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Three Positive Solutions for a Class of *m*-Point Boundary Value Problems with One-Dimensional *p*-Laplacian

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Abstract: Multi-point boundary value problems of nonlinear differential equations arise in a variety of areas of applied mathematics, physics and variational problems of control theory. With the development of the ordinary differential equations, multi-point boundary value problem is at present one of the most active fields. It has become a new important branch of mathematics. In this paper, we study the following *m*-point boundary value problem with *p*-Laplacian operator $\left(\phi_p\left(u'(t)\right)\right)' + q(t) f\left(t,u(t),u'(t)\right) = 0$, $0<t<1$, $u(0)=\sum_{i=1}^{m-2} a_i u(\xi_i)$, $u'(1)=\beta u'(0)$, where $m \ge 3$ is an integer, $a_i \in [0,1)$, $\xi_i \in (0,1)$ $(i=1,2,\cdots,m-2)$ are constants satisfying $0 \le \sum_{i=1}^{m-2} a_i < 1$ and $0<\xi_1<\xi_2<\cdots<\xi_{m-2}<1$. By using Leggett-Williams fixed point theorem, some sufficient conditions for the existence of three positive solutions are obtained. Some recent results are generalized and improved. An example is also given to illustrate the importance of the results obtained.

Keywords: *m*-Point Boundary Value Problem; Leggett-Williams Fixed Point Theorem; Cone; *p*-Laplacian Operator

一类具 *p*-Laplacian 算子的 *m* 点边值问题的三个正解

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摘 要：非线性边微分方程多点边值问题源于应用数学、物理学和控制论中的变分问题等许多领域。随着常微分方程理论的发展，多点边值问题的研究日益活跃，已逐渐成为一个新的重要数学分支。本文研究下列具 *p*-Laplacian 算子 *m* 点边值问题 $\left(\phi_p\left(u'(t)\right)\right)' + q(t) f\left(t,u(t),u'(t)\right) = 0$，$0<t<1$，$u(0)=\sum_{i=1}^{m-2} a_i u(\xi_i)$，$u'(1)=\beta u'(0)$，其中 $m \ge 3$ 为整数，$a_i \in [0,1)$，$\xi_i \in (0,1)$ $(i=1,2,\cdots,m-2)$ 为常数，满足 $0 \le \sum_{i=1}^{m-2} a_i < 1$，$0<\xi_1<\xi_2<\cdots<\xi_{m-2}<1$. 应用 Leggett-Williams 不动点定理，得到了其至少存在三个正解的充分条件，推广和改进了现有文献的一些结果。最后，给出了一个具体的例子说明所得结果的重要性。

关键词：*m* 点边值问题；Leggett-Williams 不动点定理；锥；*p*-Laplacian 算子

1. 引言

常微分方程的多点边值问题出现在应用数学和应用物理的许多领域中。例如，在数学领域，它们出现在用度量分离法解偏微分方程以求解一维自由边值问

题；在物理领域，由N部分不同密度组成的均匀截面的悬链线的振动可以转化为多点边值问题；在弹性稳定性理论中，也有许多问题可以归结为多点边值问题处理。1987年Il'in和Moiseev[1]首先对二阶线性常微分方程的多点边值问题进行了研究。此后，许多作者运用多种方法研究了非线性多点边值问题，可参考文献[2-11]。

文献[8]研究了以下m点边值问题

$$x''(t)-q(t)f(x,x')x'=0,\ 0<t<1,$$

$$u(0)=\sum_{i=1}^{m-2}b_iu(\xi_i),\quad u'(1)=\alpha u'(0),$$

其中 $\xi_i\in(0,1)$，并且 $0<\xi_1<\xi_2<\cdots<\xi_{m-2}<1$，$b_i\in[0,1)$，$\alpha>1$，该文采用了锥上的不动点定理给出了边值问题至少存在一个正解的条件。

