Pure Mathematics 理论数学, 2023, 13(7), 2080-2092
Published Online July 2023 in Hans. https://www.hanspub.org/journal/pm
https://doi.org/10.12677/pm.2023.137214

分数次极大算子及交换子在加权Morrey空间上的有界性

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收稿日期：2023年6月16日；录用日期：2023年7月19日；发布日期：2023年7月26日

摘 要

利用调和分析的实变方法以及权不等式，证明了分数次极大算子在加权Morrey空间上的强有界性和弱有界性，并且得到了分数次极大算子与BMO函数生成的交换子的加权有界性。

关键词

加权Morrey空间，分数次极大算子，交换子，BMO函数

Boundedness of Fractional Maximal Operator and its Commutator on Weighted Morrey Spaces

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Received: Jun. 16th, 2023; accepted: Jul. 19th, 2023; published: Jul. 26th, 2023

Abstract

By applying the real variable methods of harmonic analysis and the weighted inequality, the strong and weak boundedness of the fractional maximal operator is proved on the weighted Morrey space, and the weighted boundedness of commutators generated by fractional maximal operators and BMO functions is obtained.

Keywords

Weighted Morrey Space, Fractional Maximal Operator, Commutator, BMO Function

文章引用: 刘占宏. 分数次极大算子及交换子在加权 Morrey 空间上的有界性[J]. 理论数学, 2023, 13(7): 2080-2092.
DOI: 10.12677/pm.2023.137214

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1. 引言

经典 Morrey 空间在研究二阶椭圆偏微分方程解的正则性时被首次定义[1]。这类函数空间作为经典 Lebesgue 空间的推广，在调和分析本身及偏微分方程等领域有着重要应用。分数次极大算子作为现代多元调和分析基本的理论工具，其在各类函数空间上的有界性一直受到众多学者的广泛关注。1974 年，Muckenhoupt 和 Wheeden 研究了分数次极大算子在加权 Lebesgue 空间上的有界性[2]；1987 年，Chiarenza 和 Frasca 获得了分数次极大算子在 Morrey 空间上的有界性[3]；1996 年，Ding 研究了一类粗糙核极大算子交换子的有界性[4]；2016 年，Wang 和 Zhu 研究了分数次极大算子在加权变指标空间上的有界性[5]。2020 年，Duoandikoetxea 和 Rosenthal 引入一类新的加权 Morrey 空间[6]，这类空间是经典 Morrey 空间的推广并且受到了广泛的关注。2022 年，Zhou 和 Zhao 证明了分数次极大算子在这类加权 Morrey 空间上的有界性[7]。受以上启发，本文主要研究分数次极大算子及其交换子在这类加权 Morrey 空间上的加权估计。关于加权 Morrey 空间还有更多的结果[8] [9] [10] [11] [12]。在叙述本文主要结果之前，首先引入需要的概念和记号。

设 $0\le\alpha<n$，f 为 $\mathbf{R}^n$ 上局部可积函数，分数次极大算子定义为

$$M_\alpha f(x)=\sup_{x\in B}\frac{1}{|B|^{1-\frac{\alpha}{n}}}\int_B|f(y)|\mathrm{d}y, \tag{1}$$

其中，B 为 $\mathbf{R}^n$ 中的球体。相应地，给定一个局部可积函数 b，由 b 和 M_α 生成的交换子定义为

$$[b,M_\alpha]f(x)=M_\alpha(bf)(x)-b(x)M_\alpha f(x)=\sup_{x\in B}\frac{1}{|B|^{1-\frac{\alpha}{n}}}\int_B|f(y)|[b(x)-b(y)]\mathrm{d}y. \tag{2}$$

设 $1<p$，$q<\infty$，$\mathbf{R}^n$ 上的非负局部可积函数 $w(x)$ 称为 $A(p,q)$ 权，如果存在常数 $C>0$，使得

$$\sup_B\left(\frac{1}{|B|}\int_B w(x)^q\,\mathrm{d}x\right)^{\frac{1}{q}}\left(\frac{1}{|B|}\int_B w(x)^{-p'}\,\mathrm{d}x\right)^{\frac{1}{p'}}\le C<\infty. \tag{3}$$

定义 1 [13] BMO 空间定义为

$$\mathrm{BMO}=\left\{f\in L^1_{\mathrm{loc}}(\mathbf{R}^n):\|f\|_{\mathrm{BMO}}<\infty\right\}, \tag{4}$$

其中

$$\|f\|_{\mathrm{BMO}}=\sup_{x\in B}\frac{1}{|B|}\int_B|f(x)-f_B|\mathrm{d}x,$$

当 $f\in\mathrm{BMO}$ 时，对任意的 $1\le p<\infty$，有

$$\|f\|_{\mathrm{BMO}}=\|f\|_*\approx\sup_B\left(\frac{1}{|B|}\int_B|f(x)-f_B|^p\,\mathrm{d}x\right)^{\frac{1}{p}}. \tag{5}$$

定义 2 [4]设 $1\le p<\infty$，$\lambda_1,\lambda_2\in\mathbf{R}$，$f$ 为可测函数，加权 Morrey 空间定义为

$$L^{p,(\lambda_1,\lambda_2)}(w)=\left\{f:\|f\|_{L^{p,(\lambda_1,\lambda_2)}(w)}=\sup_{x\in\mathbf{R}^n,r>0}\left(\frac{1}{r^{\lambda_1}w(B(x,r))^{\frac{\lambda_2}{n}}}\int_{B(x,r)}|f(y)|^p w(y)\mathrm{d}y\right)^{\frac{1}{p}}<\infty\right\}, \tag{6}$$

其中，$B(x,r)$表示球心为 x、半径为 r 的球；$w(B(x,r))$表示的是非负局部可积函数 w 在 $B(x,r)$上的积分。

定义 3 [4]设$1\le p<\infty$，$\lambda_1,\lambda_2\in\mathbf{R}$，$f$为可测函数，弱加权 Morrey 空间定义为

$$WL^{p,(\lambda_1,\lambda_2)}(w)=\left\{f:\|f\|_{WL^{p,(\lambda_1,\lambda_2)}(w)}=\sup_{t>0,r>0}\left(\frac{t^p w(\{y\in B(x,r):|f(y)|>t\})}{r^{\lambda_1}w(B(x,r))^{\frac{\lambda_2}{n}}}\right)^{\frac{1}{p}}<\infty\right\}, \tag{7}$$

