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Boussinesq-Coriolis方程在变指数Fourier-Besov-Morrey 空间中解的整体适定性

马偌鸿，孙小春*，吴育联

西北师范大学，数学与统计学院，甘肃 兰州

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摘 要

本文考虑 Boussinesq-Coriolis 方程在变指数 Fourier-Besov-Morrey 空间 $\mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),\lambda(\cdot),q(\cdot)}$ 中的 Cauchy 问题。利用 littlewood-Paley 分解工具和 Fourier 局部化方法，我们得到了小初值 (u_0, θ_0) 整体解的存在唯一性。

关键词

Boussinesq-Coriolis方程，整体适定性，变指数Fourier-Besov-Morrey空间

Global Well-Posedness for the Boussinesq-Coriolis Equations in Variable Exponent Fourier-Besov-Morrey Spaces

Ruohong Ma, Xiaochun Sun*, Yulian Wu

College of Mathematics and Statistics, Northwest Normal University, Lanzhou Gansu

* 通讯作者

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Abstract

We consider the Cauchy problem of the three-dimensional Boussinesq equations with Coriolis force in variable exponent Fourier-Besov-Morrey spaces $\mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),\lambda(\cdot),q(\cdot)}$ in this paper. By using littlewood-Paley decomposition and the Fourier localization argument, we obtain the unique existence of the global solution for small initial data (u_0, θ_0).

Keywords

Boussinesq-Coriolis Equations, Global Well-Posedness, Variable Exponent Fourier-Besov-Morrey Spaces

1. 引言

本文研究三维 Boussinesq-Coriolis 方程:

$$\begin{cases} \partial_t u - \nu\Delta u + \Omega e_3 \times u + (u\cdot\nabla)u + \nabla p = g\theta e_3 & \text{in} \quad \mathbb{R}^3 \times (\mathbb{R}_+), \\ \partial_t \theta - \kappa\Delta\theta + (u\cdot\nabla)\theta = 0 & \text{in} \quad \mathbb{R}^3 \times (\mathbb{R}_+), \\ \mathrm{div} u = 0 & \text{in} \quad \mathbb{R}^3 \times (\mathbb{R}_+), \\ u|_{t=0} = u_0, \quad \theta|_{t=0} = \theta_0 & \text{in} \quad \mathbb{R}^3. \end{cases} \tag{1}$$

其中 $u = (u_1(x,t), u_2(x,t), u_3(x,t))$, $\theta = (\theta_1(x,t), \theta_2(x,t), \theta_3(x,t))$, $p = p(x,t)$ 表示流体的速度场,密度波动和流体所受压力. 正常数 ν, κ, g 分别表示粘性系数, 热传导系数及重力. $g\theta e_3$ 表示浮力, $\Omega e_3 \times u$ 为 Coriolis 力, 其中参数 $\Omega \in \mathbb{R}$ 为流体绕竖直单位向量 $e_3 = (0,0,1)$ 的旋转速度.

Boussinesq 方程模拟了大气锋和海洋湍流等地球物理流体以及 Rayleigh-Benard 对流, 在大气学科中具有重要的作用 [1]. 方程 (1) 是从依赖密度的流体方程中产生的, 通过使用 Boussinesq 近似, 除了涉及重力的项, 它忽略了所有项的密度依赖. 旋转引起了方程中的科里奥利力 $\Omega e_3 \times u$, 这导致

了流体的垂直刚性, 正如 Taylor - Proudman 定理所描述: 在快速旋转下, 水平速度场没有垂直剪切, 所有在同一垂直方向上的粒子一致运动. 当 $\Omega \neq 0$, Sun, Yang, Cui [2] 证明了方程 (1) 初值问题在 Besov 空间中的整体适定性. Koba, Mahalov, Yoneda [3] 研究了初值 $(u_0, \theta_0) \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3)$ 且 Prandtl 值 $P = 1$ 时, 方程 (1) 的整体适定性. 最近, Sun, Wu, Xu [4] 研究了方程 (1) 初值问题在变指数 Fourier-Besov 空间 $\mathcal{F}\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),q}$ 中的整体适定性. 更多关于 Boussinesq-Coriolis 方程的结论, 参见 [5–8].

近年来, 变指数空间受到越来越多学者关注, 将变指数空间运用到不可压 Navier-Stokes 方程的整体适定性问题, Ru [9] 证明了不可压 Navier-Stokes 方程初值问题在变指数 Fourier-Besov 空间 $\mathcal{F}\dot{B}^{s(\cdot)}_{p(\cdot),q}$ 上的整体适定性. Abidin, Chen [10] 证明了分数阶 Navier-Stokes 方程初值问题在变指数 Fourier-Besov-Morrey 空间 $\mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),\lambda(\cdot),q}$ 中的整体适定性. 更多其他方程在变指数空间中的适定性结果, 参见 [11–13].

因为变指数 Fourier-Besov-Morrey 空间 $\mathcal{F}\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),q}$ 的特殊性, 使其运用到方程的局部, 整体适定性问题时会受到很大的限制, 本文主要通过估计方法克服这类空间所受限制, 将其应用于 Boussinesq-Coriolis 方程的整体适定性问题, 从而获得更加一般性的结果.

本文运用了仿积分解, Littlewood-Paley 分解等工具. 利用 Banach 压缩映射原理以及建立先验估计等主要方法, 研究了方程 (1) 在变指数 Fourier-Besov-Morrey 空间 $\mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),\lambda(\cdot),q(\cdot)}$ 中的整体适定性.

