{n-3}x_2(s), ..., x_2(s), I_{0+}^{n-2}y_2(s), I_{0+}^{n-3}y_2(s), ..., y_2(s)\big)ds - e(t) \\
&= A(x_2,y_2)(t).
\end{aligned}
$$

即$A(x_1,y_1) \geq A(x_2,y_2)$. 用同样的方法可以得出$B(x_1,y_1) \geq B(x_2,y_2)$.

(2) 其次, 验证引理2.4 中的(i) 和(ii) 成立. 由(H_3), 对每个$\tau \in (0,1)$, $t \in [0,1]$, 存在$\varphi(\tau) \in (\tau, 1)$ 使得对每个$x, y \in C_{h,e}$, 取$\rho_0 = E^*$, 有

$$
\begin{aligned}
&A\big(\tau x + (\tau-1)e, \tau^{-1}y + (\tau^{-1}-1)e\big)(t) \\
&= \int_0^1 G(t,s)q(s)g\Big(I_{0+}^{n-2}[\tau x(s) + (\tau-1)e(s)], I_{0+}^{n-3}[\tau x(s) + (\tau-1)e(s)], ..., [\tau x(s) + (\tau-1)e(s)],
\end{aligned}
$$

$$
\begin{aligned}
&I_{0+}^{n-2}[\tau^{-1}y(s)+(\tau^{-1}-1)e(s)],I_{0+}^{n-3}[\tau^{-1}y(s)+(\tau^{-1}-1)e(s)],...,[\tau^{-1}y(s)+(\tau^{-1}-1)e(s)]\Big)ds-e(t)\\
=&\int_0^1 G(t,s)q(s)g\Big(\tau I_{0+}^{n-2}x(s)+(\tau-1)I_{0+}^{n-2}e(s),\tau I_{0+}^{n-3}x(s)+(\tau-1)I_{0+}^{n-3}e(s),...,\tau x(s)+(\tau-1)e(s),\\
&\tau^{-1}I_{0+}^{n-2}y(s)+(\tau^{-1}-1)I_{0+}^{n-2}e(s),\tau^{-1}I_{0+}^{n-3}y(s)+(\tau^{-1}-1)I_{0+}^{n-3}e(s),...,\tau^{-1}y+(\tau^{-1}-1)e(s)\Big)ds-e(t)\\
\geq&\int_0^1 G(t,s)q(s)g\Big(\tau I_{0+}^{n-2}x(s)+(\tau-1)E^*,\tau I_{0+}^{n-3}x(s)+(\tau-1)E^*,...,\tau x(s)+(\tau-1)E^*,\\
&\tau^{-1}I_{0+}^{n-2}y(s)+(\tau^{-1}-1)E^*,\tau^{-1}I_{0+}^{n-3}y(s)+(\tau^{-1}-1)E^*,...,\tau^{-1}y+(\tau^{-1}-1)E^*\Big)ds-e(t)\\
=&\int_0^1 G(t,s)q(s)g\Big(\tau I_{0+}^{n-2}x(s)+(\tau-1)\rho_0,\tau I_{0+}^{n-3}x(s)+(\tau-1)\rho_0,...,\tau x(s)+(\tau-1)\rho_0,\\
&\tau^{-1}I_{0+}^{n-2}y(s)+(\tau^{-1}-1)\rho_0,\tau^{-1}I_{0+}^{n-3}y(s)+(\tau^{-1}-1)\rho_0,...,\tau^{-1}y+(\tau^{-1}-1)\rho_0\Big)ds-e(t)\\
\geq&\varphi(\tau)\int_0^1 G(t,s)q(s)g\big(I_{0+}^{n-2}x(s),I_{0+}^{n-3}x(s),...,x(s),I_{0+}^{n-2}y(s),I_{0+}^{n-3}y(s),...,y(s)\big)ds-e(t)\\
=&\varphi(\tau)\int_0^1 G(t,s)q(s)g\big(I_{0+}^{n-2}x(s),I_{0+}^{n-3}x(s),...,x(s),I_{0+}^{n-2}y(s),I_{0+}^{n-3}y(s),...,y(s)\big)ds-e(t)+\varphi(\tau)e(t)-\varphi(\tau)e(t)\\
=&\varphi(\tau)[\int_0^1 G(t,s)q(s)g\big(I_{0+}^{n-2}x(s),I_{0+}^{n-3}x(s),...,x(s),I_{0+}^{n-2}y(s),I_{0+}^{n-3}y(s),...,y(s)\big)ds-e(t)]+(\varphi(\tau)-1)e(t)\\
=&\varphi(\tau)A(x,y)(t)+(\varphi(\tau)-1)e(t),
\end{aligned}
$$