2. 主要定理

定理 1 设$0\le\alpha<n$，M_α由式(2)所定义，那么当$1<p<\frac{\alpha}{n}$，$\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n}$，并且$w\in A(p,q)$，$\lambda_2<0$时，存在一个与 f 无关的常数 C，使得

$$\|M_\alpha(f)\|_{L^{q,\left(\frac{q(\lambda_1+\lambda_2)}{p}-\lambda_2,\lambda_2\right)}(w^q)}\le C\|f\|_{L^{p,(\lambda_1,\lambda_2)}(w^p)}.$$

定理 2 设$0\le\alpha<n$，M_α由式(2)所定义，那么当$1<p<\frac{\alpha}{n}$，$\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n}$，并且$w\in A(p,q)$，$\lambda_2<0$时，存在一个与 f 无关的常数 C，使得

$$\|M_\alpha(f)\|_{WL^{q,\left(\frac{q(\lambda_1+\lambda_2)}{p}-\lambda_2,\lambda_2\right)}(w^q)}\le C\|f\|_{L^{p,(\lambda_1,\lambda_2)}(w^p)}.$$

定理 3 设$0\le\alpha<n$，$t>\frac{n}{n-\alpha}$，$[b,M_\alpha]$由式(3)所定义，那么当$1<p_1<\frac{n}{\alpha}$，$p_1'<p_2<\infty$，$\frac{1}{q}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{\alpha}{n}$，$\frac{1}{t}=\frac{1}{p_1}-\frac{\alpha}{n}$，$\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$，并且$b\in\mathrm{BMO}(\mathbf{R}^n)$，$w\in A(p,q)$，$1+\frac{\lambda_2}{qn}\le\frac{1}{p_1}$时，存在一个与 f 无关的常数 C，使得

$$\|[b,M_\alpha](f)\|_{L^{q,\left(\frac{q(\lambda_1+\lambda_2)}{p_1}+\frac{qn}{p_2}-\lambda_2,\lambda_2\right)}(w^q)}\le C\|b\|_*\|f\|_{L^{p_1,(\lambda_1,\lambda_2)}(w^{p_1})}.$$

本文中，p'表示 p 的共轭，即$\frac{1}{p}+\frac{1}{p'}=1$；$\chi_E$代表集合 E 的特征函数；C 是和主要函数和参量无关的常数，在不同行中甚至在同一行中可以不同。

3. 定理的证明

在证明定理之前，先给出以下引理。

引理 1 [8]设$0\le\alpha<n$，$1<p<\frac{n}{\alpha}$，$\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n}$，当$w\in A(p,q)$时，存在一个与 f 无关的常数 C，使得

$$\|M_\alpha(f)\|_{L^q(w^q)}\le C\|f\|_{L^p(w^p)}.$$

引理 2 [8]设$0\le\alpha<n$，$1\le p<\frac{n}{\alpha}$，$\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n}$，当$w\in A(p,q)$时，对任意的$\lambda>0$，存在一个与 f

无关的常数 C，使得

$$\left(\int_{\{x\in\mathbf{R}^n:M_\alpha f(x)>\lambda\}} w(x)^q\,\mathrm{d}x\right)^{\frac{1}{q}} \le \frac{C}{\lambda}\left(\int_{\mathbf{R}^n}\left|f(x)w(x)\right|^p\,\mathrm{d}x\right)^{\frac{1}{p}}.$$

引理 3 [14]如果 $w\in A_q$，则对任意的 $k\in\mathbf{Z}^+$，$l>1$ 以及任意球 $B\subset\mathbf{R}^n$，当 $1\le q<\infty$ 时，有

$$w(lB)\le Cl^{nq}w(B).$$

定理 1 的证明 设 $f\in L^{p,(\lambda_1,\lambda_2)}\left(w^p\right)$ 为局部可积函数，给定任意的球 $B(x,r)$，分解 f 为 $f=f_1+f_2$，其中 $f_1=f\chi_{2B}$，$f_2=f\chi_{(2B)^c}$，则

$$\left\|M_\alpha(f)\right\|_{L^{q,\left(\frac{q(\lambda_1+\lambda_2)}{p}-\lambda_2,\lambda_2\right)}\left(w^q\right)} \le \left\|M_\alpha(f_1)\right\|_{L^{q,\left(\frac{q(\lambda_1+\lambda_2)}{p}-\lambda_2,\lambda_2\right)}\left(w^q\right)} + \left\|M_\alpha(f_2)\right\|_{L^{q,\left(\frac{q(\lambda_1+\lambda_2)}{p}-\lambda_2,\lambda_2\right)}\left(w^q\right)} := A_1+A_2.$$

对于 A_1，利用引理 1，有

$$\begin{aligned}
A_1 &= \left(\frac{1}{r^{\frac{q(\lambda_1+\lambda_2)}{p}-\lambda_2}\left(w^q(B)\right)^{\frac{\lambda_2}{n}}}\int_B\left(M_\alpha f_1(x)\right)^q w^q(x)\,\mathrm{d}x\right)^{\frac{1}{q}}\\
&\le C\left(\frac{1}{r^{\frac{q(\lambda_1+\lambda_2)}{p}-\lambda_2}\left(w^q(B)\right)^{\frac{\lambda_2}{n}}}\int_{\mathbf{R}^n}\left(M_\alpha f_1(x)\right)^q w^q(x)\,\mathrm{d}x\right)^{\frac{1}{q}}\\
&\le C\left(\frac{1}{r^{\frac{q(\lambda_1+\lambda_2)}{p}-\lambda_2}\left(w^q(B)\right)^{\frac{\lambda_2}{n}}}\right)^{\frac{1}{q}}\left(\int_{\mathbf{R}^n} f_1(x)^p\, w^p(x)\,\mathrm{d}x\right)^{\frac{1}{p}}\\
&\simeq C\left(\frac{1}{r^{\frac{q(\lambda_1+\lambda_2)}{p}-\lambda_2}\left(w^q(B)\right)^{\frac{\lambda_2}{n}}}\right)^{\frac{1}{q}}\left(\int_{2B} f(x)^p\, w^p(x)\,\mathrm{d}x\right)^{\frac{1}{p}}\\
&\le C\frac{r^{\frac{\lambda_1}{p}}w^p(2B)^{\frac{\lambda_2}{np}}}{\left(r^{\frac{q(\lambda_1+\lambda_2)}{p}-\lambda_2}w^q(B)^{\frac{\lambda_2}{n}}\right)^{\frac{1}{q}}}\|f\|_{L^{p,(\lambda_1,\lambda_2)}\left(w^p\right)}.
\end{aligned}$$

