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Finite Spectrum of a Class of Third Order Boundary Value Problems with *N* Transmission Conditions

Junwei Zhu

School of Science, Lanzhou University of Technology, Lanzhou Gansu
Email: lutjwzhu@163.com

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Abstract

The paper studied the following finite spectrum of third order boundary value problems with n transmission conditions

$$\begin{cases} (py'')' + qy = \lambda wy, t \in J = (a,c_1) \cup (c_1,c_2) \cup \cdots \cup (c_n,b), \\ AY(a) + BY(b) = 0, \\ C_iY(c_i -) + D_iY(c_i +) = 0. \end{cases}$$

For any positive integer $n, m_i, i = 0,1,\cdots,n$, there are at most $m_0 + m_1 + \cdots + m_n$ eigenvalues. The main tool used in this paper is iterative construction of the characteristic function and Rouche's theorem. The key to this analysis is the construction of discontinuous function solutions.

Keywords

Transmission Conditions, Boundary Value Problems, Characteristic Function, Rouche's Theorem

一类具有n个转移条件的三阶边值问题的有限谱

朱军伟

兰州理工大学理学院，甘肃 兰州
Email: lutjwzhu@163.com

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摘　要

本文主要研究下述具有n个转移条件的三阶边值问题的有限谱

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$$\begin{cases} (py'')' + qy = \lambda wy, t \in J = (a,c_1) \cup (c_1,c_2) \cup \cdots \cup (c_n,b), \\ AY(a) + BY(b) = 0, \\ C_i Y(c_i -) + D_i Y(c_i +) = 0. \end{cases}$$

对于任意正整数 $n, m_i, i = 0,1,\cdots,n$，经计算至多有 $m_0 + m_1 + \cdots + m_n$ 个特征值。所用的工具主要是判断函数的迭代和Rouche定理，分析的关键是不连续函数解的构造。

关键词

转移条件，边值问题，判断函数，Rouche定理

1. 引言

微分方程边值问题是微分方程理论中较为重要的研究方向。众所周知，三阶微分方程起源于应用数学和物理学的各个不同领域中，其应用范围十分广泛。例如，带有固定或变化横截面的屈曲梁的挠度、三层梁、电磁波、地球引力吹积的涨潮、工程力学等。因此，对于三阶微分方程边值问题的研究理论已然相当成熟，如文献[1] [2]。正是基于热传导和边界在滑杆上的弦振动问题，因此对于微分方程边值问题[3]有限谱的研究逐渐受到很多学者的青睐，例如对于 Sturm-Liouville 问题有限谱[4] [5] [6] [7]，带有转移条件的 Sturm-Liouville 问题有限谱[8] [9] [10] [11]，以及边界条件中含有谱参数的 Sturm-Liouville 有限谱问题的研究。且阶数已从二阶增加到四阶、六阶，甚至 2n 阶[12] [13] [14] [15]，正因为都是偶数阶，所以可以很自然的推广到高阶偶数阶问题上去。但是对于奇数阶具有有限谱的微分方程边值问题的研究仍然是比较困难的，随着 Ao 等人对于两类具有有限谱的三阶微分方程边值问题的提出[16]，使得三阶微分方程边值问题的有限谱理论成为解决实际问题的关键。理论上三阶微分方程边值问题是否具有有限谱，它的转移条件是否对于其特征值个数会产生影响，对这些问题的探讨都是非常有必要的。

目前，学者们对于偶数阶微分方程边值问题是否具有有限谱的研究做出了大量的杰出工作，如文献[17] [18] [19]。值得一提的是，2017 年，Ao 研究了下述两类具有有限谱的三阶边值问题

$$\begin{cases} (py'')' + qy = \lambda wy, t \in J = (a,b), \\ AY(a) + BY(b) = 0; \end{cases} \quad \begin{cases} (py')'' + qy = \lambda wy, t \in J = (a,b), \\ AY(a) + BY(b) = 0. \end{cases}$$

对于每一个正整数 m，该问题至多有 $2m+1$ 个特征值，他们运用的主要工具有判断函数的迭代、代数学基本定理等。

受以上杰出工作的启发，本文主要考虑下述一类具有 n 个转移条件的三阶边值问题的有限谱

$$\begin{cases} (py'')' + qy = \lambda wy, & (1) \\ AY(a) + BY(b) = 0, & (2) \\ C_i Y(c_i -) + D_i Y(c_i +) = 0, & (3) \end{cases}$$

其中 $y=y(t)$，$t\in J=(a,c_1)\cup(c_1,c_2)\cup\cdots\cup(c_n,b)$，$-\infty<a<b<+\infty$，$A,B\in M_2(\mathbb{C})$，$c_i\in(a,b)$，$C_i,D_i\in M_2(\mathbb{R})$，$\det(C_i)=\rho_i>0$，$\det(D_i)=\theta_i>0$，$i=1,2,\cdots,n$。此处 λ 为谱参数，且系数满足最小条件

$$r=\frac{1}{p},q,w\in L(J,\mathbb{C}). \tag{4}$$

其中 $L(J,\mathbb{C})$ 表示在J上Lebesgue可积的复值函数构成的集合。条件(4)是方程(1)所有初值问题在 (a,b) 上具有唯一解的充要条件，参见[20]。在本文中，设(4)恒成立，我们将证明问题(1)~(3)仍然具有有限谱。

2. 预备知识及说明

令 $u=y,v=y',z=py''$，则与方程 $(py'')'+qy=\lambda wy$ 等价的系统表示为：

$$u'=v,v'=rz,z'=(\lambda w-q)u. \tag{5}$$

可写成如下矩阵形式

$$\begin{pmatrix}u\\v\\z\end{pmatrix}'=\begin{pmatrix}0&1&0\\0&0&r\\\lambda w-q&0&0\end{pmatrix}\begin{pmatrix}u\\v\\z\end{pmatrix}. \tag{6}$$

定义 1：方程在 J 的子区间上的平凡解是指在子区间上 y 及其拟导数 $v=y',z=py''$ 都为零的解 y。

引理 1：设 $\Phi(t,\lambda)=[\phi_{ef}(t,\lambda)](e,f=1,2,t\in J)$ 为系统(5)满足初始条件 $\Phi(a,\lambda)=I$ 的基解矩阵，则 $\lambda\in\mathbb{C}$ 是问题(1)~(3)的特征值当且仅当

$$\Delta(\lambda)=\det[A+B\Phi(b,\lambda)]=0. \tag{7}$$

特别地：

$$\Delta(\lambda)=\det(A)+\det(B)+\sum_{i=1}^{4}\sum_{j=1}^{4}c_{ij}\phi_{ij}+\sum_{1\le i,j,k,l\le 4}d_{ijkl}\phi_{ij}\phi_{kl}, \tag{8}$$

