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Singularly Perturbed Problems with a Power-Law Attenuation Boundary Layer

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Abstract

Singularly Perturbed Problems with a Power-Law Layer is studied in this paper. The method of boundary layer function is used to construct the formal asymptotic expansion of the solution, and get the attenuated boundary layer functions of power-law. The existence and uniformly valid approximation of solutions are obtained by upper and lower solutions method.

Keywords

Singular Perturbation, Triple Root, Power-Law Boundary Layer, Asymptotic Solution

具有幂率衰减边界层的奇摄动问题

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摘　要

本文主要研究退化方程具有三重根的二阶奇摄动方程边值问题。运用边界层函数法构造出解的形式渐近展开式，得到以幂率形式衰减的边界层函数。最后用上下解方法得到形式解存在性和一致有效估计。

关键词

奇摄动，三重根，幂率衰减边界层，渐近解

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1. 引言

对于非线性奇摄动二阶方程的边值问题：

$$\varepsilon^2 \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = f(u,x),\ 0<x<1, \tag{1}$$

$$u(0,\varepsilon)=u^0,\ u(1,\varepsilon)=u^1. \tag{2}$$

其中，$0<\varepsilon\ll 1$，为小参数。f,φ 是连续可微的函数。

V.F. Butuzov 等在文献[1]中研究了具有指数衰减的边界层问题。在文献[2] [3]中，倪明康和 Huai-Ping Zhu 对具有幂率衰减边界层的问题进行了讨论。在文献[4] [5] [6]中，A.B. Vasil'eva 也分别对退化方程具有单根和二重根时的问题进行了研究，得到了渐近解的表达式：

$$u(x,\varepsilon)=\bar{u}(x)+\Pi(\tau)+R(\xi),\ \tau=\frac{x}{\varepsilon},\ \xi=\frac{x-1}{\varepsilon}.$$

其中

正则项 $\bar{u}(x)=\overline{u_0}(x)+\varepsilon\overline{u_1}(x)+\cdots$；

左边界层项 $\Pi(\tau,\varepsilon)=\Pi_0(\tau)+\varepsilon\Pi_1(\tau)+\cdots$；

右边界层项 $\Pi(\xi,\varepsilon)=\Pi_0(\xi)+\varepsilon\Pi_1(\xi)+\cdots$。

本文将继续对退化方程具有三重根进行讨论，设 $f(u,x,\varepsilon)$ 有如下形式

$$f(u,x,\varepsilon)=\left(u-\varphi(x)\right)^3. \tag{3}$$

并提出假设：

$$[H_1]\quad u^0>\varphi(0),\ u^1>\varphi(1),\ 0\le x\le 1. \tag{4}$$

$$[H_2]\quad \varphi''(x)>0,\ 0\le x\le 1. \tag{5}$$

2. 渐近解的构造

设问题(1)，(2)具有下述的形式渐近解：

$$u(x,\varepsilon)=\bar{u}(x,\varepsilon)+\Pi(\tau,\varepsilon),\ \tau=\frac{x}{\varepsilon},\ \xi=\frac{x-1}{\varepsilon}. \tag{6}$$

将(6)式代入(1)式，

$$\varepsilon^2\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}=\varepsilon^2\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}x^2}+\frac{\mathrm{d}^2\Pi}{\mathrm{d}\tau^2}. \tag{7}$$

$$f(u,x,\varepsilon)=f(\bar{u}(x),x,\varepsilon)+f(\bar{u}(\varepsilon\tau)+\Pi(\tau),\varepsilon\tau,\varepsilon)-f(\bar{u}(\varepsilon\tau),\varepsilon\tau,\varepsilon). \tag{8}$$

故有表达式：

$$\varepsilon^2\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}x^2}=f(\bar{u}(x),x,\varepsilon). \tag{9}$$

$$\frac{d^2\Pi}{d\tau^2}=f\left(\bar{u}\left(\varepsilon\tau\right)+\Pi\left(\tau\right),\varepsilon\tau,\varepsilon\right)-f\left(\bar{u}\left(\varepsilon\tau\right),\varepsilon\tau,\varepsilon\right). \tag{10}$$

设正则项的渐近表达式为：

$$\bar{u}\left(x,\varepsilon\right)=\overline{u_0}\left(x\right)+\varepsilon^{\frac{2}{3}}\overline{u_1}\left(x\right)+\varepsilon^{\frac{4}{3}}\overline{u_2}\left(x\right)+\cdots. \tag{11}$$

