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Oscillation Criteria of Second Order Emden-Fowler Neutral Delay Differential Equations

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Abstract: By Riccati transformation and inequality technique, the oscillation of second order Emden-Fowler neutral delay differential equations is considered, some new interval oscillation criteria are established.

Keywords: Neutral Delay Differential Equations; Oscillation; Riccati Transformation

二阶 Emden-Fowler 中立型时滞微分方程振动准则*

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摘　要：借助广义 Riccati 变换和不等式技巧，研究了一类较一般形式的二阶 Emden-Fowler 中立型时滞微分方程的振动性，给出了该类微分方程区间振动准则。

关键词：中立型时滞微分方程；振动性；Riccati 变换

1. 引言

二阶 Emden-Fowler 中立型时滞微分方程在物理学和工程技术中有重要应用，引起许多学者的关注，如文[1-3]。Abdallah[1]研究了二阶时滞中立型微分方程

$$\left[a(t)\left(x(t)+p(t)x(t-\tau)\right)'\right]'+q(t)x^{\alpha}(t-\sigma)=0,\ t\ge t_0,$$

解的振动与非振动性,得到一些好的结果。Li[2]研究了二阶半线性泛函微分方程

$$\left[r(t)\left|y'(t)\right|^{\alpha-1}y'(t)\right]'+q(t)\left|y(\tau(t))\right|^{\alpha-1}y(\tau(t))=0,\ t\ge t_0$$

解的振动性,得到一些新的区间振动准则。Xu[3]研究了 Emden-Fowler 中立型时滞微分方程

$$\left[\left|x'(t)\right|^{\gamma-1}x'(t)\right]'+q_1(t)\left|y(t-\sigma)\right|^{\alpha-1}y(t-\sigma)+q_2(t)\left|y(t-\sigma)\right|^{\beta-1}y(t-\sigma)=0,\ t\ge t_0$$

解的振动性，得到 Philos 型振动准则，改进了已有文献的结果。Yang[4]研究了二阶非线性中立型微分方程 $\left(r(t)(x(t)+p(t)x(t-\tau))'\right)'+q(t)f\left(x(t-\delta)\right)=0$, $t\ge t_0$，解的区间振动性，改进和推广了已有文献的结果。王其如[5]研究了二阶非线性微分方程 $\left(p(t)y'(t)\right)'+f\left(t,y(t),y'(t)\right)=0$, $t\ge t_0$,的振动性，得到了该类方程区间振动的一些新的准则，推广和改进了已有文献的结果。

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受文献[3-5]启发，本文研究如下形式的二阶 Emden-Fowler 中立型时滞微分方程

$$\left[r(t)\left|x'(t)\right|^{\gamma-1}x'(t)\right]'+q_1(t)\left|y(\delta_1(t))\right|^{\alpha-1}y(\delta_1(t))+q_2(t)\left|y(\delta_2(t))\right|^{\beta-1}y(\delta_2(t))=0,\quad t\ge t_0 \tag{1.1}$$

的振动性，应用广义 Riccati 变换和不等式

$$Bx-Ax^{\frac{\gamma+1}{\gamma}}\le\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}}{A^{\gamma}},\quad A>0,$$

$$B\ge 0,\gamma>0,x\in R, \tag{1.2}$$

给出微分方程(1.1)解区间振动的几个新准则。这里 $x(t)=y(t)+p(t)y(\tau(t))$，$0\le p(t)<1$，α,β,γ 是正常数，$0<\alpha<\gamma<\beta$，$t\ge t_0>0$；$r,q_1,q_2\in C([t_0,\infty),R^+)$；$\tau,\delta_1,\delta_2\in C([t_0,\infty),R)$，$\tau(t)<t,\ \delta_1(t)<t,\ \delta_2(t)<t$，$\lim_{t\to\infty}\tau(t)=\infty$，

$$\lim_{t\to\infty}\delta_1(t)=\infty,\quad \lim_{t\to\infty}\delta_2(t)=\infty,\quad \int_{t_0}^{\infty}\left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}}\mathrm{d}t=\infty。$$