$$
\begin{aligned}
&B\big(\tau x+(\tau-1)e,\tau^{-1}y+(\tau^{-1}-1)e\big)(t)\\
=&\int_0^1 G(t,s)q(s)\phi\Big(I_{0+}^{n-2}[\tau x(s)+(\tau-1)e(s)],I_{0+}^{n-3}[\tau x(s)+(\tau-1)e(s)],...,\tau x(s)+(\tau-1)e(s),\\
&I_{0+}^{n-2}[\tau^{-1}y(s)+(\tau^{-1}-1)e(s)],I_{0+}^{n-3}[\tau^{-1}y(s)+(\tau^{-1}-1)e(s)],...,\tau^{-1}y(s)+(\tau^{-1}-1)e(s)\Big)ds-e(t)\\
=&\int_0^1 G(t,s)q(s)\phi\Big(\tau I_{0+}^{n-2}x(s)+(\tau-1)I_{0+}^{n-2}e(s),\tau I_{0+}^{n-3}x(s)+(\tau-1)I_{0+}^{n-3}e(s),...,\tau x(s)+(\tau-1)e(s),\\
&\tau^{-1}I_{0+}^{n-2}y(s)+(\tau^{-1}-1)I_{0+}^{n-2}e(s),\tau^{-1}I_{0+}^{n-3}y(s)+(\tau^{-1}-1)I_{0+}^{n-3}e(s),...,\tau^{-1}y+(\tau^{-1}-1)e(s)\Big)ds-e(t)\\
\geq&\int_0^1 G(t,s)q(s)\phi\Big(\tau I_{0+}^{n-2}x(s)+(\tau-1)E^*,\tau I_{0+}^{n-3}x(s)+(\tau-1)E^*,...,\tau x(s)+(\tau-1)E^*,\\
&\tau^{-1}I_{0+}^{n-2}y(s)+(\tau^{-1}-1)E^*,\tau^{-1}I_{0+}^{n-3}y(s)+(\tau^{-1}-1)E^*,...,\tau^{-1}y+(\tau^{-1}-1)E^*\Big)ds-e(t)\\
=&\int_0^1 G(t,s)q(s)\phi\Big(\tau I_{0+}^{n-2}x(s)+(\tau-1)\rho_0,\tau I_{0+}^{n-3}x(s)+(\tau-1)\rho_0,...,\tau x(s)+(\tau-1)\rho_0,\\
&\tau^{-1}I_{0+}^{n-2}y(s)+(\tau^{-1}-1)\rho_0,\tau^{-1}I_{0+}^{n-3}y(s)+(\tau^{-1}-1)\rho_0,...,\tau^{-1}y+(\tau^{-1}-1)\rho_0\Big)ds-e(t)\\
\geq&\tau\int_0^1 G(t,s)q(s)\phi\big(I_{0+}^{n-2}x(s),I_{0+}^{n-3}x(s),...,x(s),I_{0+}^{n-2}y(s),I_{0+}^{n-3}y(s),...,y(s)\big)ds-e(t)\\
=&\tau\int_0^1 G(t,s)q(s)\phi\big(I_{0+}^{n-2}x(s),I_{0+}^{n-3}x(s),...,x(s),I_{0+}^{n-2}y(s),I_{0+}^{n-3}y(s),...,y(s)\big)ds-e(t)+\tau e(t)-\tau e(t)\\
=&\tau[\int_0^1 G(t,s)q(s)\phi\big(I_{0+}^{n-2}x(s),I_{0+}^{n-3}x(s),...,x(s),I_{0+}^{n-2}y(s),I_{0+}^{n-3}y(s),...,y(s)\big)ds-e(t)]+(\tau-1)e(t)\\
=&\tau B(x,y)(t)+(\tau-1)e(t).
\end{aligned}
$$