利用 Hölder 不等式，当 $1\le p<q<\infty$ 时，有

$$w^p(B)\le\left(w^q(B)\right)^{\frac{p}{q}}|B|^{1-\frac{p}{q}}. \tag{8}$$

通过式(8)，不难得到

$$w^p(2B)\le\left(w^q(2B)\right)^{\frac{p}{q}}r^{n\left(1-\frac{p}{q}\right)}. \tag{9}$$

显然

$$\frac{r^{\frac{\lambda_1}{p}} w^p\left(2B\right)^{\frac{\lambda_2}{np}}}{\left(r^{\frac{q(\lambda_1+\lambda_2)}{p}-\lambda_2} w^q\left(B\right)^{\frac{\lambda_2}{n}}\right)^{\frac{1}{q}}} \le \left(\frac{w^q\left(2B\right)}{w^q\left(B\right)}\right)^{\frac{\lambda_2}{qn}} \le 1, \tag{10}$$

所以

$$A_1 \le C\left\|f\right\|_{L^{p,(\lambda_1,\lambda_2)}\left(w^p\right)}. \tag{11}$$

同理，对于 A_2，利用引理 1，有

$$\begin{aligned}
A_2 &= \left(\frac{1}{r^{\frac{q(\lambda_1+\lambda_2)}{p}-\lambda_2}\left(w^q\left(B\right)\right)^{\frac{\lambda_2}{n}}}\int_B\left(M_\alpha f_2\left(x\right)\right)^q w^q\left(x\right)\mathrm{d}x\right)^{\frac{1}{q}} \\
&\le C\left(\frac{1}{r^{\frac{q(\lambda_1+\lambda_2)}{p}-\lambda_2}\left(w^q\left(B\right)\right)^{\frac{\lambda_2}{n}}}\int_{\mathbf{R}^n}\left(M_\alpha f_2\left(x\right)\right)^q w^q\left(x\right)\mathrm{d}x\right)^{\frac{1}{q}} \\
&\le C\left(\frac{1}{r^{\frac{q(\lambda_1+\lambda_2)}{p}-\lambda_2}\left(w^q\left(B\right)\right)^{\frac{\lambda_2}{n}}}\right)^{\frac{1}{q}}\left(\int_{\mathbf{R}^n} f_2\left(x\right)^p w^p\left(x\right)\mathrm{d}x\right)^{\frac{1}{p}} \\
&\simeq C\left(\frac{1}{r^{\frac{q(\lambda_1+\lambda_2)}{p}-\lambda_2}\left(w^q\left(B\right)\right)^{\frac{\lambda_2}{n}}}\right)^{\frac{1}{q}}\left(\int_{(2B)^c} f\left(x\right)^p w^p\left(x\right)\mathrm{d}x\right)^{\frac{1}{p}} \\
&\le C\left(\frac{1}{r^{\frac{q(\lambda_1+\lambda_2)}{p}-\lambda_2}\left(w^q\left(B\right)\right)^{\frac{\lambda_2}{n}}}\right)^{\frac{1}{q}}\left(\sum_{j=1}^{\infty}\int_{2^{j+1}B\backslash 2^j B} f\left(x\right)^p w^p\left(x\right)\mathrm{d}x\right)^{\frac{1}{p}} \\
&\le C\left(\frac{1}{r^{\frac{q(\lambda_1+\lambda_2)}{p}-\lambda_2}\left(w^q\left(B\right)\right)^{\frac{\lambda_2}{n}}}\right)^{\frac{1}{q}}\left(\sum_{j=1}^{\infty}\int_{2^{j+1}B} f\left(x\right)^p w^p\left(x\right)\mathrm{d}x\right)^{\frac{1}{p}} \\
&\le C\frac{r^{\frac{\lambda_1}{p}}\sum_{j=1}^{\infty} w^p\left(2^{j+1}B\right)^{\frac{\lambda_2}{np}}}{\left(r^{\frac{q(\lambda_1+\lambda_2)}{p}-\lambda_2} w^q\left(B\right)^{\frac{\lambda_2}{n}}\right)^{\frac{1}{q}}}\left\|f\right\|_{L^{p,(\lambda_1,\lambda_2)}\left(w^p\right)} \\
&\le \left(\frac{\sum_{j=1}^{\infty} w^q\left(2^{j+1}B\right)}{w^q\left(B\right)}\right)^{\frac{\lambda_2}{qn}}\left\|f\right\|_{L^{p,(\lambda_1,\lambda_2)}\left(w^p\right)}.
\end{aligned}$$

利用$\lambda_2<0$、引理3，有

$$A_2 \le C\|f\|_{L^{p,(\lambda_1,\lambda_2)}(w^p)}. \tag{12}$$

结合式(11)、(12)，定理1证毕。

定理 2 的证明 结合定理 1 的证明，我们只需考虑$p=1$的情况便可证明定理 2。对局部可积函数$f\in L^{1,(\lambda_1,\lambda_2)}(w)$，给定任意的球$B(x_0,r)$，分解$f$为$f=f_1+f_2$，其中$f_1=f\chi_{2B}$，$f_2=f\chi_{(2B)^c}$，则$M_\alpha f(x)\le M_\alpha f_1(x)+M_\alpha f_2(x)$，从而

$$\int_{\{x\in B:M_\alpha f(x)>t\}} w^q(x)\mathrm{d}x \le \int_{\left\{x\in B:M_\alpha f_1(x)>\frac{t}{2}\right\}} w^q(x)\mathrm{d}x + \int_{\left\{x\in B:M_\alpha f_2(x)>\frac{t}{2}\right\}} w^q(x)\mathrm{d}x,$$

显然，

$$\begin{aligned}\|M_\alpha(f)\|_{WL^{q,(q(\lambda_1+\lambda_2)-\lambda_2,\lambda_2)}(w^q)} &= \left(\frac{t^q\int_{\{x\in B:M_\alpha f(x)>t\}} w^q(x)\mathrm{d}x}{r^{q(\lambda_1+\lambda_2)-\lambda_2}\left(w^q(B)\right)^{\frac{\lambda_2}{n}}}\right)^{\frac{1}{q}} \\ &\le C\left(\frac{t^q\int_{\left\{x\in B:M_\alpha f_1(x)>\frac{t}{2}\right\}} w^q(x)\mathrm{d}x}{r^{q(\lambda_1+\lambda_2)-\lambda_2}\left(w^q(B)\right)^{\frac{\lambda_2}{n}}}\right)^{\frac{1}{q}} + C\left(\frac{t^q\int_{\left\{x\in B:M_\alpha f_2(x)>\frac{t}{2}\right\}} w^q(x)\mathrm{d}x}{r^{q(\lambda_1+\lambda_2)-\lambda_2}\left(w^q(B)\right)^{\frac{\lambda_2}{n}}}\right)^{\frac{1}{q}} \\ &:= B_1+B_2.\end{aligned}$$

