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时空分数阶的广义b-方程组的精确解

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摘 要

在这篇文章中，我们考虑在**Riemann−Liouville**意义下的时空分数阶广义b-方程组的精确解。我们将通过拉普拉斯变换给出这个方程组带初值条件的解析解。此外，本文将通过一个辅助方程证明这个方程组具有相同的解析解。

关键词

精确解，广义b-方程组，拉普拉斯变换，分数阶微分

Exact Solutions for the Generalized b-Family Equations with Fractional Time and Spatial Derivatives

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Abstract

In this paper, the generalized b-family equations with fractional time and spatial derivatives are considered. The fractional derivative is described in the Riemann−Liouville sense. We present the analytical solutions of the fractional equations with initial conditions by the Laplace transform of sequential fractional derivatives. Moreover, the generalized b-family equations with fractional time and spatial derivatives possess common analytical solutions which are solved by considering a simple equation.

Keywords

Exact Solutions, Generalized b-Family Equations, Laplace Transforms, Sequential Fractional Derivatives

1. 引言

考虑如下广义b–方程组

$$u_t - u_{xxt} + (b+1)uu_x = bu_xu_{xx} + uu_{xxx} + \alpha^{'}(u-u_{xx}) + \beta^{'}(u-u_{xx})_x + \gamma^{'}(u-u_{xx})_{xx}, \quad (1.1)$$

其中$\alpha^{'}$, $\beta^{'}$, $\gamma^{'}$ 和b是常数。当$\alpha^{'}=\beta^{'}=\gamma^{'}=0$时，方程(1.1)可以化为

$$u_t - u_{xxt} + (b+1)uu_x = bu_xu_{xx} + uu_{xxx}. \quad (1.2)$$

Degasperis 与Procesi [1] 证明了方程(1.2)的渐近可积性。当$b=2$时，方程(1.1)变为Camassa−Holm (CH) 浅水波方程 [2]

$$u_t - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx}. \quad (1.3)$$

当$\beta^{'}=\gamma^{'}=0$ 和$b=2$，方程(1.1) 变为带弱色散项的CH型方程：

$$u_t - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx} + \alpha^{'}(u-u_{xx}). \quad (1.5)$$

当$\alpha' = \gamma' = 0$ 和$b = 3$，方程(1.1) 变为广义的DP方程 [3] [4]：

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx} + \beta'(u - u_{xx})_x. \tag{1.7}$$

当$\alpha' = 0$ 和$b = 2$，方程变为带粘性项的CH方程 [5]：

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} + \beta'(u - u_{xx})_x + \gamma'(u - u_{xx})_{xx}. \tag{1.8}$$

近年来，分数阶微分方程在物理和工程中都有很多应用 [6] [7] [8] [9]. 许多重要的现象，例如电磁学、声学、粘弹性和材料科学，都能被分数阶微分方程描述 [10] [11] [12] [13].

在这篇文章中，我们考虑如下的时空分数阶的广义的$b-$方程组：

$$D_t^{\gamma} u - D_t^{\gamma} D_x^{\beta} D_x^{\alpha} u + (b+1) u D_x^{\alpha} u = b D_x^{\alpha} u D_x^{\beta} D_x^{\alpha} u + u D_x^{\alpha} D_x^{\beta} D_x^{\alpha} u + \alpha' m + \beta' m_x + \gamma' m_{xx}, \tag{1.11}$$

其中$m = u - D_x^{\beta} D_x^{\alpha} u$, γ 是时间微分的阶数，$0 < \alpha \leq 1$ 与$0 < \beta \leq 1$ 是空间微分的阶数。我们这里主要考虑的是Riemann−Liouville (RL)意义下的分数阶微分。这篇文章的主要目的是研究方程(1.11)的精确解。首先我们将方程化简为每一项都包含表达式$u - D_x^{\beta} D_x^{\alpha} u$的方程。然后，我们将获得方程$u - D_x^{\beta} D_x^{\alpha} u = 0$的精确解。最后，我们将通过这个辅助方程获得方程(1.11)的精确解。