定理 1.1. 设 $p(\cdot), \lambda(\cdot) \in C^{\log} \cap \mathcal{P}_0(\mathbb{R}^3)$, $2 \leq p(\cdot) \leq 6$, $1 \leq \rho \leq \infty$, $1 \leq q < 3$, 则存在一个充分小的 $\epsilon > 0$, 使得对 $\Omega \in \mathbb{R}$ 满足

$$\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),\lambda(\cdot),q}} + \|\theta_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),\lambda(\cdot),q}} < \epsilon, \tag{2}$$

则存在一个唯一的解析解

$$(u,\theta) \in \widetilde{L}^\infty(\mathbb{R}_+; \mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),\lambda(\cdot),q}) \cap \widetilde{L}^\rho(\mathbb{R}_+; \dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,\lambda(\cdot),q}) \cap \widetilde{L}^\infty(\mathbb{R}_+; \dot{\mathcal{N}}^{\frac{1}{2}}_{2,2,q}).$$

特别地, 设 $p_1(\cdot) \in C^{\log} \cap \mathcal{P}_0(\mathbb{R}^3)$,$s_1(\cdot) \in C^{\log}$ 且 $s_1(\cdot) = \frac{2}{\rho} + 2 - \frac{3}{p_1(\cdot)}$, 如果存在 $c > 0$ 使得 $2 \leq p_1(\cdot) \leq c \leq p(\cdot)$, 则上述解还满足

$$(u,\theta) \in \widetilde{L}^\infty(\mathbb{R}_+; \mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),\lambda(\cdot),q}) \cap \widetilde{L}^\rho(\mathbb{R}_+; \dot{\mathcal{N}}^{s_1(\cdot)}_{p_1(\cdot),\lambda(\cdot),q}).$$

2. 预备知识

本节给出变指数函数空间与调和分析工具的一些基本概念. Fourier 变换把物理空间变换到频率空间, 二进制分解算子是针对频率空间进行局部化得到的, 对频率空间做二进制分解, 这与 $L^p(\mathbb{R}^n)$ 上的等价范数有关. 下面介绍 Littlewood-Paley 算子.

$\mathcal{S}(\mathbb{R}^n)$ 是 Schwartz 空间, $\mathcal{S}'(\mathbb{R}^n)$ 是 $\mathcal{S}(\mathbb{R}^n)$ 的拓扑对偶空间. 若 $\psi, \varphi \in \mathcal{S}(\mathbb{R}^n)$ 是在区间 $[0,1]$

上取值的一对光滑非负函数, 且支集分别满足

$$\mathcal{B}=\left\{\xi\in\mathbb{R}^n:|\xi|\leq\frac{4}{3}\right\},$$

$$\mathcal{C}=\left\{\xi\in\mathbb{R}^n:\frac{4}{3}\leq|\xi|\leq\frac{8}{3}\right\}.$$

则有

$$\sum_{j\in\mathbb{Z}}\varphi_j(\xi)=1,\qquad\forall\xi\in\mathbb{R}^n\backslash\{0\},$$

其中

$$\varphi_j(\xi)=\varphi(2^{-j}\xi),\quad\psi(\xi)+\sum_{j\geq0}\varphi_j(\xi)=1,\qquad\forall j\in\mathbb{Z},\quad\forall\xi\in\mathbb{R}^n.$$

设 $h=\mathcal{F}^{-1}\varphi$, 频率局部化算子定义为

$$\Delta_j u=\varphi(2^{-j}D)u=2^{nj}\int_{\mathbb{R}^n}h(2^jy)u(x-y)\mathrm{d}y,$$

$$S_ju=\sum_{k\leq j-1}\Delta_k u,\qquad\forall j\in\mathbb{Z},$$

由以上定义

$$\Delta_k\Delta_j u=0,\qquad\qquad|j-k|\geq2,$$

$$\Delta_k(S_{j-1}u\Delta_j u)=0,\qquad|j-k|\geq5.$$

定义 2.1. 设 $\mathcal{P}_0(\mathbb{R}^n):=\{p(\cdot):\mathbb{R}^n\to(0,\infty)|p_-=\mathrm{essinf}_{x\in\mathbb{R}^n}p(x),p^+=\mathrm{esssup}_{x\in\mathbb{R}^n}p(x)\}$, 其中 $p(\cdot)$ 是可测函数. 变指数 Lebesgue 函数空间定义为

$$Ł^{p(\cdot)}(\mathbb{R}^n)=\{f:\mathbb{R}^n\to\mathbb{R},\ \|f\|_{L^{p(\cdot)}}<\infty\},$$

其中

$$\|f\|_{L^{p(\cdot)}}=\inf\{\mu>0:\int_{\mathbb{R}^n}(|f(x)|/\mu)^{p(x)}\mathrm{dx}\leq1\}.$$

$L^{p(\cdot)}(\mathbb{R}^n)$ 是 Banach 空间, 为了确保 Hardy-Littlewood 极大算子 M 在 $L^{p(\cdot)}(\mathbb{R}^n)$ 上有界, 假设以下条件成立:

对于正数 $C_{\log}(p)$ 和 p^{∞}, 存在一个函数 $p(\cdot):\mathbb{R}^n\to\mathbb{R}$, 使得不等式

$$|p(x)-p(y)|\le\frac{C_{\log}(p)}{\log(e+|x-y|^{-1})},\quad (x,y\in\mathbb{R}^n,x\neq y)$$

和

$$|p(x)-p_{\infty}|\le\frac{C_{\log}(p)}{\log(e+|x|)},\quad (x\in\mathbb{R}^n)$$

成立, $p(\cdot)$ 分别满足局部和整体 log-Hölder's 连续条件.

定义 2.2. 设 $p(\cdot),\lambda(\cdot)\in\mathcal{P}_0(\mathbb{R}^n)$ 且 $0<p_-\le p(\cdot)\le\lambda(\cdot)\le\infty$, 变指数 Morrey 空间定义为 $\mathbb{R}^n$ 上的可测函数空间, 且范数为

$$\|f\|_{M_{p(\cdot)}^{\lambda(\cdot)}}:=\sup_{x_0\in\mathbb{R}^n,r>0}\|r^{\frac{n}{\lambda(x)}-\frac{n}{p(x)}}f\chi_{B(x_0,r)}\|_{L^{p(\cdot)}}.$$

根据 $L^{p(\cdot)}$ 的定义, $\|f\|_{M_{p(\cdot)}^{\lambda(\cdot)}}$ 又可以表示为

$$\|f\|_{M_{p(\cdot)}^{\lambda(\cdot)}}:=\sup_{x_0\in\mathbb{R}^n,r>0}\inf\left\{\mu>0:\varrho_{p(\cdot)}(r^{\frac{n}{\lambda(x)}-\frac{n}{p(x)}}\frac{f}{\mu}\chi_{B(x_0,r)})\le 1\right\}.$$

下面介绍一些在主要结果证明过程中需要的引理.