对于B_1，利用引理2，有

$$\begin{aligned} B_1 &\le C\left(\frac{t^q\int_{\left\{x\in \mathbf{R}^n:M_\alpha f_1(x)>\frac{t}{2}\right\}} w^q(x)\mathrm{d}x}{r^{q(\lambda_1+\lambda_2)-\lambda_2}\left(w^q(B)\right)^{\frac{\lambda_2}{n}}}\right)^{\frac{1}{q}} \\ &\le C\left(\frac{\left(\int_{\mathbf{R}^n} f_1(x)w(x)\mathrm{d}x\right)^q}{r^{q(\lambda_1+\lambda_2)-\lambda_2}\left(w^q(B)\right)^{\frac{\lambda_2}{n}}}\right)^{\frac{1}{q}} \\ &\simeq \frac{\int_{2B} f(x)w(x)\mathrm{d}x}{r^{(\lambda_1+\lambda_2)-\frac{\lambda_2}{q}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}} \\ &\le C\frac{r^{\lambda_1}\left(\int_{2B} w(x)\mathrm{d}x\right)^{\frac{\lambda_2}{n}}}{r^{(\lambda_1+\lambda_2)-\frac{\lambda_2}{q}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}\|f\|_{L^{1,(\lambda_1,\lambda_2)}(w)} \\ &\le C\frac{r^{\lambda_1}\left(w(2B)\right)^{\frac{\lambda_2}{n}}}{r^{(\lambda_1+\lambda_2)-\frac{\lambda_2}{q}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}\|f\|_{L^{1,(\lambda_1,\lambda_2)}(w)} \\ &\le C\frac{r^{(\lambda_1+\lambda_2)-\frac{\lambda_2}{q}}\left(w^q(2B)\right)^{\frac{\lambda_2}{qn}}}{r^{(\lambda_1+\lambda_2)-\frac{\lambda_2}{q}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}\|f\|_{L^{1,(\lambda_1,\lambda_2)}(w)}。\end{aligned}$$

利用 $\lambda_2 < 0$、引理 3，得到

$$B_1 \le C\|f\|_{L^{p_1,(\lambda_1,\lambda_2)}(w)}. \tag{13}$$

对于 B_2，利用引理 2，有

$$B_2 \le C\left(\frac{t^q \int_{\left\{x\in\mathbf{R}^n : M_\alpha f_2(x) > \frac{t}{2}\right\}} w^q(x)\,\mathrm{d}x}{r^{q(\lambda_1+\lambda_2)-\lambda_2}\left(w^q(B)\right)^{\frac{\lambda_2}{n}}}\right)^{\frac{1}{q}}$$

$$\le C\left(\frac{\left(\int_{\mathbf{R}^n} f_2(x)w(x)\,\mathrm{d}x\right)^q}{r^{q(\lambda_1+\lambda_2)-\lambda_2}\left(w^q(B)\right)^{\frac{\lambda_2}{n}}}\right)^{\frac{1}{q}}$$

$$\simeq \frac{\int_{(2B)^c} f(x)w(x)\,\mathrm{d}x}{r^{(\lambda_1+\lambda_2)-\frac{\lambda_2}{q}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}$$

$$\le C\frac{r^{\lambda_1}\left(\sum_{j=1}^{\infty}\int_{2^{j+1}B\setminus 2^j B} w(x)\,\mathrm{d}x\right)^{\frac{\lambda_2}{n}}}{r^{(\lambda_1+\lambda_2)-\frac{\lambda_2}{q}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}\|f\|_{L^{1,(\lambda_1,\lambda_2)}(w)}$$

$$\le C\frac{r^{\lambda_1}\left(\sum_{j=1}^{\infty}\int_{2^{j+1}B} w(x)\,\mathrm{d}x\right)^{\frac{\lambda_2}{n}}}{r^{(\lambda_1+\lambda_2)-\frac{\lambda_2}{q}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}\|f\|_{L^{1,(\lambda_1,\lambda_2)}(w)}$$

$$\le C\frac{r^{\lambda_1}\left(\sum_{j=1}^{\infty} w\left(2^{j+1}B\right)\right)^{\frac{\lambda_2}{n}}}{r^{(\lambda_1+\lambda_2)-\frac{\lambda_2}{q}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}\|f\|_{L^{1,(\lambda_1,\lambda_2)}(w)}$$

$$\le C\frac{r^{(\lambda_1+\lambda_2)-\frac{\lambda_2}{q}}\left(\sum_{j=1}^{\infty} w^q\left(2^{j+1}B\right)\right)^{\frac{\lambda_2}{qn}}}{r^{(\lambda_1+\lambda_2)-\frac{\lambda_2}{q}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}\|f\|_{L^{1,(\lambda_1,\lambda_2)}(w)}.$$

利用 $\lambda_2 < 0$、引理 3，得到

$$B_2 \le C\|f\|_{L^{p_1,(\lambda_1,\lambda_2)}(w)}. \tag{14}$$

结合式(13)、(14)，定理 2 证毕。

定理 3 的证明 同理，分解 f 为 $f = f_1 + f_2$，其中 $f_1 = f\chi_{2B}$，$f_2 = f\chi_{(2B)^c}$，则

$$\left\|\left[b,M_\alpha\right](f)(x)\right\|_{L^{q,\left(\frac{q(\lambda_1+\lambda_2)}{p_1}+\frac{qn}{p_2}-\lambda_2,\lambda_2\right)}(w^q)}=\left(\frac{1}{r^{\frac{q(\lambda_1+\lambda_2)}{p_1}-\lambda_2+\frac{qn}{p_2}}\left(w^q(B)\right)^{\frac{\lambda_2}{n}}}\int_B\left|\left[b,M_\alpha(f)(x)\right]\right|^q w^q(x)\mathrm{d}x\right)^{\frac{1}{q}}$$

$$\le\frac{1}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}+\frac{n}{p_2}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}\left(\int_B\left|\left[b,M_\alpha(f_1)(x)\right]\right|^q w^q(x)\mathrm{d}x\right)^{\frac{1}{q}}$$

$$+\frac{1}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}+\frac{n}{p_2}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}\left(\int_B\left|\left[b,M_\alpha(f_2)(x)\right]\right|^q w^q(x)\mathrm{d}x\right)^{\frac{1}{q}}$$

$$:=E_1+E_2.$$

对于 E_1，有

$$E_1\le\frac{1}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}+\frac{n}{p_2}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}\left(\int_B\left|b(x)-b_B\right|^q\left|M_\alpha(f_1)(x)\right|^q w^q(x)\mathrm{d}x\right)^{\frac{1}{q}}$$

$$+\frac{1}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}+\frac{n}{p_2}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}\left(\int_B\left|M_\alpha\left(f_1(b-b_B)\right)(x)\right|^q w^q(x)\mathrm{d}x\right)^{\frac{1}{q}}:=E_{11}+E_{12}.$$

