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约当三系上非零权的相对Rota-Baxter算子

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摘　要

本文主要研究约当三系上关于非零权的相对Rota-Baxter算子。首先回顾了约当代数，约旦三系以及子系的概念，然后给出了约当三系的表示的定义，进而得出约当三系在另一个约当三系上作用的概念，以及约当三系上非零权的相对Rota-Baxter算子的定义，找到了在约当三系及其作用空间的直和上构造新约当三系的方法。最后，给出了在一个约当三系上定义与相对Rota-Baxter算子有关的新运算可得到新的约当三系的方法，并且构造了新的约当三系的表示。此外文中还给出了约当三系上构造相对Rota-Baxter算子的方法。

关键词

约当三系，相对Rota-Baxter算子，作用，伴随表示

Relative Rota-Baxter Operator of Non-Zero Weights on the Jordan Triple

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Abstract

In this paper, we mainly study the relative Rota-Baxter operators on Jordan triple systems of nonzero weights. First, we recall the concepts of Jordan algebra, Jordan triple systems, and subsystems, and then the definition of the representation of Jordan triple systems is given. Then, the concept of Jordan triple systems acting on another Jordan triple system is obtained, as well as the definition of relative Rota-Baxter operators of nonzero weights on Jordan triple systems. A method for constructing a new Jordan triple system on the direct sum of Jordan triple systems and their action spaces is found. Finally, it is shown that by defining new operations related to the rel-

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ative Rota-Baxter operator on a Jordan triple system, a new Jordan triple system can be obtained, and a new representation of the Jordan triple system is constructed. In addition, a method for constructing relative Rota-Baxter operators is also presented.

Keywords

Jordan Triple, Relative Rota-Baxter Operator, Action, Adjoint Representation

1. 引言

约当三系的定义最早是 1949 年由 Nathan Jacobson 提出的[1]，通过约当代数可以构造出约当三系。而约当代数理论最早是有量子物理学的潜在研究所激发的，在物理，特别是量子力学中起着重要的作用。对于约当三系，可以通过约当三系给出一类对称流形的代数描述，并且，约当三系可以提供几种构造李代数的方法。对于约当三系的研究，前人已经得到了很多结果。例如，在[2]中作者利用 TKK 李代数的上同调研究了约当三系的上同调理论；在[3]中作者研究了约当三系的表示，证明了有限维约当三系的通用包络是有限维的，若约当三系是它自身的根基，则约当三系的通用包络是幂零的；在[4]中作者研究了特征不为 2 的域上中心非退化约当三系的非零中心及其等价形式。

相对 Rota-Baxter 算子是 Rota-Baxter 算子在结合代数上的相对推广，与 O-算子有关，起源于经典 Yang-Baxter 方程的算子形式。在代数上，可以利用相对 Rota-Baxte 研究上同调理论，如，在[5]中作者研究了李代数上权为 1 的相对 Rota-Baxter 算子，构造了它们的上同调理论，并利用第二个上同调群来研究相对 Rota-Baxter 算子和 r-矩阵的无穷小变形。在[6]中作者给出了在李三系上的相对 Rota-Baxter 算子以及上同调理论，并利用第一个上同调群对无穷小变形进行了分类。

2. 预备知识

定义 1.1 [7] 设 J 为线性空间，J 上有双线性的代数运算“$\cdot$”：$J\times J\to J$，如果满足下列条件

$$x\cdot y=y\cdot x,$$
$$\left(\left(x\cdot x\right)\cdot y\right)\cdot x=\left(x\cdot x\right)\cdot\left(y\cdot x\right),$$

$\forall x,y\in J$，则称 $\left(J,\cdot\right)$ 为约当代数。

定义 1.2 [8]设 J 为线性空间，$\{\cdot,\cdot,\cdot\}:J\times J\times J\to J$ 是三线性映射，如果满足下列条件

$$\{x,y,z\}=\{z,y,x\} \tag{1.1}$$
$$\{x,y,\{z,u,v\}\}-\{z,u,\{x,y,v\}\}-\{\{x,y,z\},u,v\}+\{z,\{y,x,u\},v\}=0 \tag{1.2}$$