其中 $c_{i,j},1\le i,j\le 4,d_{ijkl},1\le i,j,k,l\le 4,j\ne l$ 仅仅是依赖于矩阵 A,B 的常数。

证明：引理第一部分的证明见文献[16]，式(8)直接计算可得。 □

定义 2：若对所有的 $\lambda\in\mathbb{C},\Delta(\lambda)\equiv 0$，或对任意的 $\lambda\in\mathbb{C},\Delta(\lambda)\ne 0$，则称问题(1)~(3) (或等价问题(4) (2) (3))为退化的。

3. 具有 *n* 个转移条件的三阶边值问题的有限谱

在这一部分中我们设对于区间 J 有如下分割

$$\begin{cases}a=a_0<a_1<a_2<\cdots<a_{2m_0}<a_{2m_0+1}=c_1-,\\ c_1+=c_{1,0}<c_{1,1}<c_{1,2}<\cdots<c_{1,2m_1}<c_{1,2m_1+1}=c_2-,\\ \cdots\\ c_{n-1}+=c_{n-1,0}<c_{n-1,1}<c_{n-1,2}<\cdots<c_{n-1,2m_{n-1}}<c_{n-1,2m_{n-1}+1}=c_n-,\\ c_n+=c_{n,0}<c_{n,1}<c_{n,2}<\cdots<c_{n,2m_n}<c_{n,2m_n+1}=b,\end{cases} \tag{9}$$

对于某些正整数 $m_0,m_1,m_2,\cdots,m_n$，当 $r(t)=\dfrac{1}{p(t)}=0$ 时，我们有

$$\begin{cases}\int_{a_{2k}}^{a_{2k+1}}w(t)\ne 0,\int_{a_{2k}}^{a_{2k+1}}w(t)tdt\ne 0,k=0,1,\cdots,m_0,t\in(a_{2k},a_{2k+1}),\\ \int_{c_{1,2i}}^{c_{1,2i+1}}w(t)\ne 0,\int_{c_{1,2i}}^{c_{1,2i+1}}w(t)tdt\ne 0,i=0,1,\cdots,m_1,t\in(c_{1,2i},c_{1,2i+1}),\\ \cdots\\ \int_{c_{n,2z}}^{c_{n,2z+1}}w(t)\ne 0,\int_{c_{n,2z}}^{c_{n,2z+1}}w(t)tdt\ne 0,z=0,1,\cdots,m_n,t\in(c_{n,2z},c_{n,2z+1}).\end{cases} \tag{10}$$

而 $q(t)=w(t)=0$ 时，则有

$$\begin{cases}\int_{a_{2k+1}}^{a_{2k+2}} r(t)\neq 0, \int_{a_{2k+1}}^{a_{2k+2}} r(t)t\mathrm{d}t\neq 0, k=0,1,\cdots,m_0-1, t\in(a_{2k+1},a_{2k+2}),\\ \int_{c_{1,2i+1}}^{c_{1,2i+2}} r(t)\neq 0, \int_{c_{1,2i+1}}^{c_{1,2i+2}} r(t)t\mathrm{d}t\neq 0, i=0,1,\cdots,m_1-1, t\in(c_{1,2i+1},c_{1,2i+2}),\\ \cdots\\ \int_{c_{n,2z+1}}^{c_{n,2z+2}} r(t)\neq 0, \int_{c_{n,2z+1}}^{c_{n,2z+2}} r(t)t\mathrm{d}t\neq 0, z=0,1,\cdots,m_n, t\in(c_{n,2z+1},c_{n,2z+2}),\end{cases} \tag{11}$$

引理2： 设(9)~(11)成立，对于每一个 $\lambda\in\mathbb{C}$。

$\Phi(t,\lambda)=\left[\phi_{ef}(t,\lambda)\right](t\in(a,c_1))$ 为系统(5)满足初始条件 $\Phi(a,\lambda)=I$ 的基解矩阵；

$\Psi_i(t,\lambda)=\left[\psi_{i,ef}(t,\lambda)\right]\left(t\in(c_i,c_{i+1}),c_{n+1}=b=c_{n,2m_n+1},i=1,2,\cdots,n\right)$ 为系统(5)满足初始条件 $\Psi_i(c_i+,\lambda)=I$ (此处 $\Psi_i(c_i+,\lambda)=\Psi_i(c_{i,0},\lambda)=\Phi(c_i+,\lambda)$)的基解矩阵，则有

1)

$$\Phi(a_1,\lambda)=\begin{pmatrix}1 & a_1-a_0 & 0\\ 0 & 1 & 0\\ \lambda w_0-q_0 & (\lambda w_0-q_0)(a_1-a_0) & 1\end{pmatrix}, \tag{12}$$

$$\Phi(a_3,\lambda)=\begin{pmatrix}\phi_{11}(a_3,\lambda) & \phi_{12}(a_3,\lambda) & r_0(a_3-a_1)\\ r_0(\lambda w_0-q_0) & 1+r_0(\lambda w_0-q_0)(a_1-a_0) & r_0\\ \phi_{31}(a_3,\lambda) & \phi_{32}(a_3,\lambda) & (\lambda w_1-q_1)\left[r_0(a_3-a_1)\right]+1\end{pmatrix}. \tag{13}$$

其中

$$\phi_{11}(a_3,\lambda)=(a_3-a_1)r_0(\lambda w_0-q_0)+1,$$

$$\phi_{12}(a_3,\lambda)=(a_3-a_1)\left[r_0(\lambda w_0-q_0)(a_1-a_0)+1\right]+(a_1-a_0),$$

$$\phi_{31}(a_3,\lambda)=\left[(a_3-a_1)r_0(\lambda w_0-q_0)+1\right](\lambda w_1-q_1)+(\lambda w_0-q_0),$$

$$\phi_{32}(a_3,\lambda)=(\lambda w_1-q_1)(a_3-a_1)r_0+1.$$

一般地，对于 $1\le k\le m_0$，

$$\Phi(a_{2k+1},\lambda)=\begin{pmatrix}1 & 1 & r_{k-1}\\ 0 & 1 & r_{k-1}\\ \lambda w_k-q_k & \lambda w_k-q_k & r_{k-1}(\lambda w_k-q_k)\end{pmatrix}\Phi(a_{2k-1},\lambda). \tag{14}$$

2)

$$\Psi_i(c_{i,1},\lambda)=\begin{pmatrix}1 & c_{i,1}-a_0 & 0\\ 0 & 1 & 0\\ \lambda w_{i,0}-q_{i,0} & (\lambda w_{i,0}-q_{i,0})(c_{i,1}-a_0) & 1\end{pmatrix}, \tag{15}$$

$$\Psi_i(c_{i,3},\lambda)=\begin{pmatrix}\psi_{i,11}(c_{i,3},\lambda) & \psi_{i,12}(c_{i,3},\lambda) & r_{i,0}(c_{i,3}-c_{i,1})\\ r_{i,0}(\lambda w_{i,0}-q_{i,0}) & 1+r_{i,0}(\lambda w_{i,0}-q_{i,0})(c_{i,1}-a_0) & r_{i,0}\\ \psi_{i,31}(c_{i,3},\lambda) & \psi_{i,32}(c_{i,3},\lambda) & (\lambda w_{i,1}-q_{i,1})\left[r_{i,0}(c_{i,3}-c_{i,1})\right]+1\end{pmatrix}. \tag{16}$$