(3) 第三, 证明$A(h,h)\in C_{h,e}, B(h,h)\in C_{h,e}$. 即证$A(h,h)+e\in C_h, B(h,h)+e\in C_h$. 由引理2.2, (H_4), (H_5) 可以得出

$$\begin{aligned}
A(h,h)(t)+e(t) &= \int_0^1 G(t,s)q(s)g\big(I_{0+}^{n-2}h(s), I_{0+}^{n-3}h(s), ..., h(s), I_{0+}^{n-2}h(s), I_{0+}^{n-3}h(s), ..., h(s)\big)ds \\
&= \int_0^1 G(t,s)q(s)g\big(I_{0+}^{n-2}Ms^{\alpha-n+1}, I_{0+}^{n-3}Ms^{\alpha-n+1}, \\
&\quad ..., Ms^{\alpha-n+1}, I_{0+}^{n-2}Ms^{\alpha-n+1}, I_{0+}^{n-3}Ms^{\alpha-n+1}, ..., Ms^{\alpha-n+1}\big)ds \\
&= \int_0^1 G(t,s)q(s)g\big(MI_{0+}^{n-2}s^{\alpha-n+1}, MI_{0+}^{n-3}s^{\alpha-n+1}, \\
&\quad ..., Ms^{\alpha-n+1}, MI_{0+}^{n-2}s^{\alpha-n+1}, MI_{0+}^{n-3}s^{\alpha-n+1}, ..., Ms^{\alpha-n+1}\big)ds \\
&\le \int_0^1 \frac{t^{\alpha-n+1}}{\Gamma(\alpha-n+2)}q(s)g(M,M,...,M,0,0,...,0)ds \\
&= \frac{1}{\Gamma(\alpha-n+2)}g(M,M,...,M,0,0,...,0)\int_0^1 q(s)ds\cdot t^{\alpha-n+1} \\
&= \frac{1}{M\Gamma(\alpha-n+2)}g(M,M,...,M,0,0,...,0)\int_0^1 q(s)ds\cdot h(t)
\end{aligned}$$

和

$$\begin{aligned}
A(h,h)(t)+e(t) &\ge \int_0^1 \rho(s)G(s,s)q(s)g(0,0,...,0,M,M,...,M)ds \\
&\ge \int_0^1 \rho(s)G(s,s)q(s)\delta_0\phi(0,0,...,0,M,M,...,M)ds \\
&\ge \delta_0\phi(0,0,...,0,M,M,...,M)\int_0^1 \rho(s)G(s,s)q(s)ds\cdot t^{\alpha-n+1} \\
&= \frac{1}{M}\delta_0\phi(0,0,...,0,M,M,...,M)\int_0^1 \rho(s)G(s,s)q(s)ds\cdot h(t).
\end{aligned}$$

令

$$\begin{aligned}
l_1 &= \frac{1}{M}\delta_0\phi(0,0,...,0,M,M,...,M)\int_0^1 \rho(s)G(s,s)q(s)ds, \\
L_1 &= \frac{1}{M\Gamma(\alpha-n+2)}g(M,M,...,M,0,0,...,0)\int_0^1 q(s)ds.
\end{aligned}$$

即$0<l_1\le L_1<+\infty$, $l_1h(t)\le A(h,h)(t)+e(t)\le L_1h(t)$, $t\in[0,1]$, 因此有$A(h,h)\in C_{h,e}$. 同样的方法有

$$B(h,h)(t)+e(t) = \int_0^1 G(t,s)q(s)\phi\big(I_{0+}^{n-2}h(s), I_{0+}^{n-3}h(s), ..., h(s), I_{0+}^{n-2}h(s), I_{0+}^{n-3}h(s), ..., h(s)\big)ds$$

$$
\begin{aligned}
&\leq \int_0^1 \frac{t^{\alpha-n+1}}{\Gamma(\alpha-n+2)} q(s)\phi(M,M,...,M,0,0,...,0)ds\\
&=\frac{1}{\Gamma(\alpha-n+2)}\phi(M,M,...,M,0,0,...,0)\int_0^1 q(s)ds\cdot t^{\alpha-n+1}\\
&=\frac{1}{M\Gamma(\alpha-n+2)}\phi(M,M,...,M,0,0,...,0)\int_0^1 q(s)ds\cdot h(t)
\end{aligned}
$$