$\forall x,y,z,u,v\in J$，则称 $\left(J,\{\cdot,\cdot,\cdot\}\right)$ 为约当三系。

例 1.1 [8]设 $\left(J,\cdot\right)$ 为约当代数，在 J 上定义代数运算

$$\{x,y,z\}=\left(x\cdot y\right)\cdot z+\left(z\cdot y\right)\cdot x-\left(x\cdot z\right)\cdot y,$$

$\forall x,y,z\in J$，则 $\left(J,\{\cdot,\cdot,\cdot\}\right)$ 为约当三系。

定义 1.3 [9]设 $\left(J,\{\cdot,\cdot,\cdot\}\right)$ 是约当三系，H 为 J 的子空间，如果 H 满足 $\{H,H,H\}\subseteq H$，则称 H 是 J 的子系。

3. 相对 Rota-Baxter 算子

定义 2.1 设$(J,\{\cdot,\cdot,\cdot\})$是约当三系，$V$是线性空间，$\theta_{12},\theta_{13}:J\times J\to End(V)$是双线性映射，若$\theta_{12},\theta_{13}$满足

$$\theta_{13}(a,b)=\theta_{13}(b,a)\text{，} \tag{2.1}$$

$$\theta_{12}(x,y)\theta_{12}(a,b)-\theta_{12}(a,b)\theta_{12}(x,y)-\theta_{12}(\{x,y,a\},b)+\theta_{12}(a,\{y,x,b\})=0\text{，} \tag{2.2}$$

$$\theta_{12}(x,y)\theta_{13}(a,b)+\theta_{13}(a,b)\theta_{12}(y,x)-\theta_{13}(a,\{x,y,b\}_1)-\theta_{13}(\{x,y,a\}_1,b)=0\text{，} \tag{2.3}$$

$$\theta_{12}(\{x,a,b\},y)-\theta_{12}(x,a)\theta_{12}(b,y)-\theta_{12}(b,a)\theta_{12}(x,y)+\theta_{13}(x,b)\theta_{13}(a,y)=0\text{，} \tag{2.4}$$

$$\theta_{13}(\{x,a,b\},y)-\theta_{12}(x,a)\theta_{13}(b,y)-\theta_{12}(b,a)\theta_{13}(x,y)+\theta_{13}(x,b)\theta_{12}(a,y)=0\text{，} \tag{2.5}$$

$\forall a,b,x,y\in J$，则$(\theta_{12},\theta_{13},V)$称为$J$的表示。

例 2.1 设$(J,\{\cdot,\cdot,\cdot\})$是约当三系，定义双线性映射$\sigma_{12},\sigma_{13}:J\times J\to End(J)$，其中$\sigma_{12}(x,y)z=\{x,y,z\}$，$\sigma_{13}(x,y)z=\{x,z,y\}(\forall x,y,z\in J)$，则$(\sigma_{12},\sigma_{13},J)$是$J$的表示，称为伴随表示。

证：直接验证可知$(\sigma_{12},\sigma_{13},J)$满足(2.1)-(2.5)式，故$(\sigma_{12},\sigma_{13},J)$是$J$的表示。