其中

$$\psi_{i,11}(c_{i,3},\lambda)=(c_{i,3}-c_{i,1})r_{i,0}(\lambda w_{i,0}-q_{i,0})+1,$$

$$\psi_{i,12}(c_{i,3},\lambda)=(c_{i,3}-c_{i,1})\left[r_{i,0}(\lambda w_{i,0}-q_{i,0})(c_{i,1}-a_0)+1\right]+(c_{1,1}-a_0),$$

$$\psi_{i,31}(c_{i,3},\lambda)=\left[(c_{i,3}-c_{i,1})r_{i,0}(\lambda w_{i,0}-q_{i,0})+1\right](\lambda w_{i,1}-q_{i,1})+(\lambda w_{i,0}-q_{i,0}),$$

$$\psi_{i,32}(c_{i,3},\lambda)=(\lambda w_{i,1}-q_{i,1})(c_{i,3}-c_{i,1})r_{i,0}+1.$$

更一般地，对于$1\le\kappa\le m_i(\kappa=i,j,\cdots,z)$，

$$\Psi_i(c_{i,2\kappa+1},\lambda)=\begin{pmatrix}1 & 1 & r_{i,\kappa-1}\\ 0 & 1 & r_{i,\kappa-1}\\ \lambda w_{i,\kappa}-q_{i,\kappa} & \lambda w_{i,\kappa}-q_{i,\kappa} & r_{i,\kappa-1}(\lambda w_{i,\kappa-1}-q_{i,\kappa-1})\end{pmatrix}\Psi_i(b_{i,2\kappa-1},\lambda). \tag{17}$$

证明：由(5)可知在r恒等于零的子区间v是常数，而在q,w恒等于零的子区间上z是常数。通过反复应用(5)，即可得出结论。 □

引理3：设(9)~(11)成立，对于每一个$\lambda\in\mathbb{C}$，

$\Phi(t,\lambda)=\left[\phi_{ef}(t,\lambda)\right](t\in(a,c_1))$为系统(5)满足初始条件$\Phi(a,\lambda)=I$的基解矩阵；

$\Psi_i(t,\lambda)=\left[\psi_{i,ef}(t,\lambda)\right](t\in(c_i,c_{i+1}),c_{n+1}=b=c_{n,2m_n+1},i=1,2,\cdots,n)$为系统(5)满足初始条件$\Psi_i(c_i+,\lambda)=I$(此处$\Psi_i(c_i+,\lambda)=\Psi_i(c_{i,0},\lambda)=\Phi(c_i+,\lambda)$)的基解矩阵，则有

$$\Phi(b,\lambda)=\Psi_n(b,\lambda)G_n\Psi_{n-1}(c_n-,\lambda)G_{n-1}\Psi_{n-2}(c_{n-1}-,\lambda)\cdots G_1\Phi(c_1-,\lambda),$$

其中

$$G_i=\left[g_{i,ef}\right]_{2\times2}=-D_i^{-1}C_i.$$

证明：证明方法与文献[21]中引理3.3类似，具体参见文献[21]和[22]。 □

引理4：设(9)~(11)成立，对于每一个$\lambda\in\mathbb{C}$，

$\Phi(t,\lambda)=\left[\phi_{ef}(t,\lambda)\right](t\in(a,c_1))$为系统(5)满足初始条件$\Phi(a,\lambda)=I$的基解矩阵；

$\Psi_i(t,\lambda)=\left[\psi_{i,ef}(t,\lambda)\right](t\in(c_i,c_{i+1}),c_{n+1}=b=c_{n,2m_n+1},i=1,2,\cdots,n)$为系统(5)满足初始条件$\Psi_i(c_i+,\lambda)=I$(此处$\Psi_i(c_i+,\lambda)=\Psi_i(c_{i,0},\lambda)=\Phi(c_i+,\lambda)$)的基解矩阵。对于$\Phi(b,\lambda)$则有如下结果

$$\phi_{11}(b,\lambda)=R\prod_{i=1}^{n}R_iG^*G^{**}\prod_{k=0}^{m_0-1}(\lambda w_k-q_k)\prod_{i=1}^{m_1-1}(\lambda w_{1,i}-q_{1,i})\prod_{i=2}^{n}\left[\prod_{j=1}^{m_i-1}(\lambda w_{i,j}-q_{i,j})\right]+\phi'_{11}(b,\lambda),$$

$$\phi_{12}(b,\lambda)=R\prod_{i=1}^{n}R_iG^*G^{**}\prod_{k=0}^{m_0-1}(\lambda w_k-q_k)\prod_{i=1}^{m_1-1}(\lambda w_{1,i}-q_{1,i})\prod_{i=2}^{n}\left[\prod_{j=1}^{m_i-1}(\lambda w_{i,j}-q_{i,j})\right]+\phi'_{12}(b,\lambda),$$

$$\phi_{13}(b,\lambda)=R\prod_{i=1}^{n}R_iG^*G^{**}\prod_{k=1}^{m_0-1}(\lambda w_k-q_k)\prod_{i=1}^{m_1-1}(\lambda w_{1,i}-q_{1,i})\prod_{i=2}^{n}\left[\prod_{j=1}^{m_i-1}(\lambda w_{i,j}-q_{i,j})\right]+\phi'_{13}(b,\lambda),$$

$$\phi_{21}(b,\lambda)=R\prod_{i=1}^{n}R_iG^*G^{**}\prod_{k=0}^{m_0-1}(\lambda w_k-q_k)\prod_{i=1}^{m_1-1}(\lambda w_{1,i}-q_{1,i})\prod_{i=2}^{n}\left[\prod_{j=1}^{m_i}(\lambda w_{i,j}-q_{i,j})\right]+\phi'_{21}(b,\lambda),$$

$$\phi_{22}(b,\lambda)=R\prod_{i=1}^{n}R_iG^*G^{**}\prod_{k=0}^{m_0-1}(\lambda w_k-q_k)\prod_{i=1}^{m_1-1}(\lambda w_{1,i}-q_{1,i})\prod_{i=2}^{n}\left[\prod_{j=1}^{m_i}(\lambda w_{i,j}-q_{i,j})\right]+\phi'_{22}(b,\lambda),$$

$$\phi_{23}(b,\lambda)=R\prod_{i=1}^{n}R_iG^*G^{**}\prod_{k=1}^{m_0-1}(\lambda w_k-q_k)\prod_{i=1}^{m_1-1}(\lambda w_{1,i}-q_{1,i})\prod_{i=2}^{n}\left[\prod_{j=1}^{m_i}(\lambda w_{i,j}-q_{i,j})\right]+\phi'_{23}(b,\lambda),$$