将(11)式代入(9)得：

$$\varepsilon^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\overline{u_0}\left(x\right)+\varepsilon^{\frac{2}{3}}\overline{u_1}\left(x\right)+\varepsilon^{\frac{4}{3}}\overline{u_2}\left(x\right)+\cdots\right)=\left(\overline{u_0}\left(x\right)+\varepsilon^{\frac{2}{3}}\overline{u_1}\left(x\right)+\varepsilon^{\frac{4}{3}}\overline{u_2}\left(x\right)+\cdots\right)^3. \tag{12}$$

比较上式两边的同次幂系数：

$$\varepsilon^0:0=\left(\overline{u_0}\left(x\right)-\varphi\left(x\right)\right)^3. \tag{13}$$

$$\varepsilon^2:\frac{\mathrm{d}^2\overline{u_0}\left(x\right)}{\mathrm{d}x^2}=\left(\overline{u_1}\left(x\right)\right)^3. \tag{14}$$

$$\varepsilon^{\frac{8}{3}}:\frac{\mathrm{d}^2\overline{u_1}\left(x\right)}{\mathrm{d}x^2}=3\overline{u_1}\left(x\right)\overline{u_2}\left(x\right). \tag{15}$$

由上式可解出

$$\overline{u_0}\left(x\right)=\varphi\left(x\right). \tag{16}$$

$$\overline{u_1}\left(x\right)=\left(\varphi''\left(x\right)\right)^{\frac{1}{3}}. \tag{17}$$

$$\overline{u_2}\left(x\right)=\frac{1}{9}\left(\varphi''\left(x\right)\right)^{-1}. \tag{18}$$

$i\geq 3$ 时可用类似方法得出。

再设边界层的渐近表达式为

$$\Pi\left(x,\varepsilon\right)=\Pi_0\left(x\right)+\varepsilon^{\frac{2}{3}}\Pi_1\left(x\right)+\varepsilon^{\frac{4}{3}}\Pi_2\left(x\right)+\cdots. \tag{19}$$

代入(10)得

$$\frac{\mathrm{d}^2\Pi}{\mathrm{d}\tau^2}=\left(\overline{u_0}\left(x\right)+\varepsilon^{\frac{2}{3}}\overline{u_1}\left(x\right)+\cdots+\Pi_0\left(x\right)+\varepsilon^{\frac{2}{3}}\Pi_1\left(x\right)+\cdots\right)^3-\left(\overline{u_0}\left(x\right)+\varepsilon^{\frac{2}{3}}\overline{u_1}\left(x\right)+\cdots\right)^3. \tag{20}$$

由(2)，(6)，(11)，(19)得

$$u\left(0,\varepsilon\right)=\overline{u_0}\left(0\right)+\varepsilon^{\frac{2}{3}}\overline{u_1}\left(0\right)+\cdots+\Pi_0\left(0\right)+\varepsilon^{\frac{2}{3}}\Pi_1\left(0\right)+\cdots=u^0. \tag{21}$$

得出如下形式的 $\Pi_i\left(\tau\right)$：

$$\Pi_0\left(0\right)=u^0-\varphi\left(0\right); \tag{22}$$

$$\Pi_i\left(0,\tau\right)=-u_i\left(0\right),\ i=1,2,3,\cdots \tag{23}$$

考虑到 $V_i\left(\tau\right)$ 为边界层函数，所以要求

$$\Pi_i\left(\infty\right)=0. \tag{24}$$

比较(20)式两端同次幂系数，可得Π_0函数满足的方程

$$\varepsilon^0:\frac{\mathrm{d}^2\Pi_0}{\mathrm{d}\tau^2}=\Pi_0^3,\ \Pi_0(0)=u^0-\varphi(0),\ \Pi_0(\infty)=0. \tag{25}$$

得

$$\Pi_0(\tau)=\frac{\sqrt{2}\left(u^0-\varphi(0)\right)}{\tau\left(u^0-\varphi(0)\right)+\sqrt{2}}. \tag{26}$$

