Advances in Applied Mathematics 应用数学进展, 2020, 9(2), 244-250
Published Online February 2020 in Hans. http://www.hanspub.org/journal/aam
https://doi.org/10.12677/aam.2020.92028

Hans汉斯

Approximation Properties of Bivariate (p, q)-Bernstein Operators

Pan Gao, HuiHui Liu, Xianxiang Leng

School of Mathematics and Statistics, Chaohu University, Hefei Anhui
Email: gk2816714440@163.com, 1679241827@qq.com

Received: Feb. 2nd, 2020; accepted: Feb. 17th, 2020; published: Feb. 24th, 2020

Abstract

In this paper, we introduce the bivariate (p, q)-Bernstein operator on the basis of (p, q)-Bernstein operator, and obtain the approximation theorem of the operator. The uniform convergence of the operator is verified by applying Volkov theorem, and its convergence rate is estimated. Those results further promote some of the conclusions of (p, q)-Bernstein operator.

Keywords

Bivariate (p, q)-Bernstein Operators, Rate of Convergence, Lipschitz Function

二元(p, q)-Bernstein算子的逼近性质

高 盼，刘辉辉，冷献祥

巢湖学院数学与统计学院，安徽 合肥
Email: gk2816714440@163.com, 1679241827@qq.com

收稿日期：2020年2月2日；录用日期：2020年2月17日；发布日期：2020年2月24日

摘 要

本文在(p, q)-Bernstein算子的基础上构建二元(p, q)-Bernstein算子，证明该算子的逼近定理；应用Volkov定理验证了该算子的一致收敛性，并估计其收敛速度，此结论推广了一元(p, q)-Bernstein算子的逼近结果。

关键词

二元(p, q)-Bernstein算子，收敛速度，Lipschitz函数

文章引用: 高盼, 刘辉辉, 冷献祥. 二元(p, q)-Bernstein 算子的逼近性质[J]. 应用数学进展, 2020, 9(2): 244-250.
DOI: 10.12677/aam.2020.92028

1. 引言

算子是逼近理论重要的研究对象，其中经典的算子之一为 Bernstein 算子。最早于 1912 年，Bernstein 首次提出。其后，众多研究者开始关注研究 Bernstein 算子的推广。于是，Bernstein 算子的各种变形算子纷纷被讨论，如 Szász-Mirakjan-Kantorovich 算子[1]，Baskakov 算子[2]等。

随着数学与生产生活各领域的交错发展，学者们将 q 微积分引入逼近理论，构造出大量 q 型算子。2007 年，Dalmanoglu Ö. [3]研究了 q-Bernstein-Kantorovich 算子；2011 年，Muraru C V [4]提出 q-Bernstein-Schurer 算子，并研究其逼近问题；伴随着研究的进一步深化，二元或多元算子相继被提出，故得到了大量二元算子关于逼近的相关理论，详见文献[5] [6] [7]等。

q 微积分在逼近中的发展推动了(p, q)微分学步入逼近理论。Mursaleen 于 2015 年首次在 q-Bernstein 算子的基础上提出(p, q)-Bernstein 算子[8]，实现了 q-Bernstein 算子性质的推广。自此，有关于(p, q)型算子呈现于世人面前。2016 年，Acar 在文献[9]中构建了两元(p, q)-Bernstein-Kantorovich 算子并证明该算子一些的逼近结论。由此可知，关于(p, q)型二元算子逼近问题的研究正在持续发展中。本文构建出二元(p, q)Bernstein 算子，证明算子的一些逼近相关的定理，从而更进一步推广一元算子的逼近性质，更加丰富逼近理论的完整性。

2. 知识储备

下文中出现的符号：$[n]_{p,q},[n]_{p,q}!,\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}$ 主要定义为：

$$[n]_{p,q}=\frac{p^n-q^n}{p-q},\quad n=0,1,2,\cdots$$

$$[n]_{p,q}!=\begin{cases}1, & n=0;\\ [n]_{p,q}[n-1]_{p,q}\cdots[1], & n=1,2,\cdots\end{cases}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}=\frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}$$

定义 1 [8]：设 $f\in C[0,1]$，$0<q<p\le 1$，$\forall x\in[0,1]$，则(p, q)-Bernstein 算子定义为下式：

$$B_n^{p,q}(f;x)=\sum_{k=0}^{n} b_{n,k}^{p,q}(x) f\left(\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}\right),$$