2. 引理和注释

定义2.1. 分数阶的数学定义有几种不同的形式 [14] [15]。我们主要采用在RL意义下的分数阶的定义：

$$D_t^{-q} f(t) = \frac{d^{-q} f(t)}{dt^{-q}} = \frac{1}{\Gamma(q)} \int_0^t \frac{f(x)dx}{(t-x)^{1-q}}, \tag{2.1}$$

和

$$D_t^{q} f(t) = \frac{d^n}{dt^n} \left(\frac{d^{-(n-q)} f(t)}{dt^{-(n-q)}} \right) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t \frac{f(x)dx}{(t-x)^{1-n+q}}, \tag{2.2}$$

其中q ($q > 0$ 和$q \in R$) 是微分的阶数，n 是整数并满足$n - 1 \leq q < n$ 且Γ 是欧拉伽马函数。加入q成为一个整数，我们将回到通常的定义，即

$$D_t^{q} f(t) = \left(\frac{d}{dt} \right)^q f(t), \quad (q = 1, 2, 3, \cdots). \tag{2.3}$$

其中RL分数阶微分的阶数不等于0，但是

$$D_t^{\alpha} C = \frac{C t^{-\alpha}}{\Gamma(1-\alpha)}. \tag{2.4}$$

如果$f(\tau)$ 和$g(\tau)$ 在$[a,t]$是连续的，则RL微分满足莱布尼茨法则：

$$ {}_aD^q(g(t)f(t)) = \sum_{k=0}^{\infty} \begin{pmatrix} q \\ k \end{pmatrix} g^{(k)}(t)_a D^{q-k} f(t). \tag{2.5} $$

定义2.2. 让我们来回顾一下拉普拉斯变换的基本原理。关于复变量s的函数$F(s)$定义为：

$$ F(s) = L\{f(t); s\} = \int_0^{\infty} e^{-st} f(t)dt \tag{2.11} $$

就被称为函数$f(t)$的拉普拉斯变换。他们之间存在如下的拉普拉斯逆变换：

$$ f(t) = L^{-1}\{F(s); t\} = \int_{c-i\infty}^{c+i\infty} e^{st} F(s)ds, \quad Re(s) > c_0, \tag{2.12} $$

其中c_0落在拉普拉斯积分(2.11)绝对收敛的右半平面中。

3. 方程(1.11)的精确解

我们将利用第二节的分数阶微分的定义和一些基本知识来研究方程(1.11)的精确解并给出方程(1.11)与(1.1)精确解之间的关系。

方程(1.11)在RL意义下等价于下面的方程：

$$ D_t^{\gamma}(u - D_x^{\beta} D_x^{\alpha} u) + buD_x^{\alpha} u + uD_x^{\alpha} u = bD_x^{\alpha} uD_x^{\beta} D_x^{\alpha} u + uD_x^{\alpha} D_x^{\beta} D_x^{\alpha} u + \alpha' m + \beta' m_x + \gamma' m_{xx}. \tag{3.1} $$

进一步的我们得到

$$ D_t^{\gamma}(u - D_x^{\beta} D_x^{\alpha} u) + bD_x^{\alpha} u(u - D_x^{\beta} D_x^{\alpha} u) + uD_x^{\alpha}(u - D_x^{\beta} D_x^{\alpha} u) = \alpha' m + \beta' m_x + \gamma' m_{xx}. \tag{3.2} $$

注意到$m = u - D_x^{\beta} D_x^{\alpha} u$。然后方程(3.2)能化简为

$$ D_t^{\gamma} m + bD_x^{\alpha} um + uD_x^{\alpha} m = \alpha' m + \beta' m_x + \gamma' m_{xx}. \tag{3.3} $$

方程(3.3)的每一项包含m或者它的微分项。因此我们只需要研究下面的方程：

$$ u - D_x^{\beta} D_x^{\alpha} u = 0. \tag{3.4} $$

首先我们考虑下面的线性方程：

$$ u - {}_0D_x^{\beta}{}_0 D_x^{\alpha} u = 0. \tag{3.5} $$

我们让$\alpha_1 = \alpha$, $\alpha_2 = \beta$ 和$m = 2$。因此，$\sigma_1 = \alpha$ 和$\sigma_2 = \alpha + \beta$。方程(3.5)经过拉普拉斯变换变为

$$ L\{{}_0D_x^{\beta+\alpha} u(x,t); s\} - L\{u(x,t); s\} = s^{\beta+\alpha} U(s,t) - U(s,t) - \sum_{k=0}^{1} s^{\sigma_2 - \sigma_{2-k}} [{}_0D_x^{\sigma_{2-k}-1} u(x,t)]_{x=0} = 0. \tag{3.6} $$