对于 E_{11}，利用 Hölder 不等式、引理 1，有

$$E_{11}\le\frac{1}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}+\frac{n}{p_2}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}\left(\int_B\left|b(x)-b_B\right|^{p_2}\mathrm{d}x\right)^{\frac{1}{p_2}}\left(\int_B\left|M_\alpha(f_1)(x)\right|^t w^t(x)\mathrm{d}x\right)^{\frac{1}{t}}$$

$$\le\frac{|B|^{\frac{1}{p_2}}}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}+\frac{n}{p_2}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}\left(\frac{1}{|B|}\int_B\left|b(x)-b_B\right|^{p_2}\mathrm{d}x\right)^{\frac{1}{p_2}}\left(\int_B\left|M_\alpha(f_1)(x)\right|^t w^t(x)\mathrm{d}x\right)^{\frac{1}{t}}$$

$$\le C\frac{1}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}\|b\|_*\left(\int_{2B}\left|f(x)\right|^{p_1}w^{p_1}(x)\mathrm{d}x\right)^{\frac{1}{p_1}}$$

$$\le C\|b\|_*\|f\|_{L^{p_1,(\lambda_1,\lambda_2)}(w^{p_1})}\frac{r^{\frac{\lambda_1}{p_1}}w^{p_1}(2B)^{\frac{\lambda_2}{np_1}}}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}$$

$$\le C\|b\|_*\|f\|_{L^{p_1,(\lambda_1,\lambda_2)}(w^{p_1})}\frac{w^q(2B)^{\frac{\lambda_2}{qn}}}{w^q(B)^{\frac{\lambda_2}{qn}}}.$$

利用 $\lambda_2<0$、引理 3，得到

$$E_{11}\le C\|b\|_*\|f\|_{L^{p_1,(\lambda_1,\lambda_2)}(w^{p_1})}.\tag{15}$$

对于 E_{12}，利用引理 1、Hölder 不等式，有

$$E_{12} \le \frac{1}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}+\frac{n}{p_2}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}\left(\int_{2B}\left|\left(b(x)-b_B\right)f(x)\right|^p w^p(x)\mathrm{d}x\right)^{\frac{1}{p}}$$

$$\le C\frac{1}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}+\frac{n}{p_2}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}\left(\int_{2B}\left|b(x)-b_B\right|^{p_2}\mathrm{d}x\right)^{\frac{1}{p_2}}\left(\int_{2B}\left|f(x)\right|^{p_1}w^{p_1}(x)\mathrm{d}x\right)^{\frac{1}{p_1}}$$

$$\le C\frac{r^{\frac{\lambda_1}{p_1}}w^{p_1}(2B)^{\frac{\lambda_2}{np_1}}}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}+\frac{n}{p_2}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}\left(\int_{2B}\left|b(x)-b_B\right|^{p_2}\mathrm{d}x\right)^{\frac{1}{p_2}}\|f\|_{L^{p_1,(\lambda_1,\lambda_2)}(w^{p_1})}$$

$$\le C\frac{r^{\frac{\lambda_1}{p_1}}w^{p_1}(2B)^{\frac{\lambda_2}{np_1}}}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}+\frac{n}{p_2}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}\|f\|_{L^{p_1,(\lambda_1,\lambda_2)}(w^{p_1})}\left(\left(\int_{2B}\left|b(x)-b_{2B}\right|^{p_2}\mathrm{d}x\right)^{\frac{1}{p_2}}+\left|b_{2B}-b_B\right|\left|2B\right|^{\frac{1}{p_2}}\right)$$

$$\le C\frac{r^{\frac{\lambda_1}{p_1}}w^{p_1}(2B)^{\frac{\lambda_2}{np_1}}\left|2B\right|^{\frac{1}{p_2}}}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}+\frac{n}{p_2}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}\|f\|_{L^{p_1,(\lambda_1,\lambda_2)}(w^{p_1})}\left(\left(\frac{1}{|2B|}\int_{2B}\left|b(x)-b_{2B}\right|^{p_2}\mathrm{d}x\right)^{\frac{1}{p_2}}+\left|b_{2B}-b_B\right|\right)$$

$$\le C\frac{w^q(2B)^{\frac{\lambda_2}{qn}}}{w^q(B)^{\frac{\lambda_2}{qn}}}\|f\|_{L^{p_1,(\lambda_1,\lambda_2)}(w^{p_1})}\|b\|_*.$$

注意到 $\lambda_2<0$，结合引理 3，有

$$E_{12}\le C\|b\|_*\|f\|_{L^{p_1,(\lambda_1,\lambda_2)}(w^{p_1})}. \tag{16}$$

对于 E_2，首先

$$\left|[b,M_\alpha](f_2)(x)\right|^q \le C\left(\frac{1}{|B|^{1-\frac{\alpha}{n}}}\int_B\left|b(x)-b(y)\right|\left|f_2(y)\right|\mathrm{d}y\right)^q$$

$$\le C\left(\frac{1}{|B|^{1-\frac{\alpha}{n}}}\int_B\left|f_2(y)\right|\left(\left|b(x)-b_B\right|+\left|b_B-b(y)\right|\right)\mathrm{d}y\right)^q$$

$$\le C\left(\frac{1}{|B|^{1-\frac{\alpha}{n}}}\int_B\left|f_2(y)\right|\mathrm{d}y\right)^q\left|b(x)-b_B\right|^q+C\left(\frac{1}{|B|^{1-\frac{\alpha}{n}}}\int_B\left|f_2(y)\right|\left|b(y)-b_B\right|\mathrm{d}y\right)^q,$$