其中 $b_{n,k}^{p,q}(x)=\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{[k(k-1)-n(n-1)]/2}x^k\prod_{s=0}^{n-k-1}(p^s-q^s)$。

定义 2：设 $0<q<p\le 1$，$f\in C[0,1]$，定义二元(p, q)-Bernstein 算子为：

$$B_{n_1n_2}(f;x,y)=\frac{1}{p_1^{\frac{n_1(n_1-1)}{2}}}\frac{1}{p_2^{\frac{n_2(n_2-1)}{2}}}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} b_{n_1n_2k_1k_2}(x,y)(x,y) f\left(\frac{p_1^{n_1-k_1}[k_1]_{p_1,q_1}}{[n_1]_{p_1,q_1}},\frac{p_2^{n_2-k_2}[k_2]_{p_2,q_2}}{[n_2]_{p_2,q_2}}\right)$$

其中

$$b_{n_1n_2k_1k_2}(x,y)=\begin{bmatrix}n_1\\k_1\end{bmatrix}_{p_1,q_1}\begin{bmatrix}n_2\\k_2\end{bmatrix}_{p_2,q_2}p_1^{\frac{k_1(k_1-1)}{2}}p_2^{\frac{k_2(k_2-1)}{2}}x^{k_1}y^{k_2}\prod_{s=0}^{n_1-k_1-1}\left(p_1^s-q_1^sx\right)\prod_{s=0}^{n_2-k_2-1}\left(p_2^s-q_2^sy\right).$$

引理 1 [8]：设 $0<q<p\le 1$， $x\in[0,1]$，则有

$$B_n^{p,q}(e_0;x)=1,\qquad B_n^{p,q}(e_1;x)=x,$$
$$B_n^{p,q}(e_2;x)=\frac{p^{n-1}}{[n]_{p,q}}x+\frac{q[n-1]_{p,q}}{[n]_{p,q}}x^2.$$

引理 2：设 $0<q_1<p_1\le 1,0<q_2<p_2\le 1$， $e_{ij}:I^2\times I^2,e_{ij}(x,y)=x^iy^j,0\le i+j\le 2$， $I^2=[0,1]\times[0,1]$，(i,j 为正整数)，则有下列等式成立：

$$B_{n_1n_2}(e_{00};x,y)=e_{00}(x,y);$$
$$B_{n_1n_2}(e_{10};x,y)=e_{10}(x,y);$$
$$B_{n_1n_2}(e_{01};x,y)=e_{01}(x,y);$$
$$B_{n_1n_2}(e_{11};x,y)=e_{11}(x,y);$$
$$B_{n_1n_2}(e_{20};x,y)=e_{20}(x,y)+\frac{p_1^{n_1-1}x(1-x)}{[n_1]_{p_1,q_1}};$$
$$e_{n_1n_2}(e_{02};x,y)=e_{02}(x,y)+\frac{p_2^{n_2-1}y(1-y)}{[n_2]_{p_2,q_2}}.$$

证明：根据算子定义式与引理 1，计算可得

$$B_{n_1n_2}(e_{00};x,y)$$
$$=\frac{1}{p_1^{\frac{n_1(n_1-1)}{2}}}\frac{1}{p^{\frac{n_2(n_2-1)}{2}}}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_{p_1,q_1}\begin{bmatrix}n_2\\k_2\end{bmatrix}_{p_2,q_2}p_1^{\frac{k_1(k_1-1)}{2}}p_2^{\frac{k_2(k_2-1)}{2}}x^{k_1}y^{k_2}\times\prod_{s=0}^{n_1-k_1-1}\left(p_1^s-q_1^sx\right)\prod_{s=0}^{n_2-k_2-1}\left(p_2^s-q_2^sy\right)$$
$$=\frac{1}{p_1^{\frac{n_1(n_1-1)}{2}}}\sum_{k_1=0}^{n_1}\begin{bmatrix}n_1\\k_1\end{bmatrix}_{p_1,q_1}p^{\frac{k_1(k_1-1)}{2}}x^{k_1}\prod_{s=0}^{n_1-k_1-1}\left(p_1^s-q_1^sx\right)\times\frac{1}{p_2^{\frac{n_2(n_2-1)}{2}}}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_2\\k_2\end{bmatrix}_{p_2,q_2}p^{\frac{k_2(k_2-1)}{2}}y^{k_2}\prod_{s=0}^{n_2-k_2-1}\left(p_2^s-q_2^sy\right)$$
$$=B_{n_1}(e_0;x)B_{n_2}(e_0;y)=e_{00}(x,y),$$