和

$$
\begin{aligned}
B(x,x)(t)+e(t)\geq &\int_0^1 \rho(s)G(s,s)q(s)\phi(0,0,...,0,M,M,...,M)ds\\
&\geq\phi(0,0,...,0,M,M,...,M)\int_0^1 \rho(s)G(s,s)q(s)ds\cdot t^{\alpha-n+1}\\
&=\frac{1}{M}\phi(0,0,...,0,M,M,...,M)\int_0^1 \rho(s)G(s,s)q(s)ds\cdot h(t).
\end{aligned}
$$

令

$$
\begin{aligned}
l_2&=\frac{1}{M}\phi(0,0,...,0,M,M,...,M)\int_0^1 \rho(s)G(s,s)q(s)ds,\\
L_2&=\frac{1}{M\Gamma(\alpha-n+2)}\phi(M,M,...,M,0,0,...,0)\int_0^1 q(s)ds.
\end{aligned}
$$

即$0<l_2\leq L_2<+\infty$, $l_2h(t)\leq B(h,h)(t)+e(t)\leq L_2h(t)$, $t\in[0,1]$, 因此有$B(h,h)\in C_{h,e}$.

(4) 最后, 证明引理2.4 中的(iv) 成立. 对每个$x,y\in C_{h,e}$, $t\in[0,1]$, 由(H_4) 得出

$$
\begin{aligned}
A(x,y)(t)=&\int_0^1 G(t,s)q(s)g\big(I_{0+}^{n-2}x(s),I_{0+}^{n-3}x(s),...,x(s),I_{0+}^{n-2}y(s),I_{0+}^{n-3}y(s),...,y(s)\big)ds-e(t)\\
\geq&\delta_0\int_0^1 G(t,s)q(s)\phi\big(I_{0+}^{n-2}x(s),I_{0+}^{n-3}x(s),...,x(s),I_{0+}^{n-2}y(s),I_{0+}^{n-3}y(s),...,y(s)\big)ds-e(t)-\\
&\delta_0e(t)+\delta_0e(t)\\
=&\delta_0[\int_0^1 G(t,s)q(s)\phi(I_{0+}^{n-2}x(s),I_{0+}^{n-3}x(s),...,x(s),I_{0+}^{n-2}y(s),I_{0+}^{n-3}y(s),...,y(s))ds-e(t)]+\\
&(\delta_0-1)e(t)\\
=&\delta_0B(x,y)(t)+(\delta_0-1)e(t),
\end{aligned}
$$

即$A(x,y)\geq\delta_0B(x,y)+(\delta_0-1)e$. 因此引理2.4 中的所有条件都满足. 因此定理3.1 中的结论成立. □

例 3.2 考虑如下BVP:

$$
\begin{cases}
D_{0+}^{\frac{5}{2}}u(t)+q(t)f(u(t),u'(t))-1=0, & t\in[0,1],\\
u(0)=u'(0)=u'(1)=0,
\end{cases}
\tag{7}
$$

其中$q(t)=t^{\frac{1}{2}}$, $f(u(t),u'(t))=(1+t)[(\frac{e(t)}{E^*}u(t)+e(t))^{\frac{1}{2}}+(\frac{e(t)}{E^*}u(t)+e(t))^{\frac{1}{3}}+(\frac{e(t)}{E^*}u'(t)+e(t))^{-\frac{1}{5}}+(\frac{e(t)}{E^*}u'(t)+e(t))^{-\frac{1}{6}}]$,$E^*=max\{e(t),I_{0+}^{1}e(t):t\in[0,1]\}=\frac{472\Gamma(\frac{7}{2})}{2539}$.

令$u(t)=I_{0+}^{1}x(t)$，那么方程(7) 可转化为如下形式:

$$\begin{cases}D_{0+}^{\frac{3}{2}}x(t)+q(t)f(I_{0+}^{1}x(t),x(t))-1=0, \quad t\in[0,1],\\ x(0)=x(1)=0,\end{cases} \tag{8}$$