引理 2.1. [14] 设 $p(\cdot),\lambda(\cdot)\in\mathcal{P}_0(\mathbb{R}^n)$ 且 $p(x)\le\lambda(x)$. 对于任何可测函数 f 成立

$$\|f\|_{M_{p(\cdot)}^{\lambda(\cdot)}}:=\inf\left\{\mu>0:\sup_{x_0\in\mathbb{R}^n,r>0}\varrho_{p(\cdot)}(r^{\frac{n}{\lambda(x)}-\frac{n}{p(x)}}\frac{f}{\mu}\chi_{B(x_0,r)})\le 1\right\}.$$

引理 2.2. [14] 若 f 是一个可测函数且 $p(\cdot)\in\mathcal{P}_0(\mathbb{R}^n)$, 则

$$\sup_{x_0\in\mathbb{R}^n,r>0}\varrho_{p(\cdot)}(f\chi_{B(x_0,r)})=\varrho_{p(\cdot)}(f).$$

引理 2.3. [14] 设 $p(\cdot)\in\mathcal{P}_0^n$, 则 $\|f\|_{M_{p(\cdot)}^{\lambda(\cdot)}}=\|f\|_{L^{p(\cdot)}}$.

定义 2.3. 设 $p(\cdot),q(\cdot),\lambda(\cdot)\in\mathcal{P}_0(\mathbb{R}^n)$ 且 $p(\cdot)\le\lambda(\cdot)$, 空间 $\ell^{q(\cdot)}(M_{p(\cdot)}^{\lambda(\cdot)})$ 由 $\mathbb{R}^n$ 上所有的可测函数序列 $\{f_j\}_{j\in\mathbb{Z}}$ 组成, 定义为

$$\|\{f_j\}_{j\in\mathbb{Z}}\|_{l^{q(\cdot)}(M_{p(\cdot)}^{\lambda(\cdot)})}=\inf\left\{\mu>0,\varrho_{l^{q(\cdot)}(M_{p(\cdot)}^{\lambda(\cdot)})}\left(\left\{\frac{f_j}{\mu}\right\}_{j\in\mathbb{Z}}\right)\le 1\right\}\le\infty,$$

其中

$$\varrho_{l^{q(\cdot)}(M_{p(\cdot)}^{\lambda(\cdot)})}(\{f_j\}_{j\in\mathbb{Z}})=\sum_{j\in\mathbb{Z}}\inf\left\{\mu>0:\int_{\mathbb{R}^n}\left(\frac{|r^{\frac{n}{\lambda(x)}-\frac{n}{p(x)}}f_j\chi_{B(x_0,r)}|}{\mu^{\frac{1}{q(x)}}}\right)^{p(x)}\mathrm{d}x\le 1\right\}.$$

另外, 若 $p(x) \le q(x)$ 且 $q_+ < \infty$, 则

$$\varrho_{l^{q(\cdot)}(M^{\lambda(\cdot)}_{p(\cdot)})}(\{f_j\}_{j\in\mathbb{Z}}) = \sum_{j\in\mathbb{Z}} \sup_{x_0\in\mathbb{R}^n} \|(|r^{\frac{n}{\lambda(x)}-\frac{n}{p(x)}} f_j|\chi_{B(x_0,r)})^{q(\cdot)}\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}.$$

定义 2.4. 设 $s(\cdot) \in C^{log}(\mathbb{R}^n)$, $p(\cdot), q(\cdot), \lambda(\cdot) \in C^{log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$ 且 $0 < p^- \le p(x) \le h(x) \le \infty$, 齐次变指数 Besov-Morrey 空间 $\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),\lambda(\cdot),q(\cdot)}$ 的定义为

$$\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),\lambda(\cdot),q(\cdot)} = \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),\lambda(\cdot),q(\cdot)}} < \infty\},$$

$$\|f\|_{\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),\lambda(\cdot),q(\cdot)}} := \|\{2^{js(\cdot)}\Delta_j f\}_{j\in\mathbb{Z}}\|_{\ell^{q(\cdot)}(M^{\lambda(\cdot)}_{p(\cdot)})} < \infty,$$

定义 2.5. 设 $s(\cdot) \in C^{log}(\mathbb{R}^n)$, $p(\cdot), q(\cdot), \lambda(\cdot) \in C^{log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$ 且 $0 < p^- \le p(x) \le h(x) \le \infty$, 齐次变指数 Fourier-Besov-Morrey 空间 $\mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),\lambda(\cdot),q(\cdot)}$ 定义为

$$\mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),\lambda(\cdot),q(\cdot)} = \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),\lambda(\cdot),q(\cdot)}} < \infty\},$$

$$\|f\|_{\mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),\lambda(\cdot),q(\cdot)}} := \|\{2^{js(\cdot)}\varphi_j \hat{f}\}_{j\in\mathbb{Z}}\|_{\ell^{q(\cdot)}(M^{\lambda(\cdot)}_{p(\cdot)})} < \infty.$$

类似的, Chemin-Lerner 型齐次变指数 Fourier-Besov-Morrey 空间 $L^\rho(0, T, \mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),\lambda(\cdot),q(\cdot)})$ 是所有缓增分布 u 的集合,范数定义为

$$\|u\|_{L^\rho(0,T;\mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),\lambda(\cdot),q})} := \left\| \left(\sum_{j=0}^{\infty} \|2^{js(\cdot)}\varphi_j \hat{u}\|^q_{M^{\lambda(\cdot)}_{p(\cdot)}} \right)^{\frac{1}{q}} \right\|_{L^\rho_T} < \infty.$$

混合空间 $\widetilde{L}^\rho(0, T, \mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),\lambda(\cdot),q(\cdot)})$ 上的范数定义为

$$\|u\|_{\widetilde{L}^\rho(0,T;\mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),\lambda(\cdot),q})} := \left(\sum_{j\in\mathbb{Z}} \|2^{js(\cdot)}\varphi_j \hat{u}\|^q_{L^\rho_T M^{\lambda(\cdot)}_{p(\cdot)}} \right)^{\frac{1}{q}} < \infty.$$