$$\phi_{31}(b,\lambda)=R\prod_{i=1}^{n}R_iG^{*}G^{**}\prod_{k=0}^{m_0-1}(\lambda w_k-q_k)\prod_{i=1}^{m_1-1}(\lambda w_{1,i}-q_{1,i})\prod_{i=2}^{n}\left[\prod_{j=1}^{m_i}(\lambda w_{i,j}-q_{i,j})\right]+\phi'_{31}(b,\lambda),$$

$$\phi_{32}(b,\lambda)=R\prod_{i=1}^{n}R_iG^{*}G^{**}\prod_{k=0}^{m_0-1}(\lambda w_k-q_k)\prod_{i=1}^{m_1-1}(\lambda w_{1,i}-q_{1,i})\prod_{i=2}^{n}\left[\prod_{j=1}^{m_i}(\lambda w_{i,j}-q_{i,j})\right]+\phi'_{32}(b,\lambda),$$

$$\phi_{33}(b,\lambda)=R\prod_{i=1}^{n}R_iG^{*}G^{**}\prod_{k=0}^{m_0-1}(\lambda w_k-q_k)\prod_{i=1}^{m_1-1}(\lambda w_{1,i}-q_{1,i})\prod_{i=2}^{n}\left[\prod_{j=1}^{m_i}(\lambda w_{i,j}-q_{i,j})\right]+\phi'_{33}(b,\lambda).$$

其中

$$G^{*}=g_{1,11}(\lambda w_{1,0}-q_{1,0})+g_{1,21}(\lambda w_{1,0}-q_{1,0})+g_{1,31}+g_{1,12}(\lambda w_{1,0}-q_{1,0})+g_{1,22}(\lambda w_{1,0}-q_{1,0})$$
$$+g_{1,32}+g_{1,13}(\lambda w_{1,0}-q_{1,0})+g_{1,23}(\lambda w_{1,0}-q_{1,0})+g_{1,33},$$

$$G^{**}=\prod\nolimits_{i=2}^{n}\Big[g_{i,11}(\lambda w_{i,0}-q_{i,0})+g_{i,21}(\lambda w_{i-1,m_{i-1}}-q_{i-1,m_{i-1}})+g_{i,12}(\lambda w_{i-1,m_{i-1}}-q_{i-1,m_{i-1}})$$
$$+g_{i,31}+g_{i,32}+g_{i,13}(\lambda w_{i-1,m_{i-1}}-q_{i-1,m_{i-1}})+g_{i,23}(\lambda w_{i-1,m_{i-1}}-q_{i-1,m_{i-1}})+g_{1,33}\Big],$$

$$R=\prod_{k=0}^{m_0-1}r_k,\ R_i=\prod_{j=0}^{m_i-1}r_{i,j},\ \phi'_{ef}(b,\lambda)=o\left(R\prod_{i=1}^{n}R_i\right).$$

证明： 由引理2可知

$$\Phi(c_1-,\lambda)=\Phi(a_{2m_0+1},\lambda)$$
$$=\begin{pmatrix}1&0&r_{m_0-1}\\0&1&r_{m_0-1}\\\lambda w_{m_0}-q_{m_0}&\lambda w_{m_0}-q_{m_0}&(\lambda w_{m_0}-q_{m_0})r_{m_0-1}\end{pmatrix}\Phi(a_{2m_0-1},\lambda)$$
$$=\begin{pmatrix}1&1&r_{m_0-1}\\0&1&r_{m_0-1}\\\lambda w_{m_0}-q_{m_0}&\lambda w_{m_0}-q_{m_0}&(\lambda w_{m_0}-q_{m_0})r_{m_0-1}\end{pmatrix}$$
$$\times\begin{pmatrix}1&1&r_{m_0-2}\\0&1&r_{m_0-2}\\\lambda w_{m_0-1}-q_{m_0-1}&\lambda w_{m_0-1}-q_{m_0-1}&(\lambda w_{m_0-1}-q_{m_0-1})r_{m_0-2}\end{pmatrix}\Phi(a_{2m_0-3},\lambda)$$
$$=\begin{pmatrix}\theta_{11}&\theta_{12}&\theta_{13}\\\theta_{21}&\theta_{22}&\theta_{23}\\\theta_{31}&\theta_{32}&\theta_{33}\end{pmatrix}\Phi(a_{2m_0-3},\lambda),$$

其中

$$\theta_{11}=1+r_{m_0-1}(\lambda w_{m_0-1}-q_{m_0-1})=r_{m_0-1}(\lambda w_{m_0-1}-q_{m_0-1})+o(\lambda w_{m_0-1}-q_{m_0-1}),$$

$$\theta_{12}=2+r_{m_0-1}(\lambda w_{m_0-1}-q_{m_0-1})=r_{m_0-1}(\lambda w_{m_0-1}-q_{m_0-1})+o(\lambda w_{m_0-1}-q_{m_0-1}),$$

$$\theta_{13}=2r_{m_0-2}+r_{m_0-1}r_{m_0-2}(\lambda w_{m_0-1}-q_{m_0-1})=r_{m_0-1}r_{m_0-2}(\lambda w_{m_0-1}-q_{m_0-1})+o\left(r_{m_0-1}r_{m_0-2}(\lambda w_{m_0-1}-q_{m_0-1})\right),$$

$$\theta_{21}=r_{m_0-1}(\lambda w_{m_0-1}-q_{m_0-1})=r_{m_0-1}(\lambda w_{m_0-1}-q_{m_0-1})+o(\lambda w_{m_0-1}-q_{m_0-1}),$$

$$\theta_{22}=1+r_{m_0-1}\left(\lambda w_{m_0-1}-q_{m_0-1}\right)$$
$$=r_{m_0-1}\left(\lambda w_{m_0-1}-q_{m_0-1}\right)+o\left(r_{m_0-1}\left(\lambda w_{m_0-1}-q_{m_0-1}\right)\right),$$
$$\theta_{23}=r_{m_0-2}+r_{m_0-1}r_{m_0-2}\left(\lambda w_{m_0-1}-q_{m_0-1}\right)$$
$$=r_{m_0-1}r_{m_0-2}\left(\lambda w_{m_0-1}-q_{m_0-1}\right)+o\left(r_{m_0-1}r_{m_0-2}\left(\lambda w_{m_0-1}-q_{m_0-1}\right)\right),$$
$$\theta_{31}=\lambda w_{m_0}-q_{m_0}+r_{m_0-1}\left(\lambda w_{m_0}-q_{m_0}\right)\left(\lambda w_{m_0-1}-q_{m_0-1}\right)$$
$$=r_{m_0-1}\left(\lambda w_{m_0}-q_{m_0}\right)\left(\lambda w_{m_0-1}-q_{m_0-1}\right)+o\left(r_{m_0-1}\left(\lambda w_{m_0}-q_{m_0}\right)\left(\lambda w_{m_0-1}-q_{m_0-1}\right)\right),$$
$$\theta_{32}=2\left(\lambda w_{m_0}-q_{m_0}\right)+r_{m_0-1}\left(\lambda w_{m_0}-q_{m_0}\right)\left(\lambda w_{m_0-1}-q_{m_0-1}\right)$$
$$=r_{m_0-1}\left(\lambda w_{m_0}-q_{m_0}\right)\left(\lambda w_{m_0-1}-q_{m_0-1}\right)+o\left(r_{m_0-1}\left(\lambda w_{m_0}-q_{m_0}\right)\left(\lambda w_{m_0-1}-q_{m_0-1}\right)\right),$$
$$\theta_{33}=2r_{m_0-2}\left(\lambda w_{m_0}-q_{m_0}\right)+r_{m_0-1}r_{m_0-2}\left(\lambda w_{m_0}-q_{m_0}\right)\left(\lambda w_{m_0-1}-q_{m_0-1}\right)$$
$$=r_{m_0-1}r_{m_0-2}\left(\lambda w_{m_0}-q_{m_0}\right)\left(\lambda w_{m_0-1}-q_{m_0-1}\right)+o\left(r_{m_0-1}r_{m_0-2}\left(\lambda w_{m_0}-q_{m_0}\right)\left(\lambda w_{m_0-1}-q_{m_0-1}\right)\right).$$