其中

$$\begin{aligned}f(I_{0+}^{1}x(t),x(t))=&f(x,y)=(1+t)[(\frac{e(t)}{E^*}x+e(t))^{\frac{1}{2}}+(\frac{e(t)}{E^*}x+e(t))^{\frac{1}{3}}+(\frac{e(t)}{E^*}y+e(t))^{-\frac{1}{5}}+(\frac{e(t)}{E^*}y+e(t))^{-\frac{1}{6}}],\\ g(x,x,y,y)=&(\frac{e(t)}{E^*}x+e(t))^{\frac{1}{2}}+(\frac{e(t)}{E^*}x+e(t))^{\frac{1}{3}}+(\frac{e(t)}{E^*}y+e(t))^{-\frac{1}{5}}+(\frac{e(t)}{E^*}y+e(t))^{-\frac{1}{6}},\\ \phi(x,x,y,y)=&t(\frac{e(t)}{E^*}x+e(t))^{\frac{1}{2}}+t(\frac{e(t)}{E^*}x+e(t))^{\frac{1}{3}}+t(\frac{e(t)}{E^*}y+e(t))^{-\frac{1}{5}}+t(\frac{e(t)}{E^*}y+e(t))^{-\frac{1}{6}}.\end{aligned}$$

对于任意的$x,y>-E^*$，有$f(x,y)=g(x,x,y,y)+\phi(x,x,y,y)$，$e(t)=\frac{t^{\frac{3}{2}}-t^{\frac{5}{2}}}{\Gamma(\frac{7}{2})}$，$E^*=max\{e(t),I_{0+}^{1}e(t):t\in[0,1]\}=\frac{472\Gamma(\frac{7}{2})}{2539}$. 接下来，检验定理3.1 中的所有条件是否都满足.

(i) 显然，函数$g,\phi:[-E^*,+\infty)^4\to(-\infty,+\infty)$ 是连续的.

(ii) 对于固定的$y_i\in[-E^*,+\infty)(i=1,2)$，$g(x,x,y,y)$，$\phi(x,x,y,y)$ 在$x_i\in[-E^*,+\infty)(i=1,2)$ 上是非减的; 对于固定的$x_i\in[-E^*,+\infty)(i=1,2)$，$g(x,x,y,y)$，$\phi(x,x,y,y)$ 在$y_i\in[-E^*,+\infty)(i=1,2)$ 上是非增的.

(iii) 对$\tau\in(0,1)$，$x_i,y_i\in[-E^*,+\infty)(i=1,2)$，取$\rho_0=e$，$\varphi(\tau)=\tau^{\frac{1}{2}}$，有

$$\begin{aligned}&g(\tau x+(\tau-1)\rho_0,\tau x+(\tau-1)\rho_0,\tau^{-1}y+(\tau^{-1}-1)\rho_0,\tau^{-1}y+(\tau^{-1}-1)\rho_0)\\ =&g(\tau x+(\tau-1)e,\tau x+(\tau-1)e,\tau^{-1}y+(\tau^{-1}-1)e,\tau^{-1}y+(\tau^{-1}-1)e)\\ =&\big(\tau\frac{e(t)}{E^*}x+(\tau-1)e(t)+e(t)\big)^{\frac{1}{2}}+(\tau\frac{e(t)}{E^*}x+(\tau-1)e(t)+e(t))^{\frac{1}{3}}+\big(\tau^{-1}\frac{e(t)}{E^*}y+\\ &(\tau^{-1}-1)e(t)+e(t)\big)^{-\frac{1}{5}}+\big(\tau^{-1}\frac{e(t)}{E^*}y+(\tau^{-1}-1)e(t)+e(t)\big)^{-\frac{1}{6}}\\ =&(\tau\frac{e(t)}{E^*}x+\tau e(t))^{\frac{1}{2}}+(\tau\frac{e(t)}{E^*}x+\tau e(t))^{\frac{1}{3}}+(\tau^{-1}\frac{e(t)}{E^*}y+\tau^{-1}e(t))^{-\frac{1}{5}}+(\tau^{-1}\frac{e(t)}{E^*}y+\tau^{-1}e(t))^{-\frac{1}{6}}\\ =&\tau^{\frac{1}{2}}(\frac{e(t)}{E^*}x+e(t))^{\frac{1}{2}}+\tau^{\frac{1}{3}}(\frac{e(t)}{E^*}x+e(t))^{\frac{1}{3}}+\tau^{\frac{1}{5}}(\frac{e(t)}{E^*}y+e(t))^{-\frac{1}{5}}+\tau^{\frac{1}{6}}(\frac{e(t)}{E^*}y+e(t))^{-\frac{1}{6}}\\ \geqslant&\tau^{\frac{1}{2}}[(\frac{e(t)}{E^*}x+e(t))^{\frac{1}{2}}+(\frac{e(t)}{E^*}x+e(t))^{\frac{1}{3}}+(\frac{e(t)}{E^*}y+e(t))^{-\frac{1}{5}}+(\frac{e(t)}{E^*}y+e(t))^{-\frac{1}{6}}]\\ =&\varphi(\tau)g(x,x,y,y).\end{aligned}$$

