Advances in Applied Mathematics 应用数学进展, 2023, 12(4), 1567-1573
Published Online April 2023 in Hans. https://www.hanspub.org/journal/aam
https://doi.org/10.12677/aam.2023.124161

一类带变号权Kirchhoff方程解的存在性

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收稿日期：2023年3月19日；录用日期：2023年4月15日；发布日期：2023年4月24日

摘 要

本文研究一类具有变号权的Kirchhoff方程

$$-(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx)\Delta u+u=V(x)|u|^{p-1}u \quad x\in\mathbb{R}^3,$$

解的存在性，其中 $a,b>0$，$3<p<5$，$V(x)$ 是一个连续的变号权且 $\lim_{|x|\to\infty}V(x)=V_\infty<0$。

关键词

Kirchhoff方程，非局部项，变分法，变号权

Existence of Solution for Kirchhoff Equation with Sign-Changing Weight

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文章引用: 陈莉萍. 一类带变号权Kirchhoff方程解的存在性[J]. 应用数学进展, 2023, 12(4): 1567-1573.
DOI: 10.12677/aam.2023.124161

Received: Mar. 19^{th}, 2023; accepted: Apr. 15^{th}, 2023; published: Apr. 24^{th}, 2023

Abstract

In this paper, we deal with the existence result of Kirchhoff equation with sign-changing weight

$$-(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx)\Delta u+u=V(x)|u|^{p-1}u \quad x\in\mathbb{R}^3,$$

where $a,b>0$**,** $3<p<5$**,** $V(x)$ **is a continuous and sign-changing function such that** $\lim_{|x|\to\infty}V(x)=V_\infty<0$**.**

Keywords

Kirchhoff Equation, Nonlocal Term, Variation Methods, Sign-Changing Weight

1. 引言和主要结果

在本文中, 考虑下面 Kirchhoff 方程解的存在性

$$-(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx)\Delta u+u=V(x)|u|^{p-1}u \quad x\in\mathbb{R}^3, \tag{1}$$

其中 $a,b>0$, $3<p<5$, V 是一个变号函数且满足:
(V) $V(x)\in C(\mathbb{R}^3,\mathbb{R})$ 和 $V_\infty=\lim\limits_{|x|\to\infty}V(x)<0$.

问题 (1) 来自于一般的 Kirchhoff 方程

$$-(a+b\int_{\Omega}|\nabla u|^2dx)\Delta u=f(x,u), \tag{2}$$

这与以下 Kirchhoff 型方程的静态模拟有关

$$u_{tt}-(a+b\int_{\Omega}|\nabla u|^2dx)\Delta u=f(x,u). \tag{3}$$

我们注意到, 作为对著名的 D'Alembert 波动方程关于弹性弦自由振动的一个推广, Kirchhoff 在文献 [1]中首次引入了方程 (3). 对于 Kirchhoff 类型问题的更多背景, 我们可以参考文献 [2]. 自 Lions 在文献 [3]的先驱性工作以来, 已经得到了许多关于 Kirchhoff 型问题的结果, 可以参考文献 [4–21]和其中的参考文献. 然而, 据我们所知, 很少有论文考虑具有变号权的 Kirchhoff 方程解的存在性结果. 最近, Yu 在 [22]中考虑了以下 Schrödinger-Poisson 系统

$$\begin{cases} -\Delta u+u+\phi u=a(x)|u|^{p-1}u,\ x\in\mathbb{R}^3,\\ -\Delta\phi=k(x)u^2,\ x\in\mathbb{R}^3,\end{cases} \tag{4}$$

其中 $3\le p<5$, $a(x)$ 在 $\mathbb{R}^3$ 中是一个连续的变号函数且 $\lim\limits_{|x|\to\infty}a(x)=a_\infty<0$, $k(x)$ 是连续的并且 $k(x)\in L^2(\mathbb{R}^3)$. 利用山路引理 [23], 作者证明了该问题至少有一个非平凡解. 受上述工作的启发, 我们研究 Kirchhoff 问题 (1) 非平凡解的存在性, 主要结果如下：

定理1.1 若 $V(x)$ 满足条件 (V), 方程 (1) 至少存在一个非平凡解.