因此，有

$$E_2\le C\frac{1}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}+\frac{n}{p_2}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}\left(\int_B\left(\frac{1}{|B|^{1-\frac{\alpha}{n}}}\int_B\left|f_2(y)\right|\mathrm{d}y\right)^q\left|b(x)-b_B\right|^q w^q(x)\mathrm{d}x\right)^{\frac{1}{q}}$$

$$+C\frac{1}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}+\frac{n}{p_2}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}\left(\int_B\left(\frac{1}{|B|^{1-\frac{\alpha}{n}}}\int_B\left|f_2(y)\right|\left|b(y)-b_B\right|\mathrm{d}y\right)^q w^q(x)\mathrm{d}x\right)^{\frac{1}{q}}:=E_{21}+E_{22}。$$

对于 E_{21}，利用 Minkowski 不等式、Hölder 不等式，有

$$E_{21} \le C\frac{1}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}+\frac{n}{p_2}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}\left(\int_B\left(\frac{1}{|B|^{1-\frac{\alpha}{n}}}\int_B|f_2(y)|\mathrm{d}y\right)^q|b(x)-b_B|^q w^q(x)\mathrm{d}x\right)^{\frac{1}{q}}$$

$$\le C\frac{\frac{1}{|B|^{1-\frac{\alpha}{n}}}}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}+\frac{n}{p_2}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}\left[\sup_{x\in B}w(x)\right]\sum_{j=1}^{\infty}\int_{2^{j+1}B\backslash 2^jB}|f(y)|\left(\int_B|b(x)-b_B|^q\mathrm{d}x\right)^{\frac{1}{q}}\mathrm{d}y$$

$$\le C\frac{\frac{1}{|B|^{1-\frac{\alpha}{n}}}}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}+\frac{n}{p_2}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}\left[\inf_{x\in B}w(x)\right]\sum_{j=1}^{\infty}\int_{2^{j+1}B}|f(y)|\left(\int_B|b(x)-b_B|^{p_2}\mathrm{d}x\right)^{\frac{1}{p_2}}\left(\int_B\mathrm{d}x\right)^{\frac{1}{t}}\mathrm{d}y$$

$$\le C\frac{|B|^{\frac{1}{p_2}}}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}+\frac{n}{p_2}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}\frac{1}{|B|^{1-\frac{\alpha}{n}}}\frac{w(B)}{|B|}\|b\|_*\sum_{j=1}^{\infty}\int_{2^{j+1}B}|B|^{\frac{1}{t}}|f(y)|\mathrm{d}y$$

$$\le C\frac{1}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}\frac{|B|^{\frac{1}{q}-\frac{1}{t}-\frac{1}{p_2}}}{|B|^{1-\frac{\alpha}{n}}}\|b\|_*\sum_{j=1}^{\infty}\frac{w(2^{j+1}B)}{|2^{j+1}B|}|2^{j+1}B|^{\frac{1}{t}}\int_{2^{j+1}B}|f(y)|\mathrm{d}y$$

$$\le C\frac{1}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}\frac{|B|^{\frac{1}{q}-\frac{1}{t}-\frac{1}{p_2}}}{|B|^{1-\frac{\alpha}{n}}}\sum_{j=1}^{\infty}\|b\|_*|B|^{\frac{1}{t}}\left(\int_{2^{j+1}B}|f(y)|^{p_1}w^{p_1}(y)\mathrm{d}y\right)^{\frac{1}{p_1}}|2^{j+1}B|^{1-\frac{1}{p_1}}$$

$$\le C\sum_{j=1}^{\infty}\frac{r^{\frac{\lambda_1}{p_1}}w^{p_1}\left(2^{j+1}B\right)^{\frac{\lambda_2}{np_1}}}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}\|b\|_*\frac{|B|^{\frac{1}{q}-\frac{1}{p_2}}}{|B|^{1-\frac{\alpha}{n}}}|2^{j+1}B|^{1-\frac{1}{p_1}}\left(\frac{1}{r^{\lambda_1}w^{p_1}\left(2^{j+1}B\right)^{\frac{\lambda_2}{n}}}\int_{2^{j+1}B}|f(y)|^{p_1}w^{p_1}(y)\mathrm{d}y\right)^{\frac{1}{p_1}}$$

$$\le C\|b\|_*\|f\|_{L^{p_1,(\lambda_1,\lambda_2)}(w^{p_1})}\sum_{j=1}^{\infty}\frac{w^q\left(2^{j+1}B\right)^{\frac{\lambda_2}{qn}}}{w^q(B)^{\frac{\lambda_2}{qn}}}\frac{|B|^{\frac{1}{q}-\frac{1}{p_2}}}{|B|^{1-\frac{\alpha}{n}}}|2^{j+1}B|^{1-\frac{1}{p_1}}$$

$$\le C\|b\|_*\|f\|_{L^{p_1,(\lambda_1,\lambda_2)}(w^{p_1})}\sum_{j=1}^{\infty}|2^{j+1}|^{1-\frac{1}{p_1}+\frac{\lambda_2}{qn}}|B|^{\frac{1}{q}-\frac{1}{p_1}-\frac{1}{p_2}+\frac{\alpha}{n}}$$

$$\le C\|b\|_*\|f\|_{L^{p_1,(\lambda_1,\lambda_2)}(w^{p_1})}\sum_{j=1}^{\infty}|2^{j+1}|^{1-\frac{1}{p_1}+\frac{\lambda_2}{qn}}\text{。}$$

利用 $1+\frac{\lambda_2}{qn}\le\frac{1}{p_1}$，得

$$E_{21}\le C\|b\|_*\|f\|_{L^{p_1,(\lambda_1,\lambda_2)}}. \tag{17}$$

同理，对于 E_{22}，利用 Minkowski 不等式、Hölder 不等式，有

$$E_{22} \le C \frac{1}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}+\frac{n}{p_2}} \left(w^q(B)\right)^{\frac{\lambda_2}{qn}}} \left(\int_B \left(\frac{1}{|B|^{1-\frac{\alpha}{n}}} \int_B |f_2(y)||b(y)-b_B| \mathrm{d}y \right)^q w^q(x) \mathrm{d}x \right)^{\frac{1}{q}}$$

$$\le C \frac{1}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}+\frac{n}{p_2}} \left(w^q(B)\right)^{\frac{\lambda_2}{qn}}} \frac{1}{|B|^{1-\frac{\alpha}{n}}} \sum_{j=1}^{\infty} \int_{2^{j+1}B \setminus 2^j B} |f(y)||b(y)-b_B| \left(\int_B w^q(x) \mathrm{d}x \right)^{\frac{1}{q}} \mathrm{d}y$$