$$B_{n_1n_2}(e_{10};x,y)$$
$$=\frac{1}{p^{\frac{n_1(n_1-1)}{2}}}\frac{1}{p^{\frac{n_2(n_2-1)}{2}}}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_{p_1,q_1}\begin{bmatrix}n_2\\k_2\end{bmatrix}_{p_2,q_2}p^{\frac{k_1(k_1-1)}{2}}p^{\frac{k_2(k_2-1)}{2}}x^{k_1}y^{k_2}$$
$$\times\prod_{s=0}^{n_1-k_1-1}\left(p_1^s-q_1^sx\right)\prod_{s=0}^{n_2-k_2-1}\left(p_2^s-q_2^sy\right)\frac{[k_1]_{p_1,q_1}}{p_1^{k_1-n_1}[n_1]_{p_1,q_1}}$$
$$=\frac{1}{p^{\frac{n_1(n_1-1)}{2}}}\frac{1}{p^{\frac{n_2(n_2-1)}{2}}}\sum_{k_1=0}^{n_1}\begin{bmatrix}n_1\\k_1\end{bmatrix}_{p_1,q_1}p_1^{\frac{k_1(k_1-1)}{2}}x^{k_1}\prod_{s=0}^{n_1-k_1-1}\left(p_1^s-q_1^sx\right)\frac{[k_1]_{p_1,q_1}}{p_1^{k_1-n_1}[n_1]_{p_1,q_1}}$$
$$\times\sum_{k_2=0}^{n_2}\begin{bmatrix}n_2\\k_2\end{bmatrix}_{p_2,q_2}p^{\frac{k_2(k_2-1)}{2}}y^{k_2}\prod_{s=0}^{n_2-k_2-1}\left(p_2^s-q_2^sy\right)$$
$$=B_{n_1}(e_1;x)B_{n_2}(e_0,y)=e_{10}(x,y)$$

$$
\begin{aligned}
&B_{n_1n_2}\left(e_{11};x,y\right)\\
&=\frac{1}{p^{\frac{n_1(n_1-1)}{2}}}\frac{1}{p^{\frac{n_2(n_2-1)}{2}}}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_{p_1,q_1}\begin{bmatrix}n_2\\k_2\end{bmatrix}_{p_2,q_2}p_1^{\frac{k_1(k_1-1)}{2}}p_2^{\frac{k_2(k_2-1)}{2}}x^{k_1}y^{k_2}\\
&\times\prod_{s=0}^{n_1-k_1-1}\left(p_1^s-q_1^sx\right)\prod_{s=0}^{n_2-k_2-1}\left(p_2^s-q_2^sy\right)\frac{[k_1]_{p_1,q_1}}{p_1^{k_1-n_1}[n_1]_{p_1,q_1}}\frac{[k_2]_{p_2,q_2}}{p_2^{k_2-n_2}[n_2]_{p_2,q_2}}\\
&=\frac{1}{p_1^{\frac{n_1(n_1-1)}{2}}}\sum_{k_1=0}^{n_1}\begin{bmatrix}n_1\\k_1\end{bmatrix}_{p_1,q_1}p_1^{\frac{k_1(k_1-1)}{2}}x^{k_1}\prod_{s=0}^{n_1-k_1-1}\left(p_1^s-q_1^sx\right)\frac{[k_1]_{p_1,q_1}}{p_1^{k_1-n_1}[n_1]_{p_1,q_1}}\\
&\times\frac{1}{p_1^{\frac{n_2(n_2-1)}{2}}}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_2\\k_2\end{bmatrix}_{p_2,q_2}p_2^{\frac{k_2(k_2-1)}{2}}y^{k_2}\prod_{s=0}^{n_2-k_2-1}\left(p_2^s-q_2^sy\right)\frac{[k_2]_{p_2,q_2}}{p_2^{k_2-n_2}[n_2]_{p_2,q_2}}\\
&=B_{n_1}\left(e_1,x\right)B_{n_2}\left(e_1,y\right)=e_{11}\left(x,y\right)
\end{aligned}
$$