2. 主要结果的证明

$H^1(\mathbb{R}^3)$ 是 Sobolev 空间其内积和范数如下

$$(u,v)=\int_{\mathbb{R}^3}\nabla u\nabla v+uv,\qquad \|u\|=(u,u)^{1/2}.$$

方程 (1) 的解是以下函数的临界点

$$\Gamma(u)=\frac{1}{2}(a\int_{\mathbb{R}^3}|\nabla u|^2dx+\int_{\mathbb{R}^3}|u|^2dx)+\frac{b}{4}(\int_{\mathbb{R}^3}|\nabla u|^2dx)^2-\frac{1}{p+1}\int_{\mathbb{R}^3}V(x)|u|^{p+1}dx.$$

令

$$V(x)=V^+(x)-V^-(x), \tag{5}$$

其中

$$V^+(x)=\begin{cases}V(x), & \text{如果 } V(x)\ge 0,\\ 0, & \text{如果 } V(x)<0,\end{cases} \tag{6}$$

和

$$V^-(x)=\begin{cases}0, & \text{如果 } V(x)\ge 0,\\ -V(x), & \text{如果 } V(x)<0.\end{cases} \tag{7}$$

下面, 我们将证明泛函 Γ 满足 $(PS)_c$ 条件.

引理2.1 泛函 Γ 满足 $(PS)_c$ 条件.

证明: 设 $\{u_n\} \in H^1(\mathbb{R}^3)$, 当 $n \to \infty$ 时有

$$\Gamma(u_n) \le c, \ \ \Gamma'(u_n) \to 0. \tag{8}$$

要证明这个引理, 只需证明 $\{u_n\}$ 在 $H^1(\mathbb{R}^3)$ 中有一个强收敛的子列即可. 首先证明 $\{u_n\}$ 有界. 通过 (8), 很容易得出

$$a\int_{\mathbb{R}^3}|\nabla u_n|^2dx + \int_{\mathbb{R}^3}|u_n|^2dx + b(\int_{\mathbb{R}^3}|\nabla u_n|^2dx)^2 - \int_{\mathbb{R}^3}V(x)|u_n|^{p+1}dx = o(1)\|u_n\|, \tag{9}$$

和

$$\frac{a}{2}\int_{\mathbb{R}^3}|\nabla u_n|^2dx + \frac{1}{2}\int_{\mathbb{R}^3}|u_n|^2dx + \frac{b}{4}(\int_{\mathbb{R}^3}|\nabla u_n|^2dx)^2 - \frac{1}{p+1}\int_{\mathbb{R}^3}V(x)|u_n|^{p+1}dx \le c. \tag{10}$$

利用 (9) 和 (10), 有

$$(\frac{1}{2} - \frac{1}{p+1})[\int_{\mathbb{R}^3}(a|\nabla u_n|^2 + |u_n|^2)dx] + b(\frac{1}{4} - \frac{1}{p+1})(\int_{\mathbb{R}^3}|\nabla u_n|^2dx)^2 \le c + o(1)\|u_n\|. \tag{11}$$

由于 $p > 3$, 可以得到

$$(\frac{1}{2} - \frac{1}{p+1})min\{a, 1\}\|u_n\|^2 \le c + o(1)\|u_n\|. \tag{12}$$

所以有 $\|u_n\| \le C$. 因此, 我们假设在 $H^1(\mathbb{R}^3)$ 中 $u_n \rightharpoonup u$ 和当 $n \to \infty$ 时

$$\int_{\mathbb{R}^3}|\nabla u_n|^2dx \to A \tag{13}$$

根据 (5), (9) 以及 $\|u_n\| \le C$, 有

$$\begin{aligned}&a\int_{\mathbb{R}^3}|\nabla u_n|^2dx + \int_{\mathbb{R}^3}|u_n|^2dx + b(\int_{\mathbb{R}^3}|\nabla u_n|^2dx)^2 + \int_{\mathbb{R}^3}V^-(x)|u_n|^{p+1}dx\\&= \int_{\mathbb{R}^3}V^+(x)|u_n|^{p+1}dx + o(1).\end{aligned} \tag{14}$$

由 $\Gamma'(u_n) \to 0$ 可得

$$\begin{aligned}&a\int_{\mathbb{R}^3}\nabla u\nabla vdx + \int_{\mathbb{R}^3}uvdx + bA\int_{\mathbb{R}^3}\nabla u\nabla vdx + \int_{\mathbb{R}^3}V^-(x)|u|^{p-1}uvdx\\&= \int_{\mathbb{R}^3}V^+(x)|u|^{p-1}uvdx, \quad \forall\, v \in H^1(\mathbb{R}^3).\end{aligned} \tag{15}$$