定义 2.6. 设 $u, v \in \mathcal{S}'_h$, 则有 Bony 分解定义如下

$$uv = T_u v + T_v u + R(u, v),$$

其中

$$T_u v = \sum_{j\in\mathbb{Z}} S_{j-1} u \Delta_j v, \qquad T_v u = \sum_{j\in\mathbb{Z}} S_{j-1} v \Delta_j u,$$

$$R(u, v) = \sum_{j\in\mathbb{Z}} \Delta_j u \tilde{\Delta}_j v, \quad \tilde{\Delta}_j v = \sum_{|j-k|\le 1} \Delta_k v.$$

引理 2.4. ([10]) 设 $p(\cdot),p_1(\cdot),p_2(\cdot),\lambda(\cdot),\lambda_1(\cdot),\lambda_2(\cdot)\in\mathcal{P}_0(\mathbb{R}^n)$, 且 $\frac{1}{p(\cdot)}=\frac{1}{p_1(\cdot)}+\frac{1}{p_2(\cdot)}$, $\frac{1}{\lambda(\cdot)}=\frac{1}{\lambda_1(\cdot)}+\frac{1}{\lambda_2(\cdot)}$. 则对于所有的 $f\in M^{\lambda_1(\cdot)}_{p_1(\cdot)}$ 和 $g\in M^{\lambda_2(\cdot)}_{p_2(\cdot)}$, 有

$$\|fg\|_{M^{\lambda(\cdot)}_{p(\cdot}}\le C_{p,p_1}\|f\|_{M^{\lambda_1(\cdot)}_{p_1(\cdot)}}\|g\|_{M^{\lambda_2(\cdot)}_{p_2(\cdot)}}.$$

引理 2.5. ([10]) 设 $p_0(\cdot),p_1(\cdot),\lambda_0(\cdot),\lambda_1(\cdot)\in\mathcal{P}_0(\mathbb{R}^n)$, $0<q<\infty$, $s_0(\cdot),s_1(\cdot)\in L^\infty\cap C^{log}(\mathbb{R}^n)$ 且 $s_0(\cdot)>s_1(\cdot)$. 若 $\frac{1}{q}$ 与 $s_0(x)-\frac{n}{p_0(x)}=s_1(x)-\frac{n}{p_1(x)}$ 是局部 log-Hölder 连续, 则

$$\mathcal{N}^{s_0(\cdot)}_{p_0(\cdot),\lambda_0(\cdot),q}\hookrightarrow\mathcal{N}^{s_1(\cdot)}_{p_1(\cdot),\lambda_1(\cdot),q}.$$

引理 2.6. ([14]) 对于 $p(\cdot)\in C^{\log}(\mathbb{R}^n)$ 和 $\psi\in L^1(\mathbb{R}^n)$, 假设 $\Psi(x)=\sup\limits_{y\notin B(0,|x|)}|\psi(y)|$ 是可积的, 则对于所有的 $f\in L^{p(\cdot)}(\mathbb{R}^n)$, 有

$$\|f*\psi_\varepsilon\|_{M^{\lambda(\cdot)}_{p(\cdot)}(\mathbb{R}^n)}\le C\|f\|_{M^{\lambda(\cdot)}_{p(\cdot)}(\mathbb{R}^n)}\|\Psi\|_{L^1(\mathbb{R}^n)}$$

其中 $\psi_\varepsilon=\frac{1}{\varepsilon^n}\psi(\frac{1}{\varepsilon})$ 且常数 C 仅依赖于 n.

引理 2.7. ([10]) 设 $s>0$, $1\le q,\rho,\rho_1,\rho_2,\theta(\cdot),r(\cdot),p(\cdot),\lambda(\cdot),\lambda_1(\cdot),\lambda_2(\cdot)\le\infty$, 且 $\frac{1}{\theta(\cdot)}=\frac{1}{r(\cdot)}+\frac{1}{p(\cdot)}$, $\frac{1}{\lambda(\cdot)}=\frac{1}{\lambda_1(\cdot)}+\frac{1}{\lambda_2(\cdot)}$, $\frac{1}{\rho}=\frac{1}{\rho_1}+\frac{1}{\rho_2}$. 则对于所有的 $u\in\widetilde{L}^{\rho_1}_t\dot{\mathcal{N}}^s_{p(\cdot),\lambda_2(\cdot),q}\cap L^{\rho_2}_tM^{\lambda_1}_{r(\cdot)}{}^{(\cdot)}$, 且 $v\in\widetilde{L}^{\rho_1}_t\dot{\mathcal{N}}^s_{p(\cdot),\lambda_2(\cdot),q}\cap L^{\rho_2}_tM^{\lambda_1(\cdot)}_r$, 有

$$\|uv\|_{\widetilde{L}^{\rho}_t\dot{\mathcal{N}}^s_{\theta(\cdot),\lambda(\cdot),q}}\lesssim\|u\|_{\widetilde{L}^{\rho_1}_t\dot{\mathcal{N}}^s_{p(\cdot),\lambda(\cdot),q}}\|v\|_{L^{\rho_2}_tM^{\lambda_1(\cdot)}_r}+\|v\|_{\widetilde{L}^{\rho_1}_t\dot{\mathcal{N}}^s_{p(\cdot),\lambda_2(\cdot),q}}\|u\|_{L^{\rho_2}_tM^{\lambda_1}_{r(\cdot)}{}^{(\cdot)}}.$$

3. 定理1.1的证明

为了证明方程 (1) 的整体适定性，我们考虑下面的等价积分方程 (3).

$$\begin{cases}u(t)=T_\Omega(t)u_0-\int_0^tT_\Omega(t-\tau)\mathbb{P}[(u\cdot\nabla)u]\mathrm{d}\tau+\int_0^tT_\Omega(t-\tau)\mathbb{P}g\theta e_3\mathrm{d}\tau,\\ \theta(t)=e^{t(\Delta)}\theta_0-\int_0^te^{(t-\tau)(\Delta)}[(u\cdot\nabla)\theta]\mathrm{d}\tau,\end{cases}\tag{3}$$

其中 $\mathbb{P}:=I-\nabla(-\Delta)^{-1}$ 表示霍姆赫兹投影.