微分方程(1.1)的解 $y(t)$ 称为振动的，若 $y(t)$ 既不最终为正，也不最终为负，否则称为非振动的。微分方程(1.1)称为振动的，如果它们的所有解都是振动的。

2. 主要结果与证明

为方便起见，首先给出下列记号，令

$$\mu=\min\left\{\frac{\beta-\alpha}{\beta-\gamma},\frac{\beta-\alpha}{\gamma-\alpha}\right\},\delta(t)=\min_{t\ge t_0}\{\delta_1(t),\delta_2(t)\},$$

$$Q_1(t)=\mu\left[q_1(t)\left(1-p(\delta_1(t))\right)^{\alpha}\right]^{\frac{\beta-\gamma}{\beta-\alpha}}\left[q_2(t)\left(1-p(\delta_2(t))\right)^{\beta}\right]^{\frac{\gamma-\alpha}{\beta-\alpha}}\left(\frac{\delta(t)}{t}\right)^{\gamma},$$

$$D(a,b):=\left\{u\in C^1[a,b]:u(t)\ne 0,u(a)=u(b)=0\right\}。$$

为证明方便，先给出下面的引理，其证明方法类似于文献[4]中引理 1 的证明。

引理 2.1 若 $y(t)$ 是方程(1.1)的最终正解，则存在 $t_1\in[t_0,\infty)$，使得 $x(t)>0,\ x'(t)\ge 0,$

$$\left[r(t)\left(x'(t)\right)^{\gamma}\right]'\le -q_1(t)\left[\left(1-p(\delta_1(t))\right)x(\delta_1(t))\right]^{\alpha}-q_2(t)\left[\left(1-p(\delta_2(t))\right)x(\delta_2(t))\right]^{\beta}<0,\ t\in[t_1,\infty). \tag{2.1}$$

引理 2.3 假设 $r\in C^1([t_0,\infty),R^+)$，$r'(t)\ge 0$，而且

$$\int_{t_0}^{\infty}q_1(t)\left(\left(1-p(\delta_1(t))\right)\delta_1(t)\right)^{\alpha}\mathrm{d}t=\infty \tag{2.3}$$

或者

$$\int_{t_0}^{\infty}q_2(t)\left(\left(1-p(\delta_2(t))\right)\delta_2(t)\right)^{\beta}\mathrm{d}t=\infty \tag{2.4}$$

成立。若 $y(t)$ 是方程(1.1)的最终正解，则存在 $t_1\in[t_0,\infty)$，使得 $x''(t)<0,x(t)\ge tx'(t),\frac{x(t)}{t}$ 严格递减的，$t\in[t_1,\infty)$。

定理 2.1 设 $r\in C^1([t_0,\infty),R^+)$，$r'(t)\ge 0$，而且(2.3)或(2.4)成立。若对任意 $T\ge t_0$，存在 $b>a\ge T$，$u\in D(a,b)$，且存在 $\rho\in C^1([t_0,\infty),R^+)$，有

$$\int_a^b\rho(t)\left[Q_1(t)u^{\gamma+1}(t)-\frac{r(t)}{(\gamma+1)^{\gamma+1}}\left[(\gamma+1)u'(t)+u(t)\frac{\rho'(t)}{\rho(t)}\right]^{\gamma+1}\right]\mathrm{d}t>0, \tag{2.7}$$

则方程(1.1)振动。

证明 设 $y(t)$ 是方程(1.1)的一个非振动解，不妨设最终为正。由引理 2.1 和引理 2.3，存在 $t_1\in[t_0,\infty)$，当 $t\in[t_1,\infty)$ 时，有 $x(t)>0,x'(t)\ge 0,$ $x''(t)<0,x(t)\ge tx'(t),\frac{x(t)}{t}$ 严格递减的。作 Riccati 变换

$$w(t)=\rho(t)\frac{r(t)\left(x'(t)\right)^{\gamma}}{x^{\gamma}(t)}，\quad t\ge t_1，\text{则}$$