$$
\begin{aligned}
&B_{n_1n_2}\left(e_{02};x,y\right)\\
&=\frac{1}{p_1^{\frac{n_1(n_1-1)}{2}}}\frac{1}{p_2^{\frac{n_2(n_2-1)}{2}}}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\begin{bmatrix}n_1\\k_1\end{bmatrix}_{p_1,q_1}\begin{bmatrix}n_2\\k_2\end{bmatrix}_{p_2,q_2}p_1^{\frac{k_1(k_1-1)}{2}}p_2^{\frac{k_2(k_2-1)}{2}}x^{k_1}y^{k_2}\\
&\times\prod_{s=0}^{n_1-k_1-1}\left(p_1^s-q_1^sx\right)\prod_{s=0}^{n_2-k_2-1}\left(p_2^s-q_2^sy\right)\frac{[k_2]_{p_2,q_2}^2}{p_2^{2k_2-2n_2}[n_2]_{p_2,q_2}^2}\\
&=y^2+\frac{\left(q_2[n_2-1]_{p_2,q_2}-[n_2]_{p_2,q_2}\right)y^2+p_2^{n_2-1}y}{[n_2]_{p_2,q_2}}\\
&=y^2+\frac{p_2^{n_2-1}y(1-y)}{[n_2]_{p_2,q_2}}=B_{n_1}\left(e_0,x\right)B_{n_2}\left(e_2,y\right)\\
&=e_{02}\left(x,y\right)+\frac{p_2^{n_2-1}y(1-y)}{[n_2]_{p_2,q_2}}.
\end{aligned}
$$

同理可证出 $B_{n_1n_2}\left(e_{01};x,y\right)=e_{01}\left(x,y\right)$； $B_{n_1,n_2}\left(e_{20},x,y\right)=e_{20}\left(x,y\right)+\frac{p_1^{n_1-1}x(1-x)}{[n_1]_{p_1,q_1}}$， 故引理成立。

引理 3：设 $0<q_1<p_1\le 1, 0<q_2<p_2\le 1$， $x\in[0,1]$，则有下列等式成立：

$$
\begin{aligned}
&B_{n_1n_2}\left(e_{10}-x;x,y\right)=0;\\
&B_{n_1n_2}\left(e_{01}-x;x,y\right)=0;\\
&B_{n_1n_2}\left(\left(e_{10}-x\right)^2;x,y\right)=\frac{p_1^{n_1-1}x(1-x)}{[n_1]_{p_1,q_1}};\\
&B_{n_1n_2}\left(\left(e_{01}-x\right)^2;x,y\right)=\frac{p_2^{n_2-1}y(1-y)}{[n_2]_{p_2,q_2}}.
\end{aligned}
$$

证明：根据引理 2 与算子的线性性质易得结论。

3. 主要结果

首先介绍一些记号：设 $\delta_1>0,\delta_2>0$， $f\in C\left(I^2\right)$， $I^2=[0,1]\times[0,1]$，则关于 f 的连续性模可以表示为：

$$\omega(f:\delta_1,\delta_2)=\sup\{|f(t,s)-f(x,y)|:(t,s)(x,y)\in(I_1\times I_2),|t-x|\le\delta_1,|s-y|\le\delta_2\};$$

并且 $\omega(f:\delta_1,\delta_2)$ 满足以下性质：

(i) $\omega(f:\delta_1,\delta_2)\to 0$，若 $\delta_1,\delta_2\to 0$

(ii) $f(t,s)-f(x,y)\le\omega(f:\delta_1,\delta_2)\left(1+\frac{|t-x|}{\delta_1}\right)\left(1+\frac{|s-y|}{\delta_2}\right)$

α_1,α_2 阶 Lipschitz 条件的二元函数 f：对于 $\forall(t,s),(x,y)\in I^2$，$f\in C(I^2)$，$0<\alpha_1\le 1,0<\alpha_2\le 1$，则存在常数 $M>0$，使得 $|f(t,s)-f(x,y)|\le M|t-x|^{\alpha_1}|s-y|^{\alpha_2}$；记为 $f\in Lip_M(\alpha_1,\alpha_2)$。