对于$\tau\in(0,1)$ 和$x_i,y_i\in[-E^*,+\infty)$，$(i=1,2)$，取$\rho_0=e$，有

$$\begin{aligned}&\phi(\tau x+(\tau-1)\rho_0,\tau x+(\tau-1)\rho_0,\tau^{-1}y+(\tau^{-1}-1)\rho_0,\tau^{-1}y+(\tau^{-1}-1)\rho_0)\\ =&\phi(\tau x+(\tau-1)e,\tau x+(\tau-1)e,\tau^{-1}y+(\tau^{-1}-1)e,\tau^{-1}y+(\tau^{-1}-1)e)\\ =&t[(\tau\frac{e(t)}{E^*}x+(\tau-1)e(t)+e(t))^{\frac{1}{2}}+(\tau\frac{e(t)}{E^*}x+(\tau-1)e(t)+e(t))^{\frac{1}{3}}+\big(\tau^{-1}\frac{e(t)}{E^*}y+\\ &(\tau^{-1}-1)e(t)+e(t)\big)^{-\frac{1}{5}}+(\tau^{-1}\frac{e(t)}{E^*}y+(\tau^{-1}-1)e(t)+e(t))^{-\frac{1}{6}}]\end{aligned}$$

$$
\begin{aligned}
&=t[(\tau\frac{e(t)}{E^*}x+\tau e(t))^{\frac{1}{2}}+(\tau\frac{e(t)}{E^*}x+\tau e(t))^{\frac{1}{3}}+(\tau^{-1}\frac{e(t)}{E^*}y+\tau^{-1}e(t))^{-\frac{1}{5}}+(\tau^{-1}\frac{e(t)}{E^*}y+\tau^{-1}e(t))^{-\frac{1}{6}}]\\
&=t[\tau^{\frac{1}{2}}(\frac{e(t)}{E^*}x+e(t))^{\frac{1}{2}}+\tau^{\frac{1}{3}}(\frac{e(t)}{E^*}x+e(t))^{\frac{1}{3}}+\tau^{\frac{1}{5}}(\frac{e(t)}{E^*}y+e(t))^{-\frac{1}{5}}+\tau^{\frac{1}{6}}(\frac{e(t)}{E^*}y+e(t))^{-\frac{1}{6}}]\\
&\geqslant t\{\tau^{\frac{1}{6}}[(\frac{e(t)}{E^*}x+e(t))^{\frac{1}{2}}+(\frac{e(t)}{E^*}x+e(t))^{\frac{1}{3}}+(\frac{e(t)}{E^*}y+e(t))^{-\frac{1}{5}}+(\frac{e(t)}{E^*}y+e(t))^{-\frac{1}{6}}]\}\\
&\geqslant \tau\{t[(\frac{e(t)}{E^*}x+e(t))^{\frac{1}{2}}+(\frac{e(t)}{E^*}x+e(t))^{\frac{1}{3}}+(\frac{e(t)}{E^*}y+e(t))^{-\frac{1}{5}}+(\frac{e(t)}{E^*}y+e(t))^{-\frac{1}{6}}]\}\\
&=\tau\phi(x,x,y,y).
\end{aligned}
$$

(iv) 取$\rho_0=e,\delta_0=1>0$. 那么

$$
\begin{aligned}
g(x,x,y,y)=&(\frac{e(t)}{E^*}x+\rho_0)^{\frac{1}{2}}+(\frac{e(t)}{E^*}x+\rho_0)^{\frac{1}{3}}+(\frac{e(t)}{E^*}y+\rho_0)^{-\frac{1}{5}}+(\frac{e(t)}{E^*}y+\rho_0)^{-\frac{1}{6}}\\
=&(\frac{e(t)}{E^*}x+e(t))^{\frac{1}{2}}+(\frac{e(t)}{E^*}x+e(t))^{\frac{1}{3}}+(\frac{e(t)}{E^*}y+e(t))^{-\frac{1}{5}}+(\frac{e(t)}{E^*}y+e(t))^{-\frac{1}{6}}\\
\geqslant&1\{t[(\frac{e(t)}{E^*}x+e(t))^{\frac{1}{2}}+(\frac{e(t)}{E^*}x+e(t))^{\frac{1}{3}}+(\frac{e(t)}{E^*}y+e(t))^{-\frac{1}{5}}+(\frac{e(t)}{E^*}y+e(t))^{-\frac{1}{6}}]\}\\
=&\delta_0\phi(x,x,y,y).
\end{aligned}
$$