由于

$$\Phi\left(a_{2m_0-3},\lambda\right)=\begin{pmatrix}1 & 1 & r_{m_0-3}\\ 0 & 1 & r_{m_0-3}\\ \lambda w_{m_0-2}-q_{m_0-2} & \lambda w_{m_0-2}-q_{m_0-2} & \left(\lambda w_{m_0-2}-q_{m_0-2}\right)r_{m_0-3}\end{pmatrix}\Phi\left(a_{2m_0-5},\lambda\right),$$

故

$$\Phi\left(c_1-,\lambda\right)=\begin{pmatrix}\theta_{11} & \theta_{12} & \theta_{13}\\ \theta_{21} & \theta_{22} & \theta_{23}\\ \theta_{31} & \theta_{32} & \theta_{33}\end{pmatrix}\Phi\left(a_{2m_0-3},\lambda\right)$$
$$=\begin{pmatrix}\theta_{11} & \theta_{12} & \theta_{13}\\ \theta_{21} & \theta_{22} & \theta_{23}\\ \theta_{31} & \theta_{32} & \theta_{33}\end{pmatrix}\begin{pmatrix}1 & 1 & r_{m_0-3}\\ 0 & 1 & r_{m_0-3}\\ \lambda w_{m_0-2}-q_{m_0-2} & \lambda w_{m_0-2}-q_{m_0-2} & (\lambda w_{m_0-2}-q_{m_0-2})r_{m_0-3}\end{pmatrix}\Phi\left(a_{2m_0-5},\lambda\right)$$
$$=\begin{pmatrix}\eta_{11} & \eta_{12} & \eta_{13}\\ \eta_{21} & \eta_{22} & \eta_{23}\\ \eta_{31} & \eta_{32} & \eta_{33}\end{pmatrix}\Phi\left(a_{2m_0-5},\lambda\right),$$

其中

$$\eta_{11}=\theta_{11}+\left(\lambda w_{m_0-2}-q_{m_0-2}\right)\theta_{13}$$
$$=r_{m_0-1}r_{m_0-2}\left(\lambda w_{m_0-1}-q_{m_0-1}\right)\left(\lambda w_{m_0-2}-q_{m_0-2}\right)+o\left(r_{m_0-1}r_{m_0-2}\left(\lambda w_{m_0-1}-q_{m_0-1}\right)\left(\lambda w_{m_0-2}-q_{m_0-2}\right)\right),$$
$$\eta_{12}=\theta_{11}+\theta_{12}+\left(\lambda w_{m_0-2}-q_{m_0-2}\right)\theta_{13}$$
$$=r_{m_0-1}r_{m-2}\left(\lambda w_{m_0-1}-q_{m_0-1}\right)\left(\lambda w_{m_0-2}-q_{m_0-2}\right)+o\left(r_{m_0-1}r_{m_0-2}\left(\lambda w_{m_0-1}-q_{m_0-1}\right)\left(\lambda w_{m_0-2}-q_{m_0-2}\right)\right),$$
$$\eta_{13}=r_{m_0-3}\theta_{11}+r_{m_0-3}\theta_{12}+r_{m_0-3}\left(\lambda w_{m_0-2}-q_{m_0-2}\right)\theta_{13}=r_{m_0-1}r_{m_0-2}r_{m_0-3}\left(\lambda w_{m_0-1}-q_{m_0-1}\right)\left(\lambda w_{m_0-2}-q_{m_0-2}\right)$$
$$+o\left(r_{m_0-1}r_{m_0-2}r_{m_0-3}\left(\lambda w_{m_0-1}-q_{m_0-1}\right)\left(\lambda w_{m_0-2}-q_{m_0-2}\right)\right),$$

$$\eta_{21}=\theta_{21}+\left(\lambda w_{m_0-2}-q_{m_0-2}\right)\theta_{23}=r_{m_0-1}r_{m_0-2}\left(\lambda w_{m_0-1}-q_{m_0-1}\right)\left(\lambda w_{m_0-2}-q_{m_0-2}\right)$$
$$+o\left(r_{m_0-1}r_{m_0-2}\left(\lambda w_{m_0-1}-q_{m_0-1}\right)\left(\lambda w_{m_0-2}-q_{m_0-2}\right)\right),$$

$$\eta_{22}=\theta_{21}+\theta_{22}+\left(\lambda w_{m_0-2}-q_{m_0-2}\right)\theta_{23}=r_{m_0-1}r_{m_0-2}\left(\lambda w_{m_0-1}-q_{m_0-1}\right)\left(\lambda w_{m_0-2}-q_{m_0-2}\right)$$
$$+o\left(r_{m_0-1}r_{m_0-2}\left(\lambda w_{m_0-1}-q_{m_0-1}\right)\left(\lambda w_{m_0-2}-q_{m_0-2}\right)\right),$$

$$\eta_{23}=r_{m_0-3}\theta_{21}+r_{m_0-3}\theta_{22}+r_{m_0-3}\left(\lambda w_{m_0-2}-q_{m_0-2}\right)\theta_{23}=r_{m_0-1}r_{m_0-2}r_{m_0-3}\left(\lambda w_{m_0-1}-q_{m_0-1}\right)\left(\lambda w_{m_0-2}-q_{m_0-2}\right)$$
$$+o\left(r_{m_0-1}r_{m_0-2}r_{m_0-3}\left(\lambda w_{m_0-1}-q_{m_0-1}\right)\left(\lambda w_{m_0-2}-q_{m_0-2}\right)\right),$$