定理 1：若 $p_1=p_{n_1},p_2=p_{n_2},q_1=q_{n_1},q_2=q_{n_2}$，且 $q_1\in(0,1),p_1\in(q_1,1],q_2\in(0,1),p_2\in(q_2,1]$，$\lim_{n\to\infty}p_{n_1}=1,\lim_{n\to\infty}p_{n_2}=1,\lim_{n\to\infty}q_{n_1}=1,\lim_{n\to\infty}q_{n_2}=1$，则对于 $\forall f\in C(I^2)$，都有 $\lim_{n_1,n_2\to\infty}\|B_{n_1,n_2}(f;x)-f(x)\|=0$。

证明：根据引理 2 得到

$$\|B_{n_1n_2}(e_{00}:x,y)-e_{00}\|=0;$$
$$\|B_{n_1n_2}(e_{10}:x,y)-e_{10}\|=0;$$
$$\|B_{n_1n_2}(e_{01}:x,y)-e_{01}\|=0;$$

又因为当 $n_1\to\infty,n_2\to\infty$ 时，

$$\|B_{n_1n_2}(e_{20}+e_{02})-(e_{20}+e_{02})\|=\left\|\frac{p_1^{n_1-1}x(1-x)}{[n_1]_{p_1,q_1}}+\frac{p_2^{n_2-1}y(1-y)}{[n_2]_{p_2,q_2}}\right\|\to 0.$$

故根据 Volkov 定理的内容可以得到 $\lim_{n_1,n_2\to\infty}\|B_{n_1,n_2}(f;x)-f(x)\|=0$。

定理 2：若 $f\in C(I^2),\forall(x,y)\in I^2$，有不等式

$$|B_{n_1n_2}(f:x,y)-f(x,y)|\le 4\omega(f:\delta_1,\delta_2),$$

其中 $\delta_1=\frac{1}{4[n_1]_{p_1,q_1}},\delta_2=\frac{1}{4[n_2]_{p_2,q_2}}$。

证明：根据二元函数连续模的性质，则有

$$\begin{aligned}
&|B_{n_1n_2}(f:x,y)-f(x,y)|\\
&\le B_{n_1n_2}(|f(t,s)-f(x,y)|:x,y)\\
&\le B_{n_1n_2}\left(\omega(f:\delta_1,\delta_2)\left(1+\frac{|t-x|}{\delta_1}\right)\left(1+\frac{|s-y|}{\delta_2}\right):x,y\right)\\
&=\omega(f:\delta_1,\delta_2)\left[B_{n_1n_2}(e_{10}:x,y)+B_{n_1n_2}\left(\frac{|t-x|}{\delta_1}:x,y\right)\right]\\
&\quad\cdot\left[B_{n_1n_2}(e_{01}:x,y)+B_{n_1n_2}\left(\frac{|s-y|}{\delta_2}:x,y\right)\right]\\
&=\omega(f:\delta_1,\delta_2)\left(1+\frac{1}{\delta_1}B_{n_1n_2}(|t-x|:x,y)\right)\left(1+\frac{1}{\delta_2}B_{n_1n_2}(|s-y|:x,y)\right)
\end{aligned}$$

又利用 Cauchy-Schwarz 不等式与引理 3，有

$$B_{n_1,n_2}\left(|t-x|:x,y\right)=B_{n_1,p_1,q_1}\left(|t-x|;x\right)\le\left[B_{n_1,n_2}\left((t-x)^2;x\right)\right]^{\frac{1}{2}}\le\sqrt{\frac{1}{4[n_1]_{p_1,q_1}}},$$

$$B_{n_1,n_2}\left(|s-y|;x,y\right)=B_{n_2,p_2,q_2}\left(|s-y|;y\right)\le\left[B_{n_1,n_2}\left((s-y)^2;y\right)\right]^{\frac{1}{2}}\le\sqrt{\frac{1}{4[n_2]_{p_2,q_2}}}.$$

因此，得到

$$\left|B_{n_1n_2}\left(f:x,y\right)-f\left(x,y\right)\right|\le\omega\left(f:\delta_1,\delta_2\right)\left(1+\frac{1}{\delta_1}\sqrt{\frac{1}{4[n_1]_{p_1,q_1}}}\right)\left(1+\frac{1}{\delta_2}\sqrt{\frac{1}{4[n_2]_{p_2,q_2}}}\right)$$
$$=\omega\left(f:\delta_1,\delta_2\right)\left(1+\frac{1}{\delta_1}\frac{1}{2\sqrt{[n_1]_{p_1,q_1}}}\right)\left(1+\frac{1}{\delta_2}\frac{1}{2\sqrt{[n_2]_{p_2,q_2}}}\right)$$