在 (15) 中取 $v=u$, 可以得到

$$\begin{aligned}&a\int_{\mathbb{R}^3}|\nabla u|^2dx+\int_{\mathbb{R}^3}u^2dx+bA\int_{\mathbb{R}^3}|\nabla u|^2dx+\int_{\mathbb{R}^3}V^-(x)|u|^{p+1}dx\\&=\int_{\mathbb{R}^3}V^+(x)|u|^{p+1}dx.\end{aligned}\tag{16}$$

根据假设条件 (V), 知道 $V^+(x)$ 有一个紧支集, 从而可得

$$\int_{\mathbb{R}^3}V^+(x)|u_n|^{p+1}dx=\int_{\mathbb{R}^3}V^+(x)|u|^{p+1}dx+o(1).\tag{17}$$

结合 (14), (16) 以及 (17), 可以得到

$$\begin{aligned}&a\int_{\mathbb{R}^3}|\nabla u_n|^2dx+\int_{\mathbb{R}^3}u_n^2dx+b(\int_{\mathbb{R}^3}|\nabla u_n|^2dx)^2+\int_{\mathbb{R}^3}V^-(x)|u_n|^{p+1}dx\\&=a\int_{\mathbb{R}^3}|\nabla u|^2dx+\int_{\mathbb{R}^3}u^2dx+bA^2\int_{\mathbb{R}^3}|\nabla u|^2dx+\int_{\mathbb{R}^3}V^-(x)|u|^{p+1}dx+o(1).\end{aligned}\tag{18}$$

假设 $u_n\nrightarrow u$, 则有 $\|u\|+o(1)<\|u_n\|$. 进而 $\|u\|_{L^2}+o(1)<\|u_n\|_{L^2}$ 和 $\|\nabla u\|_{L^2}+o(1)<\|\nabla u_n\|_{L^2}$ 至少有一个是成立的. 因此我们可以推断出

$$a\int_{\mathbb{R}^3}|\nabla u_n|^2dx+\int_{\mathbb{R}^3}u_n^2dx>a\int_{\mathbb{R}^3}|\nabla u|^2dx+\int_{\mathbb{R}^3}u^2dx+o(1).\tag{19}$$

根据 (13), (19) 和 Fatou 引理知

$$\begin{aligned}&a\int_{\mathbb{R}^3}|\nabla u_n|^2dx+\int_{\mathbb{R}^3}u_n^2dx+b(\int_{\mathbb{R}^3}|\nabla u_n|^2dx)^2+\int_{\mathbb{R}^3}V^-(x)|u_n|^{p+1}dx\\&>a\int_{\mathbb{R}^3}|\nabla u|^2dx+\int_{\mathbb{R}^3}u^2dx+bA^2\int_{\mathbb{R}^3}|\nabla u|^2dx+\int_{\mathbb{R}^3}V^-(x)|u|^{p+1}dx+o(1),\end{aligned}$$

这与 (18) 矛盾. 所以, 在 $H^1(\mathbb{R}^3)$ 中 $u_n\to u$.

定理1.1 的证明: 一方面, 由 Sobolev 不等式和 $3<p<5$, 我们可以得到

$$\Gamma(u)\geq\frac{1}{2}min\{a,1\}\|u\|^2-\|V\|_{L^\infty}\|u\|^{p+1},\tag{20}$$

从而存在常数 $\alpha,\rho>0$ 使得 $\Gamma|_{B_\rho}\geq\alpha>0$.

另一方面, 选择 $\varphi\in H^1(\mathbb{R}^3)$ 使得 $supp\varphi\subset suppa^+$, 则有

$$\Gamma(t\varphi)=\frac{t^2}{2}\int_{\mathbb{R}^3}(a|\nabla\varphi|^2dx+\varphi^2)dx+\frac{t^4}{4}(\int_{\mathbb{R}^3}|\nabla\varphi|^2dx)^2-\frac{t^{p+1}}{p+1}\int_{\mathbb{R}^3}V^+|\varphi|^{p+1}dx.\tag{21}$$

由于 $p>3$, 存在 t_0 使得 $\Gamma(t_0\varphi)<0$. 因此, 我们证明了函数 Γ 具有山路几何结构. 进而由引理2.1 可知方程 (1) 至少有一个非平凡解.

3. 总结

本文主要利用山路引理证明了一类带变号权 Kirchhoff 方程解的存在性. 具体来说, 将证明方程非平凡解的存在性问题转化为求解方程对应的泛函临界点问题, 接着证明了泛函满足 (PS) 条件, 最后结合 Sobolev 不等式证明了该方程至少存在一个非平凡解.

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