当$\tau\to\infty$时，$\Pi_0(0)\to 0$。且具有幂率衰减特性。

继续使用上述方法，可得到其余边界层函数，且用同样的方法可确定$R(\xi)$。

3. 形式解的一致有效性

定理：若存在函数$\alpha(x),\beta(x)$使得问题

$$Lu=\varepsilon^2u_{xx}-f(u,x)$$

$$u(0,\varepsilon)=u^0,\ u(1,\varepsilon)=u^1.$$

满足下列条件：

1) $\alpha(x)\le\beta(x)$；

2) $L\alpha(x)\ge 0, L\beta(x)\le 0$；

3) $\alpha(0)\le u^0\le\beta(0),\alpha(1)\le u^1\le\beta(1)$。

则问题存在解$u(x)$满足[7]

$$\alpha(x)\le u(x)\le\beta(x).$$

令

$$U_0=\overline{u_0}+\Pi_0+R_0=\varphi(x)+\Pi_0+R_0.$$

构造

$$\beta(x)=U_0+\lambda_1\varepsilon^{\frac{2}{3}},\ \alpha(x)=U_0-\lambda_2\varepsilon^2. \tag{27}$$

$$L\beta(x)=\frac{\mathrm{d}^2\Pi_0}{\mathrm{d}\tau^2}+\frac{\mathrm{d}^2R_0}{\mathrm{d}\xi^2}+\varepsilon^2\varphi''(x)-\left(\Pi_0+R_0+\lambda_1\varepsilon^{\frac{2}{3}}\right)^3=\varepsilon^2\varphi''(x)-\lambda_1^3\varepsilon^2-\cdots$$

当$\lambda_1^3>\varphi''(x)$时，$L\beta(x)\le 0$。

$$L\alpha(x)=\frac{\mathrm{d}^2\Pi_0}{\mathrm{d}\tau^2}+\frac{\mathrm{d}^2R_0}{\mathrm{d}\xi^2}+\varepsilon^2\varphi''(x)-\left(\Pi_0+R_0-\lambda_2\varepsilon^2\right)^3=\varepsilon^2\varphi''(x)-\lambda_2{}^3\varepsilon^6+\cdots+3\lambda_2\varepsilon^2\left(\Pi_0{}^2+R_0{}^2\right).$$

对足够大的λ_2，$L\alpha(x)\ge 0$。

$$\beta(0)=\Pi_0(0)+R_0\left(-\frac{1}{\varepsilon}\right)+\varphi(0)+\lambda_1\varepsilon^{\frac{2}{3}}=u^0-\varphi(0)+R_0\left(-\frac{1}{\varepsilon}\right)+\varphi(0)+\lambda_1\varepsilon^{\frac{2}{3}}>u^0;$$

$$\beta(1)=\Pi_0\left(\frac{1}{\varepsilon}\right)+R_0(0)+\varphi(1)+\lambda_1\varepsilon^{\frac{2}{3}}=u^1-\varphi(1)+\Pi_0\left(\frac{1}{\varepsilon}\right)+\varphi(1)+\lambda_1\varepsilon^{\frac{2}{3}}>u^1;$$

$$\alpha(0)=\Pi_0(0)+R_0\left(-\frac{1}{\varepsilon}\right)+\varphi(0)-\lambda_2\varepsilon^2=u^0-\varphi(0)+R_0\left(-\frac{1}{\varepsilon}\right)+\varphi(0)-\lambda_2\varepsilon^2<u^0;$$

$$\alpha(1)=\Pi_0\left(\frac{1}{\varepsilon}\right)+R_0(0)+\varphi(1)-\lambda_1\varepsilon^2=u^1-\varphi(1)+\Pi_0\left(\frac{1}{\varepsilon}\right)+\varphi(1)-\lambda_1\varepsilon^2<u^1.$$

则问题(1)，(2)的渐近解 $U(x)$ 满足

$$-\lambda_1\varepsilon^{\frac{2}{3}}\le-\lambda_2\varepsilon^2=\alpha-U_0\le u(x,\varepsilon)-U_0\le\beta-U_0\le\lambda_1\varepsilon^{\frac{2}{3}}.$$

亦即

$$\left|u(x,\varepsilon)-U_0\right|\le\lambda_1\varepsilon^{\frac{2}{3}}. \tag{28}$$

特别地，对于首次近似有

$$\left|u(x,\varepsilon)-\left(\varphi(x)+\Pi_0+R_0\right)\right|=O\left(\varepsilon^{\frac{2}{3}}\right). \tag{29}$$

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