最近，文献[4]研究了一类具p-Laplacian算子的非线性项带导数的三点边值问题

$$\left(\phi_p\left(u'(t)\right)\right)'+a(t)f\left(t,u(t),u'(t)\right)=0,\ 0<t<1,$$

$$u'(0)=0,\qquad u(1)=\alpha u(\eta),$$

作者利用Leggett-Williams不动点定理得到了边值问题三个正解存在的充分条件。

受以上文章的启发，本文研究下列具p-Laplacian算子的m点边值问题

$$\left(\phi_p\left(u'(t)\right)\right)'+q(t)f\left(t,u(t),u'(t)\right)=0,\ 0<t<1, \tag{1.1}$$

$$u(0)=\sum_{i=1}^{m-2}a_iu(\xi_i),\quad u'(1)=\beta u'(0), \tag{1.2}$$

其中 $\phi_p(s)=|s|^{p-2}s$，$p>1$，$\xi_i\in(0,1)$，并且 $0<\xi_1<\xi_2<\cdots<\xi_{m-2}<1$。假设

(H_1) $\beta\in[0,1)$，$a_i\in[0,1)$，且 $0\le\sum_{i=1}^{m-2}a_i<1$；

(H_2) $f\in C\left([0,1]\times[0,+\infty)\times R,(0,+\infty)\right)$；

(H_3) $q\in L^1[0,1]$，$q\ge0$，$t\in(0,1)$，且 q 在 $(0,1)$ 的任何子区间内不恒等于 0，$0<\int_0^1 q(t)\mathrm{d}t<\infty$.

近来，许多学者应用 Leggett-Williams 不动点定理，证明了一些非线性项中不含有导数项的二阶或高阶常微分方程边值问题三个正解的存在性。但对于非线性项包含导数的边值问题的研究结果并不多见(例如[2,3,7,11])，用此方法讨论边值问题(1.1)，(1.2)正解的存在性，迄今尚未见到任何结果。

2. 预备知识和引理

首先, 引入如下几个定义及引理。

定义2.1 若映射 α 满足：$\alpha:P\to[0,\infty)$ 连续且 $\alpha\left(tx+(1-t)y\right)\ge t\alpha(x)+(1-t)\alpha(y)$，$\forall x,y\in P$，$0\le t\le1$，则称 $\alpha(x)$ 为非负连续凹泛函。

定义2.2 设常数 $0<a<b$，α 是定义在锥 P 上的连续非负凹泛函，定义

$$P_a=\left\{u\in P\,\middle|\,\|u\|<a\right\},$$

$$P(\alpha,a,b)=\left\{u\in P\,\middle|\,a\le\alpha(u),\|u\|\le b\right\}.$$

令 $E=C^1[0,1]$，在范数 $\|u\|=\max\left\{\|u\|_0,\|u'\|_0\right\}$，$\|u\|_0=\max_{t\in[0,1]}|u(t)|$ 下是一个Banach空间，定义 P 是 E 中的一个锥 $P=\left\{u\in E\,\middle|\,u(0)=\sum_{i=1}^{m-2}a_iu(\xi_i),\ u(t)\text{ 在 }[0,1]\text{ 上非负，单调非减，凹的}\right\}$. 定义非负连续凹泛函 $\alpha:P\to[0,\infty)$为$\alpha(u)=\min_{\eta\le t\le1-\eta}|u(t)|$，$\eta\in\left(0,\min\left\{\frac{1}{2},\xi_1\right\}\right)$.

下面我们叙述Leggett-Williams不动点定理，这是本文主要结果的理论依据。

引理2.1[12] 设$T:\overline{P}_c\to\overline{P}_c$ 是全连续算子，α 是 P 上的非负连续凹泛函，且对任意满足 $\alpha(u)\le\|u\|$，$\forall u\in\overline{P}_c$。假设存在 a，b，d 满足 $0<a<b<d\le c$，

(S_1) $\left\{u\in P(\alpha,b,d)\,\middle|\,\alpha(u)>b\right\}\ne\phi$，且对 $u\in P(\alpha,b,d)$，有 $\alpha(Tu)>b$；

(S_2) 对 $u\in\overline{P_a}$，有 $\|Tu\|<a$；

(S_3) 对 $u\in P(\alpha,b,c)$，且 $\|Tu\|>d$，有 $\alpha(Tu)>b$.

则 T 至少存在 3 个不动点 u_1，u_2 和 u_3，满足

$\|u_1\|<a$，$b<\alpha(u_2)$，$a<\|u_3\|$，且 $\alpha(u_3)<b$.