$$\le C \frac{1}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}+\frac{n}{p_2}} \left(w^q(B)\right)^{\frac{\lambda_2}{qn}}} \frac{1}{|B|^{1-\frac{\alpha}{n}}} \left[\sup_{x\in B} w(x) \right] \sum_{j=1}^{\infty} \int_{2^{j+1}B} |f(y)||b(y)-b_B| \mathrm{d}y |B|^{\frac{1}{t}} |B|^{\left(1-\frac{q}{t}\right)\frac{1}{q}}$$

$$\le C \frac{1}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}+\frac{n}{p_2}} \left(w^q(B)\right)^{\frac{\lambda_2}{qn}}} \frac{1}{|B|^{1-\frac{\alpha}{n}}} \left[\inf_{x\in B} w(x) \right] \sum_{j=1}^{\infty} |B|^{\frac{1}{q}} \int_{2^{j+1}B} |f(y)||b(y)-b_{2^{j+1}B}| \mathrm{d}y$$

$$+ C \frac{1}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}+\frac{n}{p_2}} \left(w^q(B)\right)^{\frac{\lambda_2}{qn}}} \frac{1}{|B|^{1-\frac{\alpha}{n}}} \left[\inf_{x\in B} w(x) \right] \sum_{j=1}^{\infty} |B|^{\frac{1}{q}} \int_{2^{j+1}B} |f(y)||b_{2^{j+1}B}-b_B| \mathrm{d}y$$

$$:= F_1 + F_2。$$

对于 F_1，利用 Hölder 不等式，有

$$F_1 \le C \frac{1}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}+\frac{n}{p_2}} \left(w^q(B)\right)^{\frac{\lambda_2}{qn}}} \frac{1}{|B|^{1-\frac{\alpha}{n}}} \left[\inf_{x\in B} w(x) \right] \sum_{j=1}^{\infty} |B|^{\frac{1}{q}} \int_{2^{j+1}B} |f(y)||b(y)-b_{2^{j+1}B}| \mathrm{d}y$$

$$\le C \frac{|B|^{\frac{1}{p_2}}}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}+\frac{n}{p_2}} \left(w^q(B)\right)^{\frac{\lambda_2}{qn}}} \frac{1}{|B|^{1-\frac{\alpha}{n}+\frac{1}{p_2}}} \frac{w(B)}{|B|} \sum_{j=1}^{\infty} |B|^{\frac{1}{q}} \int_{2^{j+1}B} |f(y)||b(y)-b_{2^{j+1}B}| \mathrm{d}y$$

$$\le C \frac{1}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}} \left(w^q(B)\right)^{\frac{\lambda_2}{qn}}} \sum_{j=1}^{\infty} \frac{|B|^{\frac{1}{q}}}{|B|^{1-\frac{\alpha}{n}+\frac{1}{p_2}}} \|b\|_* \left|2^{j+1}B\right|^{\frac{1}{p_2}} \left(\int_{2^{j+1}B} |f(y)|^{p_1} w^{p_1}(y) \mathrm{d}y \right)^{\frac{1}{p_1}} \left|2^{j+1}B\right|^{1-\frac{1}{p_1}-\frac{1}{p_2}}$$

$$\le C \sum_{j=1}^{\infty} \frac{r^{\frac{\lambda_1}{p_1}} w^{p_1}\left(2^{j+1}B\right)^{\frac{\lambda_2}{np_1}}}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}} \left(w^q(B)\right)^{\frac{\lambda_2}{qn}}} \|b\|_* \frac{|B|^{\frac{1}{q}}}{|B|^{1-\frac{\alpha}{n}+\frac{1}{p_2}}} \left(\frac{1}{r^{\lambda_1} w^{p_1}\left(2^{j+1}B\right)^{\frac{\lambda_2}{n}}} \int_{2^{j+1}B} |f(y)|^{p_1} w^{p_1}(y) \mathrm{d}y \right)^{\frac{1}{p_1}} \left|2^{j+1}B\right|^{1-\frac{1}{p_1}}$$

$$\le C \|b\|_* \|f\|_{L^{p_1,(\lambda_1,\lambda_2)}(w^{p_1})} \sum_{j=1}^{\infty} \frac{w^q\left(2^{j+1}B\right)^{\frac{\lambda_2}{qn}}}{w^q(B)^{\frac{\lambda_2}{qn}}} \frac{|B|^{\frac{1}{q}}}{|B|^{1-\frac{\alpha}{n}+\frac{1}{p_2}}} \left|2^{j+1}B\right|^{1-\frac{1}{p_1}}$$

$$\le C \|b\|_* \|f\|_{L^{p_1,(\lambda_1,\lambda_2)}(w^{p_1})} \sum_{j=1}^{\infty} \left|2^{j+1}\right|^{1-\frac{1}{p_1}+\frac{\lambda_2}{qn}} |B|^{\frac{1}{q}-\frac{1}{p_1}-\frac{1}{p_2}+\frac{\alpha}{n}}$$

$$\le C \|b\|_* \|f\|_{L^{p_1,(\lambda_1,\lambda_2)}(w^{p_1})} \sum_{j=1}^{\infty} \left|2^{j+1}\right|^{1-\frac{1}{p_1}+\frac{\lambda_2}{qn}}.$$

根据 $1+\frac{\lambda_2}{qn} \le \frac{1}{p_1}$，结合引理 3，有

$$F_1 \le C\|b\|_* \|f\|_{L^{p_1,(\lambda_1,\lambda_2)}(w^{p_1})}. \tag{18}$$

对于 F_2，由于

$$\begin{aligned}\left|b_{2^{j+1}B}-b_B\right| &\le \sum_{k=0}^{j}\left|b_{2^{k+1}B}-b_{2^kB}\right| \\ &\le C\sum_{k=0}^{j}\left(\frac{1}{\left|2^{k+1}B\right|}\int_{2^{k+1}B}\left|b(y)-b_{2^{k+1}B}\right|^{p_2}\mathrm{d}y\right)^{\frac{1}{p_2}} \\ &\le C(j+1)\|b\|_*.\end{aligned}$$