(v) 取$M=1.51>\frac{1}{(\frac{5}{2}-3+1)\Gamma(\frac{5}{2}-3+3)}$,

$$
\begin{aligned}
g(M,M,0,0)<&(1.51+1)^{\frac{1}{2}}+(1.51+1)^{\frac{1}{3}}+1+1=4.9433<+\infty,\\
\phi(M,M,0,0)<&(1.51+1)^{\frac{1}{2}}+(1.51+1)^{\frac{1}{3}}+1+1=4.9433<+\infty,\\
\phi(0,0,M,M)>&0.
\end{aligned}
$$

因此, 定理*3.1* 中的假设全部满足. 结合引理*2.3* , *BVP 7* 存在唯一的非平凡解. 此外, 设置以下序列:

$$
\begin{aligned}
\omega_n(t)=&\int_0^1 G(t,s)\Big((\frac{e(s)}{E^*}\omega_{n-1}(s)+e(s))^{\frac{1}{2}}+(\frac{e(s)}{E^*}\omega_{n-1}(s)+e(s))^{\frac{1}{3}}+(\frac{e(s)}{E^*}\tau_{n-1}(s)+\\
&e(s))^{-\frac{1}{5}}+(\frac{e(s)}{E^*}\tau_{n-1}(s)+e(s))^{-\frac{1}{6}}\Big)ds+\int_0^1 G(t,s)\Big(t(\frac{e(s)}{E^*}\omega_{n-1}(s)+e(s))^{\frac{1}{2}}+\\
&t(\frac{e(s)}{E^*}\omega_{n-1}(s)+e(s))^{\frac{1}{3}}+t(\frac{e(s)}{E^*}\tau_{n-1}(s)+e(s))^{-\frac{1}{5}}+s(\frac{e(s)}{E^*}\tau_{n-1}(s)+\\
&e(s))^{-\frac{1}{6}}\Big)ds+\frac{t^{\frac{3}{2}}-t^{\frac{5}{2}}}{\Gamma(\frac{7}{2})},\ n=1,2,\cdots
\end{aligned}
$$

和

$$
\begin{aligned}
\tau_n(t)=&\int_0^1 G(t,s)\Big((\frac{e(s)}{E^*}\tau_{n-1}(s)+e(s))^{\frac{1}{2}}+(\frac{e(s)}{E^*}\tau_{n-1}(s)+e(s))^{\frac{1}{3}}+(\frac{e(s)}{E^*}\omega_{n-1}(s)+\\
&e(s))^{-\frac{1}{5}}+(\frac{e(s)}{E^*}\omega_{n-1}(s)+e(s))^{-\frac{1}{6}}\Big)ds+\int_0^1 G(t,s)\Big(t(\frac{e(s)}{E^*}\tau_{n-1}(s)+e(s))^{\frac{1}{2}}+
\end{aligned}
$$

$$t(\frac{e(s)}{E^*}\tau_{n-1}(s)+e(s))^{\frac{1}{3}}+t(\frac{e(s)}{E^*}\omega_{n-1}(s)+e(s))^{-\frac{1}{5}}+s(\frac{e(s)}{E^*}\omega_{n-1}(s)+$$
$$e(s))^{-\frac{1}{6}}\Big)ds+\frac{t^{\frac{3}{2}}-t^{\frac{5}{2}}}{\Gamma(\frac{7}{2})},\ n=1,2,\cdots,$$

对于任意给定的初值$\omega_0,\tau_0\in C_{h,e}$，凡$t\in[0,1]$，我们有$\{\omega_n(t)\}$ 和$\{\tau_n(t)\}$ 都一致收敛于x^*.因此，定理*3.1* 中的假设成立. 结合引理2.3, BVP 7 存在唯一的非平凡解.

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