引理2.2 假设 $(H_1)-(H_3)$ 成立，对 $x\in C^{1+}[0,1]=:\left\{x\in C^1[0,1]:x(t)\ge0\right\}$，边值问题

$$\left(\phi_p\left(u'(t)\right)\right)'+q(t)f\left(t,x(t),x'(t)\right)=0,\ 0<t<1, \tag{2.1}$$

$$u(0)=\sum_{i=1}^{m-2}a_iu(\xi_i),\quad u'(1)=\beta u'(0), \tag{2.2}$$

有唯一解

$$u(t)=\int_0^t\phi_p^{-1}\left(\int_s^1 q(\tau)f(\tau,x(\tau),x'(\tau))\mathrm{d}\tau+\frac{\phi_p(\beta)}{1-\phi_p(\beta)}\int_0^1 q(\tau)f(\tau,x(\tau),x'(\tau))\mathrm{d}\tau\right)\mathrm{d}s+\frac{1}{1-\sum_{i=1}^{m-2}a_i}\cdot\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\phi_p^{-1}\left(\int_s^1 q(\tau)f(\tau,x(\tau),x'(\tau))\mathrm{d}\tau+\frac{\phi_p(\beta)}{1-\phi_p(\beta)}\int_0^1 q(\tau)f(\tau,x(\tau),x'(\tau))\mathrm{d}\tau\right)\mathrm{d}s.$$

证明 由式(2.1)有

$$\left(\phi_p\left(u'(t)\right)\right)'=-q(t)f\left(t,x(t),x'(t)\right),$$

对 $t\in[0,1]$，从 0 到 t 积分，得

$$u'(t)=\phi_p^{-1}\left(\phi_p\left(u'(0)\right)-\int_0^t q(\tau)f\left(\tau,x(\tau),x'(\tau)\right)\mathrm{d}\tau\right),$$

对 $t\in[0,1]$，再从 0 到 t 积分，得

$$u(t)=u(0)+\left(\int_0^t\phi_p^{-1}\left(\phi_p\left(u'(0)\right)-\int_0^s q(\tau)f\left(\tau,x(\tau),x'(\tau)\right)\mathrm{d}\tau\right)\mathrm{d}s,\right.$$

由式(2.2)得出唯一解为

$$u(t)=\int_0^t\phi_p^{-1}\left(\int_s^1 q(\tau)f(\tau,x(\tau),x'(\tau))\mathrm{d}\tau+\frac{\phi_p(\beta)}{1-\phi_p(\beta)}\int_0^1 q(\tau)f(\tau,x(\tau),x'(\tau))\mathrm{d}\tau\right)\mathrm{d}s+\frac{1}{1-\sum_{i=1}^{m-2}a_i}\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\phi_p^{-1}\left(\int_s^1 q(\tau)f(\tau,x(\tau),x'(\tau))\mathrm{d}\tau+\frac{\phi_p(\beta)}{1-\phi_p(\beta)}\int_0^1 q(\tau)f(\tau,x(\tau),x'(\tau))\mathrm{d}\tau\right)\mathrm{d}s.$$

引理2.3 假设 $(H_1)-(H_3)$ 成立，对 $x\in C^{1+}[0,1]$，边值问题(2.1)和(2.2)的唯一解 $u(t)$ 是凹的，并且 $u'(t)\ge 0$，$u(t)\ge 0$，其中 $t\in[0,1]$.

证明 由(H_2)，(H_3)，易知 $\left(\phi_p(u')\right)'(t)=-q(t)f\left(t,x(t),x'(t)\right)\le 0$，所以 $\left(\phi_p(u')\right)$ 是单调非增的，从而 $u'(t)$ 也是非增的，故 $u(t)$ 是凹的。由 $u(t)$ 的凹性和边界条件 $u'(1)=\beta u'(0)$，可知 $u'(t)\ge 0$，$t\in[0,1]$. 那么 $u(\xi_i)\ge u(0)$，$i=1,2,\cdots,m-2$，因此

$$u(0)=\sum_{i=1}^{m-2}a_iu(\xi_i)\ge\sum_{i=1}^{m-2}a_iu(0),$$

由 $1-\sum_{i=1}^{m-2}a_i>0$，可知 $u(0)\ge 0$，则 $u(t)\ge 0$，$t\in[0,1]$.

引理 2.4 设 $u\in P$，则 $\max_{0\le t\le 1}|u(t)|\le\overline{M}\max_{0\le t\le 1}|u'(t)|$，其中 $\overline{M}=1+\frac{\sum_{i=1}^{m-2}a_i\xi_i}{1-\sum_{i=1}^{m-2}a_i}$.