$$\eta_{31}=\theta_{31}+\left(\lambda w_{m_0-2}-q_{m_0-2}\right)\theta_{33}=r_{m_0-1}r_{m_0-2}\left(\lambda w_{m_0}-q_{m_0}\right)\left(\lambda w_{m_0-1}-q_{m_0-1}\right)\left(\lambda w_{m_0-2}-q_{m_0-2}\right)$$
$$+o\left(r_{m_0-1}r_{m_0-2}\left(\lambda w_{m_0}-q_{m_0}\right)\left(\lambda w_{m_0-1}-q_{m_0-1}\right)\left(\lambda w_{m_0-2}-q_{m_0-2}\right)\right),$$

$$\eta_{32}=\theta_{31}+\theta_{32}+\left(\lambda w_{m_0-2}-q_{m_0-2}\right)\theta_{33}=r_{m_0-1}r_{m_0-2}\left(\lambda w_{m_0}-q_{m_0}\right)\left(\lambda w_{m_0-1}-q_{m_0-1}\right)\left(\lambda w_{m_0-2}-q_{m_0-2}\right)$$
$$+o(r_{m_0-1}r_{m_0-2}\left(\lambda w_{m_0}-q_{m_0}\right)\left(\lambda w_{m_0-1}-q_{m_0-1}\right)\left(\lambda w_{m_0-2}-q_{m_0-2}\right),$$

$$\eta_{33}=r_{m_0-3}\theta_{31}+r_{m_0-3}\theta_{32}+r_{m_0-3}\left(\lambda w_{m_0-2}-q_{m_0-2}\right)\theta_{33}=r_{m_0-1}r_{m_0-2}r_{m_0-3}\left(\lambda w_{m_0}-q_{m_0}\right)\left(\lambda w_{m_0-1}-q_{m_0-1}\right)\left(\lambda w_{m_0-2}-q_{m_0-2}\right)$$
$$+o\left(r_{m_0-1}r_{m_0-2}r_{m_0-3}\left(\lambda w_{m_0}-q_{m_0}\right)\left(\lambda w_{m_0-1}-q_{m_0-1}\right)\left(\lambda w_{m_0-2}-q_{m_0-2}\right)\right),$$

$$\cdots$$

重复上述方法，我们得到

$$\Phi\left(c_1-,\lambda\right)=\begin{pmatrix}\xi_{11}&\xi_{12}&\xi_{13}\\\xi_{21}&\xi_{22}&\xi_{23}\\\xi_{31}&\xi_{32}&\xi_{33}\end{pmatrix}\Phi\left(a_1,\lambda\right),$$

其中

$$\xi_{11}=\prod_{k=1}^{m_0-1}r_k\prod_{k=1}^{m_0-1}\left(\lambda w_k-q_k\right)+o\left(\prod_{k=1}^{m_0-1}r_k\prod_{k=1}^{m_0-1}\left(\lambda w_k-q_k\right)\right),$$

$$\xi_{12}=\prod_{k=1}^{m_0-1}r_k\prod_{k=1}^{m_0-1}\left(\lambda w_k-q_k\right)+o\left(\prod_{k=1}^{m_0-1}r_k\prod_{k=1}^{m_0-1}\left(\lambda w_k-q_k\right)\right),$$

$$\xi_{13}=\prod_{k=0}^{m_0-1}r_k\prod_{k=1}^{m_0-1}\left(\lambda w_k-q_k\right)+o\left(\prod_{k=0}^{m_0-1}r_k\prod_{k=1}^{m_0-1}\left(\lambda w_k-q_k\right)\right),$$

$$\xi_{21}=\prod_{k=1}^{m_0-1}r_k\prod_{k=1}^{m_0-1}\left(\lambda w_k-q_k\right)+o\left(\prod_{k=1}^{m_0-1}r_k\prod_{k=1}^{m_0-1}\left(\lambda w_k-q_k\right)\right),$$

$$\xi_{22}=\prod_{k=1}^{m_0-1}r_k\prod_{k=1}^{m_0-1}\left(\lambda w_k-q_k\right)+o\left(\prod_{k=1}^{m_0-1}r_k\prod_{k=1}^{m_0-1}\left(\lambda w_k-q_k\right)\right),$$

$$\xi_{23}=\prod_{k=0}^{m_0-1}r_k\prod_{k=1}^{m_0-1}\left(\lambda w_k-q_k\right)+o\left(\prod_{k=0}^{m_0-1}r_k\prod_{k=1}^{m_0-1}\left(\lambda w_k-q_k\right)\right),$$

$$\xi_{31}=\prod_{k=1}^{m_0-1}r_k\prod_{k=1}^{m_0-1}\left(\lambda w_k-q_k\right)+o\left(\prod_{k=1}^{m_0-1}r_k\prod_{k=1}^{m_0-1}\left(\lambda w_k-q_k\right)\right),$$

$$\xi_{32}=\prod_{k=1}^{m_0-1}r_k\prod_{k=1}^{m_0-1}(\lambda w_k-q_k)+o\left(\prod_{k=1}^{m_0-1}r_k\prod_{k=1}^{m_0-1}(\lambda w_k-q_k)\right),$$

$$\xi_{33}=\prod_{k=0}^{m_0-1}r_k\prod_{k=1}^{m_0-1}(\lambda w_k-q_k)+o\left(\prod_{k=0}^{m_0-1}r_k\prod_{k=1}^{m_0-1}(\lambda w_k-q_k)\right).$$

且

$$\Phi(a_1,\lambda)=\begin{pmatrix}1 & a_1-a_0 & 0\\ 0 & 1 & 0\\ \lambda w_0-q_0 & (\lambda w_0-q_0)(a_1-a_0) & 1\end{pmatrix},$$

故

$$\begin{aligned}\Phi(c_1-,\lambda)&=\begin{pmatrix}\xi_{11} & \xi_{12} & \xi_{13}\\ \xi_{21} & \xi_{22} & \xi_{23}\\ \xi_{31} & \xi_{32} & \xi_{33}\end{pmatrix}\Phi(a_1,\lambda)\\ &=\begin{pmatrix}\xi_{11} & \xi_{12} & \xi_{13}\\ \xi_{21} & \xi_{22} & \xi_{23}\\ \xi_{31} & \xi_{32} & \xi_{33}\end{pmatrix}\begin{pmatrix}1 & a_1-a_0 & 0\\ 0 & 1 & 0\\ \lambda w_0-q_0 & (\lambda w_0-q_0)(a_1-a_0) & 1\end{pmatrix}\\ &=\begin{pmatrix}\phi_{11}(c_1-,\lambda) & \phi_{12}(c_1-,\lambda) & \phi_{13}(c_1-,\lambda)\\ \phi_{21}(c_1-,\lambda) & \phi_{22}(c_1-,\lambda) & \phi_{23}(c_1-,\lambda)\\ \phi_{31}(c_1-,\lambda) & \phi_{32}(c_1-,\lambda) & \phi_{33}(c_1-,\lambda)\end{pmatrix}.\end{aligned}$$