这里我们假定

$$\left[{}_0D_x^{\beta-1}({}_0D_x^{\alpha}u(x,t))\right]_{x=0}=A(t),\quad \left[{}_0D_x^{\alpha-1}u(x,t)\right]_{x=0}=B(t). \tag{3.7}$$

因此方程(3.6)化简为

$$(s^{\beta+\alpha}-1)U(s,t)=A(t)+B(t)s^{\beta}, \tag{3.8}$$

就有

$$U(s,t)=\frac{A(t)}{s^{\beta+\alpha}-1}+\frac{B(t)s^{\beta}}{s^{\beta+\alpha}-1}. \tag{3.9}$$

进一步，我们就有

$$\frac{1}{s^{\beta+\alpha}-1}=\int_0^{\infty}e^{-sx}x^{\alpha+\beta-1}E_{\alpha+\beta,\alpha+\beta}(x^{\alpha+\beta})dx \tag{3.10}$$

和

$$\frac{s^{\beta}}{s^{\beta+\alpha}-1}=\int_0^{\infty}e^{-sx}x^{\alpha-1}E_{\alpha+\beta,\alpha}(x^{\alpha+\beta})dx. \tag{3.11}$$

将(3.10) 和(3.11) 代入(3.9)，我们得到

$$U(s,t)=A(t)\int_0^{\infty}e^{-sx}x^{\alpha+\beta-1}E_{\alpha+\beta,\alpha+\beta}(x^{\alpha+\beta})dx+B(t)\int_0^{\infty}e^{-sx}x^{\alpha-1}E_{\alpha+\beta,\alpha}(x^{\alpha+\beta})dx. \tag{3.12}$$

利用拉普拉斯逆变换，方程(3.12)变为

$$u(x,t)=A(t)x^{\alpha+\beta-1}E_{\alpha+\beta,\alpha+\beta}(x^{\alpha+\beta})+B(t)x^{\alpha-1}E_{\alpha+\beta,\alpha}(x^{\alpha+\beta}). \tag{3.13}$$

当$\alpha+\beta=1$，方程(3.13)变为

$$u(x,t)=A(t)E_{1,1}(x)+B(t)x^{\alpha-1}E_{1,\alpha}(x)=A(t)e^{x}+B(t)x^{\alpha-1}E_{1,\alpha}(x). \tag{3.14}$$

对于$\alpha=1$ 与$\beta=1$，方程(3.13) 化简为

$$u(x,t)=A(t)xE_{2,2}(x^2)+B(t)E_{2,1}(x^2). \tag{3.15}$$

利用定义(2.2)，显然双曲正弦与余弦是Mittag−Leffler函数的特殊情形。经过简单的计算，我们有

$$E_{2,1}(x^2)=E_2(x^2)=\sum_{k=0}^{\infty}\frac{x^{2k}}{\Gamma(2k+1)}=\sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!}=cosh(x), \tag{3.16}$$

与

$$E_{2,2}(x^2)=\sum_{k=0}^{\infty}\frac{x^{2k}}{\Gamma(2k+2)}=\frac{1}{x}\sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!}=\frac{sinh(x)}{x}. \tag{3.17}$$

将(3.16) 和(3.17) 代入(3.15)，我们得到

$$u(x,t) = A(t)sinh(x) + B(t)cosh(x), \tag{3.18}$$

这同样是方程(1.1)的解，也在文献 [16]中给出来了。

4. 结论

这篇文章给出了连续分数阶微分的拉普拉斯变换来获得时空分数阶的广义$b-$方程组的精确解。我们避开了直接求解该方程的精确解的难处。利用拉普拉斯变换获得辅助方程在RL意义下的精确解。当我们将分数阶的参数变为整数的时候，我们的结果就与先前的研究结果一致。

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