证明 由 $u\in P$，根据 u 是凹的并且 $u'(t)\ge 0$，可知

$$u(1)-u(0)\le u'(0)=\max_{0\le t\le 1}|u'(t)|.$$

另一方面，由微分学中值定理可知

$$\begin{aligned}\left(1-\sum_{i=1}^{m-2}a_i\right)u(0)&=u(0)-\sum_{i=1}^{m-2}a_iu(0)\\&=\sum_{i=1}^{m-2}a_iu(\xi_i)-\sum_{i=1}^{m-2}a_iu(0)\\&=\sum_{i=1}^{m-2}a_i\left(u(\xi_i)-u(0)\right)\\&=\sum_{i=1}^{m-2}a_i\xi_iu'(\eta_i),\end{aligned}$$

其中 $\eta_i\in(0,\xi_i)$，因此

$$u(0)=\frac{\sum_{i=1}^{m-2}a_i\xi_iu'(\eta_i)}{1-\sum_{i=1}^{m-2}a_i}\le\frac{\sum_{i=1}^{m-2}a_i\xi_i}{1-\sum_{i=1}^{m-2}a_i}\max_{0\le t\le 1}|u'(t)|,$$

由此可得

$$\max_{0\le t\le 1}|u(t)|=u(1)\le\left(1+\frac{\sum_{i=1}^{m-2}a_i\xi_i}{1-\sum_{i=1}^{m-2}a_i}\right)u'(0)=\overline{M}\max_{0\le t\le 1}|u'(t)|.$$

注：由引理 2.4 可知，$\alpha(u)\le\|u\|$，$\|u\|\le\overline{M}\|u'\|_0$.

引理 2.5 假设 $(H_1)-(H_3)$ 成立，定义算子 T：

$$
\begin{aligned}
(Tu)(t)=\int_0^t\phi_p^{-1}\Bigg(&\int_s^1 q(\tau)\,f\big(\tau,u(\tau),u'(\tau)\big)\mathrm{d}\tau\\
&+\frac{\phi_p(\beta)}{1-\phi_p(\beta)}\int_0^1(\tau)\,f\big(\tau,u(\tau),u'(\tau)\big)\mathrm{d}\tau\Bigg)\mathrm{d}s\\
&+\frac{1}{1-\sum_{i=1}^{m-2}a_i}\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\phi_p^{-1}\Bigg(\int_s^1(\tau)\,f\big(\tau,u(\tau),u'(\tau)\big)\mathrm{d}\tau\\
&+\frac{\phi_p(\beta)}{1-\phi_p(\beta)}\int_0^1(\tau)\,f\big(\tau,u(\tau),u'(\tau)\big)\mathrm{d}\tau\Bigg)\mathrm{d}s,
\end{aligned}
$$

则算子 $T:P\to P$ 是全连续的。

证明 根据算子 T 的定义及引理 2.3，容易证明 $T(P)\subset P$。类似于文献[6,10]的证明，易证算子 $T:P\to P$ 是全连续的。

3. 主要结果

为了方便，引用下面记号：

$$
K=\overline{M}\phi_p^{-1}\left(\frac{1}{1-\phi_p(\beta)}\int_0^1 q(\tau)\mathrm{d}\tau\right),
$$

$$
\begin{aligned}
L=\eta\phi_p^{-1}&\left(\frac{1}{1-\phi_p(\beta)}\int_\eta^1 q(\tau)\mathrm{d}\tau\right)\\
&+\frac{1}{1-\sum_{i=1}^{m-2}a_i}\sum_{i=1}^{m-2}a_i\xi_i\phi_p^{-1}\\
&\cdot\left(\int_{\xi_i}^1 q(\tau)\mathrm{d}\tau+\frac{\phi_p(\beta)}{1-\phi_p(\beta)}\int_\eta^1 q(\tau)\mathrm{d}\tau\right),
\end{aligned}
$$

$$
M=\max\left\{\frac{1-\sum_{i=1}^{m-2}a_i(1-\xi_i)}{\eta+\sum_{i=1}^{m-2}a_i(\xi_i-\eta)},\overline{M}\right\}.
$$