因此

$$\begin{aligned}\phi_{11}(c_1-,\lambda)&=\prod_{k=0}^{m_0-1}r_k\prod_{k=0}^{m_0-1}(\lambda w_k-q_k)+o\left(\prod_{k=0}^{m_0-1}r_k\prod_{k=0}^{m_0-1}(q_k-\lambda w_k)\right),\\ \phi_{12}(c_1-,\lambda)&=\prod_{k=0}^{m_0-1}r_k\prod_{k=0}^{m_0-1}(\lambda w_k-q_k)+o\left(\prod_{k=0}^{m_0-1}r_k\prod_{k=0}^{m_0-1}(q_k-\lambda w_k)\right),\\ \phi_{13}(c_1-,\lambda)&=\prod_{k=0}^{m_0-1}r_k\prod_{k=1}^{m_0-1}(\lambda w_k-q_k)+o\left(\prod_{k=0}^{m_0-1}r_k\prod_{k=1}^{m_0}(q_k-\lambda w_k)\right),\\ \phi_{21}(c_1-,\lambda)&=\prod_{k=0}^{m_0-1}r_k\prod_{k=0}^{m_0-1}(\lambda w_k-q_k)+o\left(\prod_{k=0}^{m_0-1}r_k\prod_{k=0}^{m_0-1}(q_k-\lambda w_k)\right),\\ \phi_{22}(c_1-,\lambda)&=\prod_{k=0}^{m_0-1}r_k\prod_{k=0}^{m_0-1}(\lambda w_k-q_k)+o\left(\prod_{k=0}^{m_0-1}r_k\prod_{k=0}^{m_0-1}(q_k-\lambda w_k)\right),\\ \phi_{23}(c_1-,\lambda)&=\prod_{k=0}^{m_0-1}r_k\prod_{k=1}^{m_0-1}(\lambda w_k-q_k)+o\left(\prod_{k=0}^{m_0-1}r_k\prod_{k=1}^{m_0-1}(q_k-\lambda w_k)\right),\\ \phi_{31}(c_1-,\lambda)&=\prod_{k=0}^{m_0-1}r_k\prod_{k=0}^{m_0-1}(\lambda w_k-q_k)+o\left(\prod_{k=0}^{m_0-1}r_k\prod_{k=0}^{m_0-1}(q_k-\lambda w_k)\right),\\ \phi_{32}(c_1-,\lambda)&=\prod_{k=0}^{m_0-1}r_k\prod_{k=0}^{m_0-1}(\lambda w_k-q_k)+o\left(\prod_{k=0}^{m_0-1}r_k\prod_{k=0}^{m_0-1}(q_k-\lambda w_k)\right),\\ \phi_{33}(c_1-,\lambda)&=\prod_{k=0}^{m_0-1}r_k\prod_{k=1}^{m_0-1}(\lambda w_k-q_k)+o\left(\prod_{k=0}^{m_0-1}r_k\prod_{k=1}^{m_0-1}(q_k-\lambda w_k)\right).\end{aligned}\tag{18}$$

重复上述方法，可得

$$
\begin{aligned}
\psi_{i,11}(c_{i+1}-,\lambda) &= \prod_{i=0}^{m_i-1} r_{i,j}\prod_{j=0}^{m_i-1}(\lambda w_{i,j}-q_{i,j}) + o\left(\prod_{i=0}^{m_i-1} r_{i,j}\prod_{j=0}^{m_i-1}(\lambda w_{i,j}-q_{i,j})\right),\\
\psi_{i,12}(c_{i+1}-,\lambda) &= \prod_{i=0}^{m_i-1} r_{i,j}\prod_{j=0}^{m_i-1}(\lambda w_{i,j}-q_{i,j}) + o\left(\prod_{i=0}^{m_i-1} r_{i,j}\prod_{j=0}^{m_i-1}(\lambda w_{i,j}-q_{i,j})\right),\\
\psi_{i,13}(c_{i+1}-,\lambda) &= \prod_{i=0}^{m_i-1} r_{i,j}\prod_{j=1}^{m_i-1}(\lambda w_{i,j}-q_{i,j}) + o\left(\prod_{i=0}^{m_i-1} r_{i,j}\prod_{j=1}^{m_i-1}(\lambda w_{i,j}-q_{i,j})\right),\\
\psi_{i,21}(c_{i+1}-,\lambda) &= \prod_{i=0}^{m_i-1} r_{i,j}\prod_{j=0}^{m_i-1}(\lambda w_{i,j}-q_{i,j}) + o\left(\prod_{i=0}^{m_i-1} r_{i,j}\prod_{j=0}^{m_i-1}(\lambda w_{i,j}-q_{i,j})\right),\\
\psi_{i,22}(c_{i+1}-,\lambda) &= \prod_{i=0}^{m_i-1} r_{i,j}\prod_{j=0}^{m_i-1}(\lambda w_{i,j}-q_{i,j}) + o\left(\prod_{i=0}^{m_i-1} r_{i,j}\prod_{j=0}^{m_i-1}(\lambda w_{i,j}-q_{i,j})\right),\\
\psi_{i,23}(c_{i+1}-,\lambda) &= \prod_{i=0}^{m_i-1} r_{i,j}\prod_{j=1}^{m_i-1}(\lambda w_{i,j}-q_{i,j}) + o\left(\prod_{i=0}^{m_i-1} r_{i,j}\prod_{j=1}^{m_i-1}(\lambda w_{i,j}-q_{i,j})\right),\\
\psi_{i,31}(c_{i+1}-,\lambda) &= \prod_{i=0}^{m_i-1} r_{i,j}\prod_{j=0}^{m_i-1}(\lambda w_{i,j}-q_{i,j}) + o\left(\prod_{i=0}^{m_i-1} r_{i,j}\prod_{j=0}^{m_i-1}(\lambda w_{i,j}-q_{i,j})\right),\\
\psi_{i,32}(c_{i+1}-,\lambda) &= \prod_{i=0}^{m_i-1} r_{i,j}\prod_{j=0}^{m_i-1}(\lambda w_{i,j}-q_{i,j}) + o\left(\prod_{i=0}^{m_i-1} r_{i,j}\prod_{j=0}^{m_i-1}(\lambda w_{i,j}-q_{i,j})\right),\\
\psi_{i,33}(c_{i+1}-,\lambda) &= \prod_{i=0}^{m_i-1} r_{i,j}\prod_{j=1}^{m_i-1}(\lambda w_{i,j}-q_{i,j}) + o\left(\prod_{i=0}^{m_i-1} r_{i,j}\prod_{j=1}^{m_i-1}(\lambda w_{i,j}-q_{i,j})\right).
\end{aligned}
\tag{19}
$$

结合引理3可知

$$\Phi(b,\lambda)=\Psi_n(b,\lambda)G_n\Psi_{n-1}(c_n-,\lambda)G_{n-1}\Psi_{n-2}(c_{n-1}-,\lambda)\cdots G_1\Phi(c_1-,\lambda),$$