定义 2.2 设$(J_1,\{\cdot,\cdot,\cdot\}_1),(J_2,\{\cdot,\cdot,\cdot\}_2)$是约当三系，$(\theta_{12},\theta_{13},J_2)$是$(J_1,\{\cdot,\cdot,\cdot\}_1)$的表示。若$\forall x_1,y_1\in J_1,x_2,y_2,z_2\in J_2$，有

$$\theta_{12}(x_1,y_1)z_2\in C(J_2)\text{，}$$

$$\theta_{13}(x_1,y_1)z_2\in C(J_2)\text{，}$$

$$\theta_{12}(x_1,y_1)\{x_2,y_2,z_2\}_2=0\text{，}$$

$$\theta_{13}(x_1,y_1)\{x_2,y_2,z_2\}_2=0\text{，}$$

则称θ_{12},θ_{13}为$(J_1,\{\cdot,\cdot,\cdot\}_1)$在$(J_2,\{\cdot,\cdot,\cdot\}_2)$上的一个作用。这里$C(J_2)=\{u\in J_2\mid\{u,v,w\}_2=0,\{v,u,w\}_2=0,\forall v,w\in J_2\}$。

设$(J,\{\cdot,\cdot,\cdot\})$是约当三系，定义$J^1=\{J,J,J\}$。

例 2.2 $(J,\{\cdot,\cdot,\cdot\})$是约当三系，如果$J$满足$J^1\subseteq C(J)$，则伴随表示$(\sigma_{12},\sigma_{13},J)$是$J$在自身上的一个作用。

证：由例 2.1 可知，伴随表示$(\sigma_{12},\sigma_{13},J)$是$J$的表示。因为$J^1\subseteq C(J)$，所以$\forall x_1,x_2,x_3,x_4,x_5\in J$，

$$\sigma_{12}(x_1,x_2)x_3=\{x_1,x_2,x_3\}\in C(J)\text{，}$$

$$\sigma_{13}(x_1,x_2)x_3=\{x_1,x_3,x_2\}\in C(J)\text{，}$$

$$\sigma_{12}(x_1,x_2)\{x_3,x_4,x_5\}=\{x_1,x_2,\{x_3,x_4,x_5\}\}=0\text{，}$$

$$\sigma_{13}(x_1,x_2)\{x_3,x_4,x_5\}=\{x_1,\{x_3,x_4,x_5\},x_2\}=0\text{，}$$

因此，σ_{12},σ_{13}是J在自身上的一个作用。

定义 2.3 $(J_1,\{\cdot,\cdot,\cdot\}_1),(J_2,\{\cdot,\cdot,\cdot\}_2)$是约当三系，$\theta_{12},\theta_{13}$是$(J_1,\{\cdot,\cdot,\cdot\}_1)$在$(J_2,\{\cdot,\cdot,\cdot\}_2)$上的作用，$T:J_2\to J_1$是线性映射，如果$T$满足

$$\{Tu,Tv,Tw\}_1=T\left(\theta_{12}(Tw,Tv)u+\theta_{13}(Tu,Tw)v+\theta_{12}(Tu,Tv)w+\lambda\{u,v,w\}_2\right)\text{，} \tag{2.6}$$

$\forall u,v,w\in J_2$，则T称为关于作用θ_{12},θ_{13}的从$(J_2,\{\cdot,\cdot,\cdot\}_2)$到$(J_1,\{\cdot,\cdot,\cdot\}_1)$的权为$\lambda$的相对 Rota-Baxter 算子。

命题 2.1 $(J_1,\{\cdot,\cdot,\cdot\}_1),(J_2,\{\cdot,\cdot,\cdot\}_2)$ 是约当三系，θ_{12},θ_{13} 是 $(J_1,\{\cdot,\cdot,\cdot\}_1)$ 在 $(J_2,\{\cdot,\cdot,\cdot\}_2)$ 上的作用。在 $J_1\oplus J_2$ 上定义运算 $\{\cdot,\cdot,\cdot\}_\theta$，其中

$$\{x+u,y+v,z+w\}_\theta=\{x,y,z\}_1+\theta_{12}(z,y)u+\theta_{13}(x,z)v+\theta_{12}(x,y)w+\lambda\{u,v,w\}_2,$$