以下我们将应用 Leggett-Williams 不动点定理给出边值问题(1.1)，(1.2)三个正解的存在性。

定理 3.1 设 $(H_1)-(H_3)$ 成立，存在常数 $0<a<b<d/M<d\le c$，且满足以下条件：

(H4) $f(t,u,v)\le\phi_p\left(\dfrac{c}{K}\right)$,

$(t,u,v)\in[0,1]\times[0,c]\times[0,c]$；

(H5) $f(t,u,v)\phi_p\left(\dfrac{a}{K}\right)$,

$(t,u,v)\in[0,1]\times[0,a]\times[0,a]$；

(H6) $f(t,u,v)\ge\phi_p\left(\dfrac{b}{L}\right)$,

$(t,u,v)\in[\eta,1]\times[b,d]\times[0,d]$；

(H7)

$$
\min_{(t,u,v)\in[\eta,1]\times[b,c]\times[0,c]}f(t,u,v)\phi_p(L)\ge\max_{(t,u,v)\in[0,1]\times[0,c]\times[0,c]}f(t,u,v)\left(\frac{1}{1-\phi_p(\beta)}\int_0^1 q(\tau)\mathrm{d}\tau\right),
$$

则边值问题(1.1)，(1.2)至少存在三个正解 u_1，u_2 和 u_3，使得

$$
\|u_1\|<a,\quad b<\alpha(u_2),\quad a<\|u_3\|,\quad \alpha(u_3)<b.
$$

证明 我们分三步完成证明。

第一步，证明 $T\overline{P}_c\subset\overline{P}_c$，$T\overline{P}_a\subset P_a$.

由引理 2.5 可知 $T\overline{P}_c\subset P$. $\forall u\in\overline{P}_c$，如果 $0\le t\le 1$，有 $0\le u(t)\le c$，$0\le u'(t)\le c$，由(H_4)得

$$
\begin{aligned}
\|Tu\|_0&=\max_{0\le t\le 1}\left|(Tu)(t)\right|=(Tu)(1)\\
&=\int_0^1\phi_p^{-1}\Bigg(\int_s^1 q(\tau)\,f\big(\tau,u(\tau),u'(\tau)\big)\mathrm{d}\tau\\
&\quad+\frac{\phi_p(\beta)}{1-\phi_p(\beta)}\int_0^1 q(\tau)\,f\big(\tau,u(\tau),u'(\tau)\big)\mathrm{d}\tau\Bigg)\mathrm{d}s\\
&\quad+\frac{1}{1-\sum_{i=1}^{m-2}a_i}\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\phi_p^{-1}\Bigg(\int_s^1 q(\tau)\,f\big(\tau,u(\tau),u'(\tau)\big)\mathrm{d}\tau\\
&\quad+\frac{\phi_p(\beta)}{1-\phi_p(\beta)}\int_0^1 q(\tau)\,f\big(\tau,u(\tau),u'(\tau)\big)\mathrm{d}\tau\Bigg)\mathrm{d}s\\
&\le\int_0^1\phi_p^{-1}\Bigg(\int_0^1 q(\tau)\,f\big(\tau,u(\tau),u'(\tau)\big)\mathrm{d}\tau\\
&\quad+\frac{\phi_p(\beta)}{1-\phi_p(\beta)}\int_0^1 q(\tau)\,f\big(\tau,u(\tau),u'(\tau)\big)\mathrm{d}\tau\Bigg)\mathrm{d}s\\
&\quad+\frac{1}{1-\sum_{i=1}^{m-2}a_i}\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\phi_p^{-1}\Bigg(\int_0^1 q(\tau)\,f\big(\tau,u(\tau),u'(\tau)\big)\mathrm{d}\tau\\
&\quad+\frac{\phi_p(\beta)}{1-\phi_p(\beta)}\int_0^1 q(\tau)\,f\big(\tau,u(\tau),u'(\tau)\big)\mathrm{d}\tau\Bigg)\mathrm{d}s\\
&=\overline{M}\phi_p^{-1}\left(\frac{1}{1-\phi_p(\beta)}\int_0^1 q(\tau)\,f\big(\tau,u(\tau),u'(\tau)\big)\mathrm{d}\tau\right)\\
&\le\overline{M}\phi_p^{-1}\left(\frac{1}{1-\phi_p(\beta)}\int_0^1 q(\tau)\mathrm{d}\tau\,\phi_p\left(\frac{c}{K}\right)\right)=\frac{c}{K}K=c,
\end{aligned}
$$