$$\begin{aligned}T_\Omega(t)u&=\mathcal{F}^{-1}\left[\cos\left(\Omega\frac{\xi_3}{|\xi|}t\right)e^{-t|\xi|^2}I+\sin\left(\Omega\frac{\xi_3}{|\xi|}t\right)e^{-t|\xi|^2}R(\xi)\right]*u\\&=\mathcal{F}^{-1}\left[\cos\left(\Omega\frac{\xi_3}{|\xi|}t\right)I+\sin\left(\Omega\frac{\xi_3}{|\xi|}t\right)R(\xi)\right]*(e^{t\Delta}u).\end{aligned}$$

这里无散度向量场 $u\in\mathcal{S}'(\mathbb{R}^3)$, I 是 $M_{3\times3}(\mathbb{R})$ 中的单位矩阵, $R(\xi)$ 是斜对称矩阵且其定义为

$$R(\xi):=\frac{1}{|\xi|}\begin{pmatrix}0&\xi_3&-\xi_2\\-\xi_3&0&\xi_1\\\xi_2&-\xi_1&0\end{pmatrix},\quad\xi\in\mathbb{R}^3\backslash\{0\}.$$

对于 $T_\Omega(\cdot)$ 的显式推导, 参见文献 [15].

为了证明方程 (1) 的整体适定性, 我们需要对方程 (3) 建立如下线性估计.

引理 3.1. 设 $p(\cdot),p_1(\cdot),\lambda(\cdot)\in C_{\log}\cap\mathcal{P}_0(\mathbb{R}^3)$, $2\le p_1(\cdot)\le c\le p(\cdot)\le 6$, $s_1(\cdot)=\frac{2}{\rho}+2-\frac{3}{p_1(\cdot)}$ 且$1\le q,\rho\le\infty$,则对任意的 $\Omega\in\mathbb{R}$, 且 $f\in\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),\lambda(\cdot),q}$ 有

$$\|T_\Omega(t)f\|_{\widetilde{L}^\rho\left(0,\infty;\mathcal{F}\dot{\mathcal{N}}^{s_1(\cdot)}_{p_1(\cdot),\lambda(\cdot),q}\right)}\lesssim\|f\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),\lambda(\cdot),q}},$$

证明 通过引理 2.4 以及傅里叶乘子 $T_\Omega(t)f$ 的有界性，可得

$$\begin{aligned}
&\|T_\Omega(t)f\|_{\widetilde{L}^\rho\left(\mathbb{R}_+;\mathcal{F}\dot{\mathcal{N}}^{s_1(\cdot)}_{p_1(\cdot),\lambda(\cdot),q}\right)}\\
&=\left\|\|2^{js_1(\cdot)}\varphi_j\mathcal{F}[T_\Omega(t)f]\|_{L^\rho(\mathbb{R}_+;M^{\lambda(\cdot)}_{p_1(\cdot)})}\right\|_{l^q}\\
&\lesssim\left\|\|2^{js_1(\cdot)}\varphi_je^{-t|\cdot|^2}\widehat{f}\|_{L^\rho(\mathbb{R}_+;M^{\lambda(\cdot)}_{p_1(\cdot)})}\right\|_{l^q}\\
&\lesssim\left\|\sum_{l=0,\pm1}\left\|2^{j(2-\frac{3}{c})}\varphi_j\widehat{f}\right\|_{M^{\lambda(\cdot)}_{p(\cdot)}}\left\|r^{\frac{-3(c-p_1(\cdot))}{cp_1(\cdot)}}2^{j(\frac{2}{\rho}-\frac{3}{c}-\frac{3}{p_1(\cdot)})}\varphi_{j+l}e^{-t2^{2(j+l)}}\right\|_{L^\rho(\mathbb{R}_+;L^{\frac{cp_1(\cdot)}{c-p_1(\cdot)}})}\right\|_{l^q}\\
&\lesssim\left\|\sum_{l=0,\pm1}\left\|2^{j(2-\frac{3}{p(\cdot)})}\varphi_j\widehat{f}\right\|_{M^{\lambda(\cdot)}_{p(\cdot)}}\right\|_{l^q}\\
&=\|f\|_{\mathcal{F}\dot{\mathcal{N}}^{s(\cdot)+\frac{3}{p'(\cdot)}}_{p(\cdot),\lambda(\cdot),q}}.
\end{aligned}$$

其中我们用到了下面的估计

$$\begin{aligned}
&\left\|r^{\frac{-3(c-p_1(\cdot))}{cp_1(\cdot)}}2^{j(\frac{2}{\rho}+\frac{3}{c}-\frac{3}{p_1(\cdot)})}\varphi_{j+l}e^{-t2^{2(j+l)}}\right\|_{L^\rho(\mathbb{R}_+;L^{\frac{cp_1(\cdot)}{c-p_1(\cdot)}})}\\
&=\|r^{\frac{-3(c-p_1(\cdot))}{cp_1(\cdot)}}2^{j\frac{2}{\rho}}e^{-t2^{2(j+l)}}\|_{L^\rho(\mathbb{R}_+)}\|2^{j(\frac{3}{c}-\frac{3}{p_1(\cdot)})}\varphi_{j+l}\|_{L^{\frac{cp_1(\cdot)}{c-p_1(\cdot)}}}\\
&\lesssim\|2^{j(\frac{3}{c}-\frac{3}{p_1(\cdot)})}\varphi_{j+l}\|_{L^{\frac{cp_1(\cdot)}{c-p_1(\cdot)}}}\\
&\lesssim\inf\{\lambda>0:\textstyle\int_{\mathbb{R}^3}|2^{j(\frac{3}{c}-\frac{3}{p_1(x)})}\varphi_{j+l}|^{\frac{cp_1(x)}{c-p_1(x)}}\mathrm{d}x\le1\}\\
&\lesssim\inf\{\lambda>0:\textstyle\int_{\mathbb{R}^3}|\varphi_{j+l}|^{\frac{cp_1(x)}{c-p_1(x)}}2^{-3j}\mathrm{d}x\le1\}\\
&\lesssim\inf\{\lambda>0:\textstyle\int_{\mathbb{R}^3}|\varphi_l|^{\frac{cp_1(2^jx)}{c-p_1(2^jx)}}\mathrm{d}x\le1\}\\
&\lesssim C.
\end{aligned}$$

运用引理 3.1, 以及 Banach 压缩映射原理, 可以得到方程 (1) 的存在性和唯一性.