综合(18)和(19)，结论得证。 □

定理3：设 $m,n\in\mathbb{N}$， $g_{1,12}g_{i,21}\neq 0$，且(9)~(11)成立，则问题(1)~(3)至多有 $m_0+m_1+\cdots+m_n$ 个特征值。

证明：由引理1可知

$$\Delta(\lambda)=\det(A)+\det(B)+\sum_{i=1}^{4}\sum_{j=1}^{4}c_{ij}\phi_{ij}+\sum_{1\le i,j,k,l\le 4}d_{ijkl}\phi_{ij}\phi_{kl},$$

再由引理4可知， $\phi_{11}(b,\lambda),\phi_{12}(b,\lambda),\phi_{13}(b,\lambda)$ 关于 λ 的次数都 $m_0+m_1+\cdots+m_n$， 而 $\phi_{21}(b,\lambda),\phi_{22}(b,\lambda)$，$\phi_{23}(b,\lambda),\phi_{31}(b,\lambda),\phi_{32}(b,\lambda),\phi_{33}(b,\lambda)$ 关于 λ 的次数都为 $m_0+m_1+\cdots+m_n+n-1$，由代数学基本定理可知，$\Delta(\lambda)=0$ 时，至多有 $m_0+m_1+\cdots+m_n$ 个特征值。其它情况判断函数 $\Delta(\lambda)$ 关于 λ 的次数必定小于或等于 $m_0+m_1+\cdots+m_n$。定理得证。 □

参考文献

[1] Greenberg, M. (1987) Third Order Linear Differential Equations. Reidel, Dordrecht.

[2] Wu, Y.Y. and Zhao, Z.Q. (2011) Positive Solutions for Third-Order Boundary Value Problems with Change of Signs. *Applied Mathematics and Computation*, **218**, 2744-2749. https://doi.org/10.1016/j.amc.2011.08.015

[3] Atkinson, F.V. (1964) Discrete and Continuous Boundary Problems. Academic Press, New York/London. https://doi.org/10.1063/1.3051875

[4] Kong, Q., Wu, H. and Zettl, A. (2001) Sturm-Liouville Problems with Finite Spectrum. *Journal of Mathematical Analysis and Applications*, **263**, 748-762. https://doi.org/10.1006/jmaa.2001.7661

[5] Kong, Q. and Zettl, A. (1996) Eigenvalues of Regular Sturm-Liouville Problems. *Journal of Differential Equations*,

131, 1-19. https://doi.org/10.1006/jdeq.1996.0154

[6] Yang, C.F. and Yang, X.P. (2009) An Interior Inverse Problem for the Sturm-Liouville Operator with Discontinuous Conditions. *Applied Mathematics Letters*, **22**, 1315-1319. https://doi.org/10.1016/j.aml.2008.12.001

[7] Kong, Q., Wu, H. and Zettl, A. (1999) Dependence of the *n*th Sturm-Liouville Eigenvalue on the Problem. *Journal of Differential Equations*, **156**, 328-354. https://doi.org/10.1006/jdeq.1998.3613

[8] Mukhtarov, O. and Aydemir, S.K. (2015) Eigenfunction Expansion for Sturm-Liouville Problems with Transmission Conditions at One Interior Point. *Acta Mathematica Scientia*, **35**, 639-649. https://doi.org/10.1016/S0252-9602(15)30010-2

[9] Ao, J.J., Sun, J. and Zhang, M.Z. (2011) The Finite Spectrum of Sturm-Liouville Problems with Transmission Conditions. *Applied Mathematics and Computation*, **218**, 1166-1173. https://doi.org/10.1016/j.amc.2011.05.033

[10] Chanane, B. (2010) Sturm-Liouville Problems with Impulse Effects. *Applied Mathematics and Computation*, **190**, 610-626. https://doi.org/10.1016/j.amc.2007.01.092

[11] Mukhtarov, O.S., Kadakal, M. and Muhtarov, F.S. (2004) Eigenvalues and Nomarlized Eigenfunctions of Discontinuous Sturm-Liouville Problem with Transmission Conditions. *Reports on Mathematical Physics*, **54**, 41-56. https://doi.org/10.1016/S0034-4877(04)80004-1

[12] Ao, J.J., Sun, J. and Zettl, A. (2015) Finite Spectrum of 2*n*th Order Boundary Value Problems. *Applied Mathematics Letters*, **42**, 1-8. https://doi.org/10.1016/j.aml.2014.10.003

[13] Zhang, X. and Sun, J. (2013) A Class of Fourth-Order Differential Operator with Eigenparameter-Dependent Boundary and Transmission Conditions. *Acta Mathematicae Applicatae Sinica*, **26**, 205-219.

[14] Ao, J.J., Bo, F.Z. and Sun, J. (2014) Fourth Order Boundary Value Problems with Finite Spectrum. *Applied Mathematics and Computation*, **244**, 952-958. https://doi.org/10.1016/j.amc.2014.07.054

[15] Mirzaei, H. and Ghanbari, K. (2015) Matrix Representation of a Sixth Order Sturm-Liouville Problem and Related Inverse Problem with Finite Spectrum. *Bulletin of the Iranian Mathematical Society*, **41**, 1031-1043.

[16] Ao, J.J. (2017) On Two Classes of Third Boundary Value Problems with Finite Spectrum. *Bulletin of the Iranian Mathematical Society*, **43**, 1089-1099.

[17] Yang, C. and Sun, J. (2012) Existence of Multiple Positive Solutions for Fourth-Order Boundary Value Problems in Banach Spaces. *Boundary Value Problems*, **107**, 1-13. https://doi.org/10.1186/1687-2770-2012-13

[18] Ge, S., Wang, W.J. and Ao, J. (2014) Matrix Representations of Fourth Order Boundary Value Problems with Periodic Boundary Conditions. *Applied Mathematics and Computation*, **227**, 601-609. https://doi.org/10.1016/j.amc.2013.11.059

[19] Ao, J.J. and Sun, J. (2014) Matrix Representations of Sturm-Liouville Problems with Coupled Eigenparameter-Dependent Boundary Conditions. *Applied Mathematics and Computation*, **244**, 142-148. https://doi.org/10.1016/j.amc.2014.06.096

[20] Everitt, W.N. and Race, D. (1978) On Necessary and Sufficient Conditions for the Existence of Carathéodory Solutions of Ordinary Differential Equations. *Quaestiones Mathematicae*, **2**, 507-512. https://doi.org/10.1080/16073606.1978.9631549

[21] Xu, M.Z., Wang, W.Y. and Ao, J.J. (2018) Finite Spectrum of Sturm-Liouville Problems with *n* Transmission Conditions. *Iranian Journal of Science and Technology*, *Transactions A*: *Science*, **42**, 811-817. https://doi.org/10.1007/s40995-016-0072-1

[22] Zettl, A. (2005) Sturm-Liouville Theory, Mathematical Surveys and Monographs. American Mathematical Society, Providence, 121.