$$\left\|(Tu)'\right\|_0=\max_{0\le t\le 1}\left|(Tu)'(t)\right|=(Tu)'(0)=\phi_p^{-1}\left(\int_0^1 q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau+\frac{\phi_p(\beta)}{1-\phi_p(\beta)}\int_0^1 q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)$$
$$=\phi_p^{-1}\left(\frac{1}{1-\phi_p(\beta)}\int_0^1 q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\le\phi_p^{-1}\left(\frac{1}{1-\phi_p(\beta)}\int_0^1 q(\tau)\mathrm{d}\tau\,\phi_p\left(\frac{c}{K}\right)\right)=\frac{c}{K}\frac{K}{\overline{\overline{M}}}=\frac{c}{\overline{\overline{M}}}\le c.$$

因此，$\|Tu\|\le c$，即 $T\overline{P}_c\subset\overline{P}_c$. 同理，由$(H_5)$可以证明对 $u\in\overline{P_a}$，有 $\|Tu\|<a$.

第二步，证明 $\{u\in P(\alpha,b,d)\,|\,\alpha(u)>b\}\ne\phi$，且对 $u\in P(\alpha,b,d)$，有 $\alpha(Tu)>b$. 取

$$u_0(t)=\frac{1}{2}\left[\frac{b}{\eta+\sum_{i=1}^{m-2}a_i(\xi_i-\eta)}+\frac{d}{1-\sum_{i=1}^{m-2}a_i(1-\xi_i)}\right]\left[\left(1-\sum_{i=1}^{m-2}a_i\right)t+\sum_{i=1}^{m-2}a_i\xi_i\right],$$

易验证 $u_0(t)\in P$，且

$$\alpha(u_0)=\min_{\eta\le t\le 1-\eta}|u_0(t)|=u_0(\eta)=\frac{1}{2}\left[\frac{b}{\eta+\sum_{i=1}^{m-2}a_i(\xi_i-\eta)}+\frac{d}{1-\sum_{i=1}^{m-2}a_i(1-\xi_i)}\right]\left[\eta+\sum_{i=1}^{m-2}a_i(\xi_i-\eta)\right]$$
$$>\frac{1}{2}\left[\frac{b}{\eta+\sum_{i=1}^{m-2}a_i(\xi_i-\eta)}+\frac{Mb}{1-\sum_{i=1}^{m-2}a_i(1-\xi_i)}\right]\left[\eta+\sum_{i=1}^{m-2}a_i(\xi_i-\eta)\right]$$
$$\ge\frac{b}{2}+\frac{\eta+\sum_{i=1}^{m-2}a_i(\xi_i-\eta)}{1-\sum_{i=1}^{m-2}a_i(1-\xi_i)}\frac{1-\sum_{i=1}^{m-2}a_i(1-\xi_i)}{\eta+\sum_{i=1}^{m-2}a_i(\xi_i-\eta)}\frac{b}{2}=b,$$

$$\|u_0\|_0=\max_{0\le t\le 1}|u_0(t)|=u_0(1)=\frac{1}{2}\left[\frac{b}{\eta+\sum_{i=1}^{m-2}a_i(\xi_i-\eta)}+\frac{d}{1-\sum_{i=1}^{m-2}a_i(1-\xi_i)}\right]\left[1-\sum_{i=1}^{m-2}a_i(1-\xi_i)\right]$$
$$<\frac{1}{2}\left[\frac{d}{M\eta+M\sum_{i=1}^{m-2}a_i(\xi_i-\eta)}+\frac{d}{1-\sum_{i=1}^{m-2}a_i(1-\xi_i)}\right]\left[1-\sum_{i=1}^{m-2}a_i(1-\xi_i)\right]$$
$$\le\frac{1-\sum_{i=1}^{m-2}a_i(1-\xi_i)}{\eta+\sum_{i=1}^{m-2}a_i(\xi_i-\eta)}\frac{\eta+\sum_{i=1}^{m-2}a_i(\xi_i-\eta)}{1-\sum_{i=1}^{m-2}a_i(1-\xi_i)}\frac{d}{2}+\frac{d}{2}=d,$$

$$\|u_0'\|_0=\max_{0\le t\le 1}|u_0'(t)|=u_0'(0)=\frac{1}{2}\left[\frac{b}{\eta+\sum_{i=1}^{m-2}a_i(\xi_i-\eta)}+\frac{d}{1-\sum_{i=1}^{m-2}a_i(1-\xi_i)}\right]\left(1-\sum_{i=1}^{m-2}a_i\right)\le\|u_0\|_0<d,$$

因此$\left\{u_0\in P(\alpha,b,d)\,\middle|\,\alpha(u)>b\right\}\neq\phi$.