定理 1.1 的证明 设 $M>0$, $\delta>0$, 且有

$$\begin{aligned}
D=\Bigg\{(u,\theta):\|u\|_{\widetilde{L}^\rho(\mathbb{R}_+;\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),\lambda(\cdot),q})}+\|\theta\|_{\widetilde{L}^\rho(\mathbb{R}_+;\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),\lambda(\cdot),q})}\le L,\\
\|u\|_{\widetilde{L}^\rho(\mathbb{R}_+;\dot{N}^{\frac{2}{\rho}+\frac{1}{2}}_{2,\lambda(\cdot),q})\cap\widetilde{L}^\infty(\mathbb{R}_+;\dot{N}^{\frac{1}{2}}_{2,2,q})}+\|\theta\|_{\widetilde{L}^\rho(\mathbb{R}_+;\dot{N}^{\frac{2}{\rho}+\frac{1}{2}}_{2,\lambda(\cdot),q})\cap\widetilde{L}^\infty(\mathbb{R}_+;\dot{N}^{\frac{1}{2}}_{2,2,q})}\le\delta\Bigg\},
\end{aligned}$$

我们定义

$$X=\widetilde{L}^{\infty}(\mathbb{R}_+;\mathcal{F}\dot{N}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),\lambda(\cdot),q})\cap\widetilde{L}^{\rho}(\mathbb{R}_+;\dot{N}^{\frac{2}{\rho}+\frac{1}{2}}_{2,\lambda(\cdot),q})\cap\widetilde{L}^{\infty}(\mathbb{R}_+;\dot{N}^{\frac{1}{2}}_{2,2,q}).$$

考虑映射

$$\mathfrak{M}:(u,\theta)\to(T_\Omega(t)u_0,e^{t\Delta}\theta_0)-\left(\int_0^t T_\Omega(t-\tau)\mathbb{P}[(u\cdot\nabla)u]\mathrm{d}\tau,\int_0^t e^{(t-\tau)\Delta}[(u\cdot\nabla)\theta]\mathrm{d}\tau\right)$$
$$+\left(\int_0^t T_\Omega(t-\tau)\mathbb{P}g\theta e_3\mathrm{d}\tau,0\right).$$

其具备度量

$$d((u,\theta),(u_1,\theta_1))=\|u-u_1\|_{\widetilde{L}^{\rho}(\mathbb{R}_+;\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),\lambda(\cdot),q})\cap\widetilde{L}^{\rho}(\mathbb{R}_+;\dot{N}^{\frac{2}{\rho}+\frac{1}{2}}_{2,\lambda(\cdot),q})\cap\widetilde{L}^{\infty}(\mathbb{R}_+;\dot{N}^{\frac{1}{2}}_{2,2,q})}$$
$$+\|\theta-\theta_1\|_{\widetilde{L}^{\rho}(\mathbb{R}_+;\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),\lambda(\cdot),q})\cap\widetilde{L}^{\rho}(\mathbb{R}_+;\dot{N}^{\frac{2}{\rho}+\frac{1}{2}}_{2,\lambda(\cdot),q})\cap\widetilde{L}^{\infty}(\mathbb{R}_+;\dot{N}^{\frac{1}{2}}_{2,2,q})}.$$

根据引理 3.1, 有

$$\|T_\Omega(t)u_0\|_{\widetilde{L}^{\rho}\left(\mathbb{R}_+;\mathcal{F}\dot{\mathcal{N}}^{s_1(\cdot)}_{p_1(\cdot),\lambda(\cdot),q}\right)}\lesssim\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),\lambda(\cdot),q}},$$

当 $\Omega=0$ 时, 有

$$\|e^{t\Delta}\theta_0\|_{\widetilde{L}^{\rho}\left(\mathbb{R}_+;\mathcal{F}\dot{\mathcal{N}}^{s_1(\cdot)}_{p_1(\cdot),\lambda(\cdot),q}\right)}\lesssim\|\theta_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),\lambda(\cdot),q}},$$

相似地, 可得

$$\|T_\Omega(t)u_0\|_{\widetilde{L}^{\rho}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,\lambda(\cdot),q}\right)}\lesssim\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),\lambda(\cdot),q}},$$

$$\|e^{t\Delta}\theta_0\|_{\widetilde{L}^{\rho}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,\lambda(\cdot),q}\right)}\lesssim\|\theta_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),\lambda(\cdot),q}},$$

若 $\rho=\infty$ 且 $p_1(\cdot)=p(\cdot)$, 得到 $T_\Omega(t)u_0$ 和 $e^{t\Delta}\theta_0$ 的估计如下

$$\|T_\Omega(t)u_0\|_{\widetilde{L}^{\infty}\left(\mathbb{R}_+;\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),\lambda(\cdot),q}\right)}\lesssim\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),\lambda(\cdot),q}},$$

$$\|T_\Omega(t)u_0\|_{\widetilde{L}^{\infty}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac{1}{2}}_{2,2,q}\right)}\lesssim\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),\lambda(\cdot),q}},$$

$$\|e^{t\Delta}\theta_0\|_{\widetilde{L}^{\infty}\left(\mathbb{R}_+;\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),\lambda(\cdot),q}\right)}\lesssim\|\theta_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),\lambda(\cdot),q}},$$

$$\|e^{t\Delta}\theta_0\|_{\widetilde{L}^{\infty}\left(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac{1}{2}}_{2,2,q}\right)}\lesssim\|\theta_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),\lambda(\cdot),q}}.$$