因此，利用 Hölder 不等式，有

$$\begin{aligned}F_2 &\le C\frac{1}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}+\frac{n}{p_2}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}\frac{1}{|B|^{1-\frac{\alpha}{n}}}\left[\inf_{x\in B}w(x)\right]\sum_{j=1}^{\infty}|B|^{\frac{1}{q}}\int_{2^{j+1}B}|f(y)|\left|b_{2^{j+1}B}-b_B\right|\mathrm{d}y \\
&\le C\frac{1}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}+\frac{n}{p_2}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}\frac{1}{|B|^{1-\frac{\alpha}{n}}}\frac{w(B)}{|B|}\sum_{j=1}^{\infty}|B|^{\frac{1}{q}}\left|b_{2^{j+1}B}-b_B\right|\int_{2^{j+1}B}|f(y)|\mathrm{d}y \\
&\le C\frac{|B|^{\frac{1}{p_2}}}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}+\frac{n}{p_2}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}\frac{|B|^{\frac{1}{q}}}{|B|^{1-\frac{\alpha}{n}+\frac{1}{p_2}}}\sum_{j=1}^{\infty}(j+1)\|b\|_*\left(\int_{2^{j+1}B}|f(y)|^{p_1}w^{p_1}(y)\mathrm{d}y\right)^{\frac{1}{p_1}}\left|2^{j+1}B\right|^{1-\frac{1}{p_1}} \\
&\le C\|b\|_*\|f\|_{L^{p_1,(\lambda_1,\lambda_2)}(w^{p_1})}\sum_{j=1}^{\infty}(j+1)\frac{|B|^{\frac{1}{q}}}{|B|^{1-\frac{\alpha}{n}+\frac{1}{p_2}}}\frac{r^{\frac{\lambda_1}{p_1}}w^{p_1}\left(2^{j+1}B\right)^{\frac{\lambda_2}{np_1}}}{r^{\frac{(\lambda_1+\lambda_2)}{p_1}-\frac{\lambda_2}{q}}\left(w^q(B)\right)^{\frac{\lambda_2}{qn}}}\left|2^{j+1}B\right|^{1-\frac{1}{p_1}} \\
&\le C\|b\|_*\|f\|_{L^{p_1,(\lambda_1,\lambda_2)}(w^{p_1})}\sum_{j=1}^{\infty}(j+1)\frac{|B|^{\frac{1}{q}}}{|B|^{1-\frac{\alpha}{n}+\frac{1}{p_2}}}\frac{w^q\left(2^{j+1}B\right)^{\frac{\lambda_2}{qn}}}{w^q(B)^{\frac{\lambda_2}{qn}}}\left|2^{j+1}B\right|^{1-\frac{1}{p_1}} \\
&\le C\|b\|_*\|f\|_{L^{p_1,(\lambda_1,\lambda_2)}(w^{p_1})}\sum_{j=1}^{\infty}(j+1)\left|2^{j+1}\right|^{1-\frac{1}{p_1}+\frac{\lambda_2}{qn}}|B|^{\frac{1}{q}-\frac{1}{p_1}-\frac{1}{p_2}+\frac{\alpha}{n}} \\
&\le C\|b\|_*\|f\|_{L^{p_1,(\lambda_1,\lambda_2)}(w^{p_1})}\sum_{j=1}^{\infty}\left|2^{j+1}\right|^{1-\frac{1}{p_1}+\frac{\lambda_2}{qn}}.\end{aligned}$$

根据$1+\dfrac{\lambda_2}{qn}\le\dfrac{1}{p_1}$，利用引理 3，有

$$F_2 \le C\|b\|_* \|f\|_{L^{p_1,(\lambda_1,\lambda_2)}(w^{p_1})}. \tag{19}$$

结合式(15)~(19)，定理 3 证毕。

参考文献

[1] Morrey, C.B. (1938) On the Solutions of Quasi-Linear Elliptic Partial Differential Equations. *Transactions of the American Mathematical Society*, **43**, 126-166. https://doi.org/10.1090/S0002-9947-1938-1501936-8

[2] Muckenhoupt, B. and Wheeden, R. (1974) Weighted Norm Inequalities for Fractional Integrals. *Transactions of the American Mathematical Society*, **192**, 261-274. https://doi.org/10.1090/S0002-9947-1974-0340523-6

[3] Chiarenza, F. and Frasca, M. (1987) Morrey Spaces and Hardy-Littlewood Maximal Function. *Rendiconti del Semina-*

rio Matematico della Università di Padova, **7**, 273-279.

[4] 丁勇. 一类粗糙极大算子交换子的加权有界性[J]. 科学通报, 1996, 41(5): 385-388.

[5] Wang, Z.J. and Zhu, Y.P. (2016) Boundness of Fractional Maximal Operator on Weighted Lebesgue Spaces with Variable Exponents. *Journal of Nan Jing University Mathematical Biquarterly*, **33**, 469-584.

[6] Duoandikoetxea, J. and Rosenthal, M. (2020) Boundedness Properties in a Family of Weighted Morrey Spaces with Emphasis on Power Weights. *Journal of Functional Analysis*, **279**, Article ID: 108687. https://doi.org/10.1016/j.jfa.2020.108687

[7] Zhou, J. and Zhao, F.Y. (2022) Boundedness of the Fractional Hardy-Littlewood Maximal Operator on Weighted Morrey Spaces. *Analysis and Mathematical Physics*, **12**, Article No. 87. https://doi.org/10.1007/s13324-022-00695-5

[8] Tanaka, H. (2010) Morrey Spaces and Fractional Operators. *Journal of the Australian Mathematical Society*, **88**, 247-259. https://doi.org/10.1017/S1446788709000457

[9] Perez, C. (1994) Two Weighted Inequalities for Potential and Fractional Type Maximal Operators. *Indiana University Mathematics Journal*, **43**, 663-683. https://doi.org/10.1512/iumj.1994.43.43028

[10] Sawano, Y., Sugano, S. and Tanaka, H. (2011) Generalized Fractional Integral Operators and Fractional Maximal Operators in the Framework of Morrey Spaces. *Transactions of the American Mathematical Society*, **363**, 6481-6503. https://doi.org/10.1090/S0002-9947-2011-05294-3

[11] Nakamura, S. (2016) Generalized Weighted Morrey Spaces and Classical Operators. *Mathematische Nachrichten*, **289**, 2235-2262. https://doi.org/10.1002/mana.201500260

[12] 陶双平, 高荣. 多线性分数次积分和极大算子在 Morrey 空间上的加权估计[J]. 山东大学学报(理学版), 2018, 53(6): 30-37.

[13] Stein, E.M. and Murphy, T.S. (1993) Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals. Princeton University Press, Princeton, 115-125. https://doi.org/10.1515/9781400883929

[14] Yu, X., Zhang, H.H. and Zhao, G.P. (2016) Weighted Boundedness of Some Integral Operators on Weighted λ-Central Morrey Spaces. *Applied Mathematics*: *A Journal of Chinese Universities*, **31**, 331-342. https://doi.org/10.1007/s11766-016-3348-5