$$w'(t)=\rho'(t)\frac{r(t)\left(x'(t)\right)^{\gamma}}{\left(x(t)\right)^{\gamma}}+\rho(t)\left(\frac{r(t)\left(x'(t)\right)^{\gamma}}{x^{\gamma}(t)}\right)'$$

$$=\rho'(t)\frac{r(t)\left(x'(t)\right)^{\gamma}}{x^{\gamma}(t)}+\rho(t)\frac{\left[r(t)\left(x'(t)\right)^{\gamma}\right]'}{x^{\gamma}(t)}-\gamma\rho(t)\frac{r(t)\left(x'(t)\right)^{\gamma-1}\left(x'(t)\right)^{2}}{\left(x(t)\right)^{\gamma-1}\left(x(t)\right)^{2}}$$

$$\le\frac{\rho'(t)}{\rho(t)}w(t)-\rho(t)q_1(t)\left(1-p\left(\delta_1(t)\right)\right)^{\alpha}\frac{\left(x\left(\delta_1(t)\right)\right)^{\alpha}}{\left(x(t)\right)^{\gamma}}$$

$$-\rho(t)q_2(t)\left(1-p\left(\delta_2(t)\right)\right)^{\beta}\frac{\left(x\left(\delta_2(t)\right)\right)^{\beta}}{\left(x(t)\right)^{\gamma}}-\frac{\gamma}{\left(r(t)\rho(t)\right)^{\frac{1}{\gamma}}}\left(w(t)\right)^{\frac{\gamma+1}{\gamma}},$$

由不等式$|ab|\le\frac{1}{p}|a|^{p}+\frac{1}{q}|b|^{q},a,b\in R,p>1,q>1,\frac{1}{p}+\frac{1}{q}=1$，可得

$$\frac{\beta-\gamma}{\beta-\alpha}q_1(t)\left(1-p\left(\delta_1(t)\right)\right)^{\alpha}\frac{\left(x\left(\delta_1(t)\right)\right)^{\alpha}}{\left(x(t)\right)^{\gamma}}+\frac{\gamma-\alpha}{\beta-\alpha}q_2(t)\left(1-p\left(\delta_2(t)\right)\right)^{\beta}\frac{\left(x\left(\delta_2(t)\right)\right)^{\beta}}{\left(x(t)\right)^{\gamma}}$$

$$\ge\left[q_1(t)\left(1-p\left(\delta_1(t)\right)\right)^{\alpha}\frac{\left(x\left(\delta_1(t)\right)\right)^{\alpha}}{\left(x(t)\right)^{\gamma}}\right]^{\frac{\beta-\gamma}{\beta-\alpha}}\left[q_2(t)\left(1-p\left(\delta_2(t)\right)\right)^{\beta}\frac{\left(x\left(\delta_2(t)\right)\right)^{\beta}}{\left(x(t)\right)^{\gamma}}\right]^{\frac{\gamma-\alpha}{\beta-\alpha}}$$

$$=\left(q_1(t)\left(1-p\left(\delta_1(t)\right)\right)^{\alpha}\right)^{\frac{\beta-\gamma}{\beta-\alpha}}\left(q_2(t)\left(1-p\left(\delta_2(t)\right)\right)^{\beta}\right)^{\frac{\gamma-\alpha}{\beta-\alpha}}\left(\frac{\left(x\left(\delta_1(t)\right)\right)^{\alpha}}{\left(x(t)\right)^{\gamma}}\right)^{\frac{\beta-\gamma}{\beta-\alpha}}\left(\frac{\left(x\left(\delta_2(t)\right)\right)^{\beta}}{\left(x(t)\right)^{\gamma}}\right)^{\frac{\gamma-\alpha}{\beta-\alpha}}$$

$$\ge\left(q_1(t)\left(1-p\left(\delta_1(t)\right)\right)^{\alpha}\right)^{\frac{\beta-\gamma}{\beta-\alpha}}\left(q_2(t)\left(1-p\left(\delta_2(t)\right)\right)^{\beta}\right)^{\frac{\gamma-\alpha}{\beta-\alpha}}\left(\frac{x\left(\delta(t)\right)}{x(t)}\right)^{\gamma},$$