设 $\overline{p}=\frac{6p_1(\cdot)}{6-p_1(\cdot)}$, 运用引理 2.3,2.4,2.7, 和 Hausdorff-Young's 不等式, 有

$$
\begin{aligned}
&\left\|\int_0^t T_\Omega(t-\tau)\mathbb{P}[(u\cdot\nabla)u]\mathrm{d}\tau\right\|_{\widetilde{L}^\rho(\mathbb{R}_+;\mathcal{F}\dot{\mathcal{N}}^{s_1(\cdot)}_{p_1(\cdot),\lambda(\cdot),q})}\\
=&\left\|\left\{\left\|2^{js_1(\cdot)}\varphi_j\mathcal{F}\left\{\int_0^t T_\Omega(t-\tau)\mathbb{P}[(u\cdot\nabla)u]\mathrm{d}\tau\right\}\right\|_{L^\rho(\mathbb{R}_+;M^{\lambda(\cdot)}_{p_1(\cdot)})}\right\}_{j\in\mathbb{Z}}\right\|_{l^q}\\
\lesssim&\left\|\left\{\left\|\int_0^t 2^{js_1(\cdot)}\varphi_j e^{-(t-\tau)|\cdot|^2}[\widehat{(u\cdot\nabla)}u]\mathrm{d}\tau\right\|_{L^\rho(\mathbb{R}_+;M^{\lambda(\cdot)}_{p_1(\cdot)})}\right\}_{j\in\mathbb{Z}}\right\|_{l^q}\\
\lesssim&\left\|\left\|\int_0^t \|r^{-\frac{3}{\overline{p}}}2^{j(s_1(\cdot)+1)}\varphi_j e^{-(t-\tau)|\cdot|^2}\|_{L^{\overline{p}}}\|\Delta_j\widehat{(u\otimes u)}\|_{M_6^{\lambda(\cdot)}}\mathrm{d}\tau\right\|_{L^\rho(\mathbb{R}_+)}\right\|_{l^q}\\
\lesssim&\left\|\left\|\int_0^t 2^{j(\frac{5}{2}+\frac{2}{\rho})}\|r^{-\frac{3}{\overline{p}}}2^{-3j\frac{1}{\overline{p}}}\varphi_j e^{-(t-\tau)|\cdot|^2}\|_{L^{\overline{p}}}\|\Delta_j(u\otimes u)\|_{M^{\lambda(\cdot)}_{\frac{6}{5}}}\mathrm{d}\tau\right\|_{L^\rho(\mathbb{R}_+)}\right\|_{l^q}\\
\lesssim&\left\|\left\|\int_0^t 2^{j(\frac{5}{2}+\frac{2}{\rho})}e^{-(t-\tau)2^{2j}}\|r^{-\frac{3}{\overline{p}(\cdot)}}2^{-3j\frac{1}{\overline{p}(\cdot)}}\varphi_j\|_{L^{\overline{p}(\cdot)}}\|\Delta_j(u\otimes u)\|_{M^{\lambda(\cdot)}_{\frac{6}{5}}}\mathrm{d}\tau\right\|_{L^\rho(\mathbb{R}_+)}\right\|_{l^q}\\
\lesssim&\left\|\left\|2^{j(\frac{5}{2}+\frac{2}{\rho}-2)}\|\Delta_j(u\otimes u)\|_{M^{\lambda(\cdot)}_{\frac{6}{5}}}\right\|_{L^\rho(\mathbb{R}_+)}\|e^{-t2^{2j}}2^{2j}\|_{L^1(\mathbb{R}_+)}\right\|_{l^q}\\
\lesssim&\|u\|_{\widetilde{L}^\rho(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac{1}{2}+\frac{2}{\rho}}_{2,\lambda(\cdot),q})}\|u\|_{L^\infty(\mathbb{R}_+;L^3)}\\
\lesssim&\|u\|_{\widetilde{L}^\rho(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac{1}{2}+\frac{2}{\rho}}_{2,\lambda(\cdot),q})}\|u\|_{\widetilde{L}^\infty(\mathbb{R}_+;\dot{B}^{\frac{1}{2}}_{2,q})}\\
\lesssim&\|u\|_{\widetilde{L}^\rho(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac{1}{2}+\frac{2}{\rho}}_{2,\lambda(\cdot),q})}\|u\|_{\widetilde{L}^\infty(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac{1}{2}}_{2,2,q})}.
\end{aligned}
$$

相似地, 可得

$$
\begin{aligned}
&\left\|\int_0^t T_\Omega(t-\tau)\mathbb{P}g\theta e_3\mathrm{d}\tau\right\|_{\widetilde{L}^\rho(\mathbb{R}_+;\mathcal{F}\dot{\mathcal{N}}^{s_1(\cdot)}_{p_1(\cdot),\lambda(\cdot),q})}\lesssim\|\theta\|_{\widetilde{L}^\rho(\mathbb{R}_+;\dot{N}^{\frac{2}{\rho}+\frac{1}{2}}_{2,\lambda(\cdot),q})},\\
&\left\|\int_0^t e^{(t-\tau)\Delta}[(u\cdot\nabla)\theta]\mathrm{d}\tau\right\|_{\widetilde{L}^\rho(\mathbb{R}_+;\mathcal{F}\dot{\mathcal{N}}^{s_1(\cdot)}_{p_1(\cdot),\lambda(\cdot),q})}\lesssim\|u\|_{\widetilde{L}^\rho(\mathbb{R}_+;\dot{N}^{\frac{2}{\rho}+\frac{1}{2}}_{2,\lambda(\cdot),q})}\|\theta\|_{\widetilde{L}^\infty(\mathbb{R}_+;\dot{N}^{\frac{1}{2}}_{2,2,q})}.
\end{aligned}
$$

另外, 通过上面的方法以及 Plancherel 定理, 有

$$
\begin{aligned}
&\left\|\int_0^t T_\Omega(t-\tau)\mathbb{P}[(u\cdot\nabla)u]\mathrm{d}\tau\right\|_{\widetilde{L}^\rho(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,\lambda(\cdot),q})\cap\widetilde{L}^\infty(0,\infty;\dot{\mathcal{N}}^{\frac{1}{2}}_{2,2,q})}\\
=&\left\|\int_0^t T_\Omega(t-\tau)\mathbb{P}[(u\cdot\nabla)u]\mathrm{d}\tau\right\|_{\widetilde{L}^\rho(\mathbb{R}_+;\mathcal{F}\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,\lambda(\cdot),q})\cap\widetilde{L}^\infty(\mathbb{R}_+;\mathcal{F}\dot{\mathcal{N}}^{\frac{1}{2}}_{2,2,q})}\\
\lesssim&\|u\|_{\widetilde{L}^\rho(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,\lambda(\cdot),q})\cap\widetilde{L}^\infty(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac{1}{2}}_{2,2,q})}\|u\|_{\widetilde{L}^\infty(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac{1}{2}}_{2,2,q})},
\end{aligned}
$$