当$u\in P(\alpha,b,d)$，如果$\eta\le t\le 1$，有$b\le u(t)\le d$，$0\le u'(t)\le d$，由(H_6)得

$$\begin{aligned}\alpha(Tu)&=\min_{\eta\le t\le 1-\eta}\left|(Tu)(t)\right|=(Tu)(\eta)=\int_0^\eta\phi_p^{-1}\left(\int_s^1 q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau+\frac{\phi_p(\beta)}{1-\phi_p(\beta)}\int_0^1 q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s\\&+\frac{1}{1-\sum_{i=1}^{m-2}a_i}\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\phi_p^{-1}\left(\int_s^1 q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\mathrm{d}\tau+\frac{\phi_p(\beta)}{1-\phi_p(\beta)}\int_0^1 q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s\\&>\int_0^\eta\phi_p^{-1}\left(\int_\eta^1 q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau+\frac{\phi_p(\beta)}{1-\phi_p(\beta)}\int_\eta^1 q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s\\&+\frac{1}{1-\sum_{i=1}^{m-2}a_i}\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\phi_p^{-1}\left(\int_{\xi_i}^1 q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau+\frac{\phi_p(\beta)}{1-\phi_p(\beta)}\int_\eta^1 q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s\\&=\eta\phi_p^{-1}\left(\frac{1}{1-\phi_p(\beta)}\int_\eta^1 q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)+\frac{1}{1-\sum_{i=1}^{m-2}a_i}\sum_{i=1}^{m-2}a_i\xi_i\phi_p^{-1}\left(\int_{\xi_i}^1 q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right.\\&\left.+\frac{\phi_p(\beta)}{1-\phi_p(\beta)}\int_\eta^1 q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\ge\eta\phi_p^{-1}\left(\frac{1}{1-\phi_p(\beta)}\int_\eta^1 q(\tau)\mathrm{d}\tau\phi_p\left(\frac{b}{L}\right)\right)\\&+\frac{1}{1-\sum_{i=1}^{m-2}a_i}\sum_{i=1}^{m-2}a_i\xi_i\phi_p^{-1}\left(\int_{\xi_i}^1 q(\tau)\mathrm{d}\tau\phi_p\left(\frac{b}{L}\right)+\frac{\phi_p(\beta)}{1-\phi_p(\beta)}\int_\eta^1 q(\tau)\mathrm{d}\tau\phi_p\left(\frac{b}{L}\right)\right)\\&=\frac{b}{L}L=b,\end{aligned}$$

因此，当$u\in P(\alpha,b,d)$时，$\alpha(Tu)>b$.

第三步，证明对$u\in P(\alpha,b,d)$，且$\|Tu\|>d$，有$\alpha(Tu)>b$. 根据(H_7)和引理 2.4 之注，类似第二步，可得

$$\begin{aligned}\alpha(Tu)&=\min_{\eta\le t\le 1-\eta}\left|(Tu)(t)\right|=(Tu)(\eta)>\eta\phi_p^{-1}\left(\frac{1}{1-\phi_p(\beta)}\int_\eta^1 q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\\&+\frac{1}{1-\sum_{i=1}^{m-2}a_i}\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\phi_p^{-1}\left(\int_{\xi_i}^1 q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau+\frac{\phi_p(\beta)}{1-\phi_p(\beta)}\int_\eta^1 q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s\\&\ge\phi_p^{-1}\left(\min_{(t,u,v)\in[\eta,1]\times[b,c]\times[0,c]}f(t,u,v)\right)L=\phi_p^{-1}\left(\min_{(t,u,v)\in[\eta,1]\times[b,c]\times[0,c]}f(t,u,v)\phi_p(L)\right)\\&\ge\phi_p^{-1}\left(\max_{(t,u,v)\in[0,1]\times[0,c]\times[0,c]}f(t,u,v)\left(\frac{1}{1-\phi_p(\beta)}\int_0^1 q(\tau)\mathrm{d}\tau\right)\right)\\&\ge\phi_p^{-1}\left(\frac{1}{1-\phi_p(\beta)}\int_0^1 q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\\&=(Tu)'(0)\ge\frac{1}{\overline{M}}\|Tu\|>\frac{d}{\overline{M}}\ge\frac{d}{M}>b,\end{aligned}$$