由于$\frac{x(t)}{t}$严格递减，所以$\frac{x\left(\delta(t)\right)}{x(t)}\ge\frac{\delta(t)}{t}$。因此

$$w'(t)\le\frac{\rho'(t)}{\rho(t)}w(t)-\rho(t)Q_1(t)-\frac{\gamma}{\left(r(t)\rho(t)\right)^{\frac{1}{\gamma}}}\left(w(t)\right)^{\frac{\gamma+1}{\gamma}},$$

因而有

$$\int_a^b u^{\gamma+1}(t)\rho(t)Q_1(t)\mathrm{d}t \le \int_a^b u^{\gamma+1}(t)\left\{-w'(t)+\frac{\rho'(t)}{\rho(t)}w(t)-\gamma\frac{(w(t))^{\frac{\gamma+1}{\gamma}}}{(r(t)\rho(t))^{\frac{1}{\gamma}}}\right\}\mathrm{d}t$$

$$=\int_a^b\left[\left[(\gamma+1)u^{\gamma}(t)u'(t)+u^{\gamma+1}(t)\frac{\rho'(t)}{\rho(t)}\right]w(t)-\gamma u^{\gamma+1}(t)\frac{(w(t))^{\frac{\gamma+1}{\gamma}}}{(r(t)\rho(t))^{\frac{1}{\gamma}}}\right]\mathrm{d}t$$

令 $A=\dfrac{\gamma u^{\gamma+1}(t)}{(r(t)\rho(t))^{\frac{1}{\gamma}}}, B=(\gamma+1)u^{\gamma}(t)u'(t)+u^{\gamma+1}(t)\dfrac{\rho'(t)}{\rho(t)}$，利用不等式(1.2)，得

$$\int_a^b\rho(t)\left[Q_1(t)u^{\gamma+1}(t)-\frac{r(t)}{(\gamma+1)^{\gamma+1}}\left[(\gamma+1)u'(t)+u(t)\frac{\rho'(t)}{\rho(t)}\right]^{\gamma+1}\right]\mathrm{d}t\le 0,$$

这与条件(2.7)矛盾，证毕。

特别地，取 $\rho(t)\equiv 1$ 时，可以得出下面的推论。

推论 2.1 设 $0\le p(t)<1$，$r\in C^1([t_0,\infty),R^+)$，$r'(t)\ge 0$，而且(2.3)或(2.4)成立。若对任意的 $T\ge t_0$，存在 $b>a\ge T$，$u\in D(a,b)$，有 $\int_a^b\left[Q_1(t)u^{\gamma+1}(t)-r(t)(u'(t))^{\gamma+1}\right]\mathrm{d}t>0$，则方程(1.1)振动。

与定理 2.1 和推论 2.1 类似，可以得出下面的定理 2.2 和推论 2.2。

3. 应用举例

考虑二阶 Emden-Fowler 中立型时滞微分方程

$$\left((x'(t))^3\right)'+ty(t-2)+t^2|y(t-3)|^3y(t-3)=0,$$

$$x(t)=y(t)+\frac{1}{2}y(t-1), t>15。$$

易验证满足推论 2.1 的条件。对任意的 $T\ge 0$，存在充分大的 k，考虑区间 $[a,b]=[k\pi,(k+1)\pi]$，取 $u(t)=\sin t$，由于

$$\int_{k\pi}^{k\pi+\pi}\left[Q_1(t)u^4(t)-r(t)(u'(t))^4\right]\mathrm{d}t>\frac{3}{2}\left(\frac{1}{2}\right)^3(15)^{\frac{5}{3}}\left(\frac{4}{5}\right)^3\int_{k\pi}^{k\pi+\pi}\left[\sin^4(t)-\cos^4(t)\right]\mathrm{d}t=0$$

由推论 2.1，得该方程振动。

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