$$
\begin{aligned}
&\left\|\int_0^t T_\Omega(t-\tau)\mathbb{P}g\theta e_3\mathrm{d}\tau\right\|_{\widetilde{L}^\rho(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,\lambda(\cdot),q})\cap\widetilde{L}^\infty(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac{1}{2}}_{2,2,q})}\\
\lesssim&\|\theta\|_{\widetilde{L}^\rho(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,\lambda(\cdot),q})\cap\widetilde{L}^\infty(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac{1}{2}}_{2,2,q})},\\
&\left\|\int_0^t e^{(t-\tau)\Delta}[(u\cdot\Delta)\theta]\mathrm{d}\tau\right\|_{\widetilde{L}^\rho(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,\lambda(\cdot),q})\cap\widetilde{L}^\infty(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac{1}{2}}_{2,2,q})}\\
\lesssim&\|u\|_{\widetilde{L}^\rho(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,\lambda(\cdot),q})\cap\widetilde{L}^\infty(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac{1}{2}}_{2,2,q})}\|\theta\|_{\widetilde{L}^\infty(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac{1}{2}}_{2,2,q})}.
\end{aligned}
$$

最后, 证明方程 (1) 存在性和唯一性

$$\begin{aligned}&\|\mathfrak{M}(u,\theta)\|_X\\&=\|\mathfrak{M}(u)\|_X+\|\mathfrak{M}(\theta)\|_X\\&\lesssim\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),\lambda(\cdot),q}}+\|\theta_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),\lambda(\cdot),q}}+\|u\|_{\widetilde{L}^\rho(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac12}_{2,\lambda(\cdot),q})\bigcap\widetilde{L}^\infty(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac12}_{2,2,q})}\|u\|_{\widetilde{L}^\infty(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac12}_{2,2,q})}\\&\quad+\|u\|_{\widetilde{L}^\rho(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac12}_{2,\lambda(\cdot),q})\bigcap\widetilde{L}^\infty(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac12}_{2,2,q})}\|\theta\|_{\widetilde{L}^\infty(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac12}_{2,2,q})}+\|\theta\|_{\widetilde{L}^\rho(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac12}_{2,\lambda(\cdot),q})\bigcap\widetilde{L}^\infty(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac12}_{2,2,q})}.\end{aligned}$$

令 $\delta=L=2\left(\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),\lambda(\cdot),q}}+\|\theta_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),\lambda(\cdot),q}}\right)<2C_\epsilon$, 如果 ϵ 足够小, 有

$$\|\mathfrak{M}(u,\theta)\|_X\le\frac{\delta}{2}+\frac{\delta}{2}=\delta.$$

相似地，有

$$d(\mathfrak{M}(u,\theta),\mathfrak{M}(u_1,\theta_1))\le\frac12 d((u,\theta),(u_1,\theta_1)).$$

由 Banach 压缩映射原理可知, 当 ϵ 足够小时, 方程 (1) 存在唯一的整体解，且满足

$$(u,\theta)\in\widetilde{L}^\infty(\mathbb{R}_+;\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),\lambda(\cdot),q})\cap\widetilde{L}^\rho(\mathbb{R}_+;\mathcal{F}\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac12}_{2,\lambda(\cdot),q})\cap\widetilde{L}^\infty(\mathbb{R}_+;\mathcal{F}\dot{\mathcal{N}}^{\frac12}_{2,2,q}).$$

另一方面，令

$$\begin{aligned}Y=\widetilde{L}^\rho(\mathbb{R}_+;\mathcal{F}\dot{\mathcal{N}}^{s_1}_{p_1(\cdot),\lambda(\cdot),q})\cap\widetilde{L}^\rho(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac12}_{2,\lambda(\cdot),q})\\\cap\widetilde{L}^\infty(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac12}_{2,2,q})\cap\widetilde{L}^\infty(\mathbb{R}_+;\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),\lambda(\cdot),q}),\end{aligned}$$

有

$$\begin{aligned}&\|\mathfrak{M}(u,\theta)\|_Y\\&=\|\mathfrak{M}(u)\|_Y+\|\mathfrak{M}(\theta)\|_Y\\&\lesssim\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),\lambda(\cdot),q}}+\|\theta_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),\lambda(\cdot),q}}+\|u\|_{\widetilde{L}^\rho(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac12}_{2,\lambda(\cdot),q})\bigcap\widetilde{L}^\infty(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac12}_{2,2,q})}\|u\|_{\widetilde{L}^\infty(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac12}_{2,2,q})}\\&\quad+\|u\|_{\widetilde{L}^\rho(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac12}_{2,\lambda(\cdot),q})\bigcap\widetilde{L}^\infty(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac12}_{2,2,q})}\|\theta\|_{\widetilde{L}^\infty(\mathbb{R}_+;\dot{N}^{\frac12}_{2,2,q})}+\|\theta\|_{\widetilde{L}^\rho(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac{2}{\rho}+\frac12}_{2,\lambda(\cdot),q})\bigcap\widetilde{L}^\infty(\mathbb{R}_+;\dot{\mathcal{N}}^{\frac12}_{2,2,q})}.\end{aligned}$$

令 $\delta=L=2\left(\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),\lambda(\cdot),q}\bigcap\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p_1(\cdot)}}_{p_1(\cdot),\lambda(\cdot),q}}+\|\theta_0\|_{\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),\lambda(\cdot),q}\bigcap\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p_1(\cdot)}}_{p_1(\cdot),\lambda(\cdot),q}}\right)<2C_\epsilon$, 如果 ϵ 足够小, 则

$$\|\mathfrak{M}(u,\theta)\|_Y\le\frac{\delta}{2}+\frac{\delta}{2}=\delta.$$

相似地, 有

$$d(\mathfrak{M}(u,\theta),\mathfrak{M}(u_1,\theta_1))\le\frac12 d((u,\theta),(u_1,\theta_1)).$$

由 Banach 压缩映射原理可知, 当 ϵ 足够小时, 方程 (1) 存在唯一的全局解，且满足

$$(u,\theta)\in\widetilde{L}^\rho(\mathbb{R}_+;\mathcal{F}\dot{\mathcal{N}}^{s_1(\cdot)}_{p_1(\cdot),\lambda(\cdot),q})\cap\widetilde{L}^\infty(\mathbb{R}_+;\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),\lambda(\cdot),q}).$$

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