引理 2.1 的所有条件均满足，所以边值问题(1.1)，(1.2)至少存在三个正解u_1，u_2和u_3，使得

$\|u_1\|<a$，$b<\alpha(u_2)$，$a<\|u_3\|$，$\alpha(u_3)<b$

定理3.1证明完毕。

4. 应用举例

本部分将给出一个例子来说明我们定理的可行性。

取 $p=3$，$q(t)=1$，$m=4$，$\beta=\frac{\sqrt{15}}{4}$，$\xi_1=\frac{1}{3}$，$\xi_2=\frac{2}{3}$，$a_1=\frac{1}{2}$，$a_2=\frac{1}{4}$. 考虑下面边值问题：

$$\left(|u'(t)|u'(t)\right)'+f\left(t,x(t),x'(t)\right)=0,\ 0<t<1, \tag{4.1}$$

$$u(0)=\frac{1}{2}u\left(\frac{1}{3}\right)+\frac{1}{4}u\left(\frac{2}{3}\right),\quad u'(1)=\frac{\sqrt{15}}{4}u'(0), \tag{4.2}$$

其中

$$f(t,u,v)=\begin{cases}3u^7+\dfrac{2+\sin v+t}{10^4}, & 0\le u\le 1, v\le\dfrac{\pi}{2},\\ 3u^7+\dfrac{3+t}{10^4}, & 0\le u\le 1, v\ge\dfrac{\pi}{2},\\ 3\sqrt[20]{u}+\dfrac{2+\sin v+t}{10^4}, & u\ge 1, v\le\dfrac{\pi}{2},\\ 3\sqrt[20]{u}+\dfrac{3+t}{10^4}, & u\ge 1, v\ge\dfrac{\pi}{2},\end{cases}$$

选取 $a=\frac{1}{4}$，$b=1$，$c=20$，$d=10$，$\eta=\frac{1}{4}$. 通过直接计算，有 $K=\frac{28}{3}$，$L\approx 5.436$，$M=\overline{M}=\frac{7}{3}$，可以验证 f 满足(H_2)，且有

(1) 当 $(t,u,v)\in[0,1]\times[0,20]\times[0,20]$ 时，

$$f(t,u,v)<3.001<\phi_3\left(\frac{c}{K}\right)\approx 4.592,\quad 0\le u\le 1,$$

$$f(t,u,v)\le 3.485<\phi_3\left(\frac{c}{K}\right)\approx 4.592,\quad u\ge 1;$$

(2) 当 $(t,u,v)\in[0,1]\times[0,\frac{1}{4}]\times[0,\frac{1}{4}]$ 时，

$$f(t,u,v)\le 0.0005<\phi_3\left(\frac{a}{K}\right)\approx 0.0007;$$

(3) 当 $(t,u,v)\in[\frac{1}{4},1]\times[1,10]\times[0,10]$ 时，

$$f(t,u,v)>3>\phi_3\left(\frac{b}{L}\right)\approx 0.034;$$

(4) 当 $(t,u,v)\in[\frac{1}{4},1]\times[1,20]\times[0,20]$ 时，

$$\min_{(t,u,v)\in[\frac{1}{4},1]\times[1,20]\times[0,20]} f(t,u,v)>3,\quad \phi_3(L)=29.55,$$

$$\max_{(t,u,v)\in[\frac{1}{4},1]\times[1,20]\times[0,20]} f(t,u,v)\approx 3.485,$$

$$\frac{1}{1-\phi_3(\beta)}\int_0^1 q(\tau)\mathrm{d}\tau=16,$$

$$3\times 29.55=88.65>3.485\times 16=55.76.$$

故定理 3.1 的条件均满足，则边值问题(4.1)和(4.2)至少有三个解 u_1，u_2 和 u_3，使得

$$\|u_1\|<\frac{1}{4},\quad 1<\alpha(u_2),\quad \frac{1}{4}<\|u_3\|,\quad \alpha(u_3)<1.$$

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