取 $\delta_1=\frac{1}{2\sqrt{[n_1]_{p_1,q_1}}},\delta_2=\frac{1}{2\sqrt{[n_2]_{p_2,q_2}}}$，即 $\left|B_{n_1n_2}\left(f:x,y\right)-f\left(x,y\right)\right|\le 4\omega\left(f:\delta_1,\delta_2\right)$ 成立。

定理 3： 设 $C\left(I^2\right)=\left\{f\left(x,y\right)\in C\left(I^2\right)\right\}$ 且 $\frac{\partial_f}{\partial_x},\frac{\partial_f}{\partial_y}\in C\left(I^2\right)$，则有

$$\left|B_{n_1n_2}\left(f:x,y\right)-f\left(x,y\right)\right|\le\left\|f_x'\right\|\delta_1+\left\|f_y'\right\|\delta_2,$$

其中 $\delta_1=\frac{1}{2\sqrt{[n_1]_{p_1,q_1}}},\delta_2=\frac{1}{2\sqrt{[n_2]_{p_2,q_2}}}$

证明： 因为 $\forall\left(x,y\right)\in I^2, f\left(t,s\right)-f\left(x,y\right)=\int_x^t f_u'\left(u,s\right)\mathrm{d}u+\int_s^y f_v'\left(x,v\right)\mathrm{d}v$，可得

$$\left|B_{n_1n_2}\left(f:x,y\right)-f\left(x,y\right)\right|\le B_{n_1n_2}\left(\left|\int_x^t f_u'\left(u,s\right)\mathrm{d}u\right|:x,y\right)+B_{n_1n_2}\left(\left|\int_s^y f_v'\left(x,v\right)\mathrm{d}v\right|:x,y\right),$$

又因为

$$\int_x^t f_u'\left(u,s\right)\mathrm{d}u\le\left\|f_x'\right\|_I|t-x|,\int_s^y f_v'\left(x,v\right)\mathrm{d}v\le\left\|f_y'\right\|_I|y-s|$$

利用算子作用与柯西－施瓦茨不等式计算有

$$\begin{aligned}&\left|B_{n_1n_2}\left(f:x,y\right)-f\left(x,y\right)\right|\\&\le\left\|f_x'\right\|_I B_{n_1n_2}\left(|t-x|;x,y\right)+\left\|f_y'\right\|_I B_{n_1n_2}\left(|y-s|;x,y\right)\\&\le\left\|f_x'\right\|_I B_{n_1}\left(|t-x|;x\right)+\left\|f_y'\right\|_I B_{n_2}\left(|y-s|;y\right)\\&\le\left\|f_x'\right\|\left\{B_{n_1}\left(t-x\right)^2;x\right\}^{\frac{1}{2}}\left\{B_{n_1}\left(1,x\right)\right\}^{\frac{1}{2}}+\left\|f_y'\right\|\left\{B_{n_2}\left(y-s\right)^2;y\right\}^{\frac{1}{2}}\left\{B_{n_2}\left(1,y\right)\right\}^{\frac{1}{2}}\\&\le\left\|f_x'\right\|\frac{1}{2\sqrt{[n_1]_{p_1,q_1}}}+\left\|f_y'\right\|\frac{1}{2\sqrt{[n_2]_{p_2,q_2}}}\\&=\left\|f_x'\right\|\delta_1+\left\|f_y'\right\|\delta_2.\end{aligned}$$

定理 4： 若 $f\in Lip_M\left(\alpha_1,\alpha_2\right)$，则存在一个常数 $M>0$，有下式成立：

$$\left|B_{n_1n_2}\left(f:x,y\right)-f\left(x,y\right)\right|\le M\delta_1^{\alpha_1}\left(x\right)\delta_2^{\alpha_2}\left(y\right)$$

其中 $\delta_1\left(x\right)=B_{n_1}\left(\left(t-x\right)^2,x\right)^{\frac{1}{2}},\delta_2\left(y\right)=B_{n_2}\left(\left(s-y\right)^2,y\right)^{\frac{1}{2}}$。

证明：由 $f \in Lip_M(\alpha_1,\alpha_2)$，计算可得

$$\begin{aligned}&\left|B_{n_1n_2}(f:x,y)-f(x,y)\right|\\&\le B_{n_1n_2}\left(\left|f(t,s)-f(x,y)\right|:x,y\right)\\&\le MB_{n_1n_2}\left(\left|t-x\right|^{\alpha_1}\left|s-y\right|^{\alpha_2};x,y\right)\\&\le MB_{n_1}\left(\left|t-x\right|^{\alpha_1};x\right)B_{n_2}\left(\left|s-y\right|^{\alpha_2};y\right)\end{aligned}$$

利用 Hölder 不等式，取 $p=\frac{2}{\alpha_1},q=\frac{2}{2-\alpha_1},p^*=\frac{2}{\alpha_2},q^*=\frac{2}{2-\alpha_2}$

$$\begin{aligned}&\left|B_{n_1n_2}(f:x,y)-f(x,y)\right|\\&\le MB_{n_1}\left((t-x)^2,x\right)^{\frac{\alpha_1}{2}}B_{n_1}(1,x)^{\frac{2}{2-\alpha_1}},B_{n_2}\left((s-y)^2,y\right)^{\frac{\alpha_2}{2}}B_{n_1}(1,y)^{\frac{2}{2-\alpha_2}}\\&\le MB_{n_1}\left((t-x)^2,x\right)^{\frac{\alpha_1}{2}}B_{n_2}\left((s-y)^2,y\right)^{\frac{\alpha_2}{2}}\\&\le B\delta_1^{\alpha_1}(x)\delta_2^{\alpha_2}(y).\end{aligned}$$

取 $\delta_1(x)=B_{n_1}\left((t-x)^2,x\right)^{\frac{1}{2}},\delta_2(y)=B_{n_2}\left((s-y)^2,y\right)^{\frac{1}{2}}$ 即定理成立。

致　谢

本文的写作感谢查星星老师的指导。

基金项目

巢湖学院国家级大学生创新创业训练计划资助项目(201910380035)，巢湖学院省级大学生创新创业训练计划资助项目(S201910380068)。

参考文献

[1] Totik, V. (1983) Approximation by Szász-Mirakjan-Kantorovich Operators in L^p ($p > 1$). *Analysis Mathematica*, **9**, 147-167. https://doi.org/10.1007/BF01982010

[2] Pethe, S. (1984) On the Baskakov Operator. *Indian Journal of Mathematics*, **26**, 43-48.

[3] Dalmanoglu, Ö. (2007) Approximation by Kantorovich Type *q*-Bernstein Operators. *Proceedings of the 12th WSEAS International Conference on Applied Mathematics*, Cairo, Egypt, 29-31 December 2007, 113-117.

[4] Muraru, C.V. (2011) Note on *q*-Bernstein-Schurer Operators. *Studia Universitatis Babes-Bolyai Mathematica*, **56**, 489-495.

[5] Simeonov, P. and Zafiris, V. (2008) Geometric Applications of Bivariate *q*-Bernstein and *q*-Orthogonal Polynomials. *American Conference on Applied Mathematics*, **3**, 25-28.

[6] Agrawal, P.N., Finta, Z. and Kumar, A.S. (2015) Bivariate *q*-Bernstein-Schurer-Kantorovich Operators. *Results in Mathematics*, **67**, 365-380. https://doi.org/10.1007/s00025-014-0417-z

[7] Pop, O.T. and Fărcaş, M.D. (2009) About the Bivariate Operators of Durrmeyer-Type. *Demonstratio Mathematica*, **42**, 97-107. https://doi.org/10.1515/dema-2009-0110

[8] Mursaleen, M., Ansari. K.J. and Khan, A. (2016) Erratum to "On (*p*, *q*)-Analogue of Bernstein Operators [Appl. Math. Comput. 266 (2015) 874-882]. *Applied Mathematics and Computation*, **278**, 70-71. https://doi.org/10.1016/j.amc.2016.02.008

[9] Acar, T., Aral, A. and Mohiuddine, S.A. (2016) Approximation by Bivariate (*p*, *q*)-Bernstein-Kantorovich Operators. *Iranian Journal of Science and Technology*, *Transactions A*: *Science*, **42**, 655-662. https://doi.org/10.1007/s40995-016-0045-4