$\forall x,y,z\in J_1,u,v,w\in J_2$，则 $(J_1\oplus J_2,\{\cdot,\cdot,\cdot\}_\theta)$ 是约当三系，称为 $(J_1,\{\cdot,\cdot,\cdot\}_1)$ 和 $(J_2,\{\cdot,\cdot,\cdot\}_2)$ 的半直积。

证：显然，在 $J_1\oplus J_2$ 上定义的运算对三个变量都是线性的，因此若要证 $(J_1\oplus J_2,\{\cdot,\cdot,\cdot\}_\theta)$ 是约当三系，只需在 $J_1\oplus J_2$ 上验证(1.1)-(1.2)式成立。$\forall x_1,y_1,z_1,a_1,b_1\in J_1,x_2,y_2,z_2,a_2,b_2\in J_2$，直接计算(1.1)式左式得

$$\begin{aligned}&\{x_1+x_2,y_1+y_2,z_1+z_2\}_\theta-\{z_1+z_2,y_1+y_2,x_1+x_2\}_\theta\\&=\{x_1,y_1,z_1\}_1+\theta_{13}(x_1,z_1)y_2+\theta_{12}(z_1,y_1)x_2+\theta_{12}(x_1,y_1)z_2+\lambda\{x_2,y_2,z_2\}_2\\&\quad-\{z_1,y_1,x_1\}_1-\theta_{13}(z_1,x_1)y_2-\theta_{12}(x_1,y_1)z_2-\theta_{12}(z_1,y_1)x_2-\lambda\{z_2,y_2,x_2\}_2\end{aligned},$$

由于 $(J_1,\{\cdot,\cdot,\cdot\}_1),(J_2,\{\cdot,\cdot,\cdot\}_2)$ 是约当三系且 θ_{13} 是对称的，故在 $J_1\oplus J_2$ 上(1.1)式成立。

$$\begin{aligned}&\{x_1+x_2,y_1+y_2,\{z_1+z_2,a_1+a_2,b_1+b_2\}_\theta\}_\theta-\{z_1+z_2,a_1+a_2,\{x_1+x_2,y_1+y_2,b_1+b_2\}_\theta\}_\theta\\&-\{\{x_1+x_2,y_1+y_2,z_1+z_2\}_\theta,a_1+a_2,b_1+b_2\}_\theta+\{z_1+z_2,\{y_1+y_2,x_1+x_2,a_1+a_2\}_\theta,b_1+b_2\}_\theta\\&=\{x_1+x_2,y_1+y_2,\{z_1,a_1,b_1\}_1+\theta_{13}(z_1,b_1)a_2+\theta_{12}(b_1,a_1)z_2+\theta_{12}(z_1,a_1)b_2+\lambda\{z_2,a_2,b_2\}_2\}_\theta\\&\quad-\{z_1+z_2,a_1+a_2,\{x_1,y_1,b_1\}_1+\theta_{13}(x_1,b_1)y_2+\theta_{12}(b_1,y_1)x_2+\theta_{12}(x_1,y_1)b_2+\lambda\{x_2,y_2,b_2\}_2\}_\theta\\&\quad-\{\{x_1,y_1,z_1\}_1+\theta_{13}(x_1,z_1)y_2+\theta_{12}(z_1,y_1)x_2+\theta_{12}(x_1,y_1)z_2+\lambda\{x_2,y_2,z_2\}_2,a_1+a_2,b_1+b_2\}_\theta\\&\quad+\{z_1+z_2,\{y_1,x_1,a_1\}_1+\theta_{13}(y_1,a_1)x_2+\theta_{12}(a_1,x_1)y_2+\theta_{12}(y_1,x_1)a_2+\lambda\{y_2,x_2,a_2\}_2,b_1+b_2\}_\theta\\&=\{x_1,y_1,\{z_1,a_1,b_1\}_1\}_1+\theta_{13}(x_1,\{z_1,a_1,b_1\}_1)y_2+\theta_{12}(\{z_1,a_1,b_1\}_1,y_1)x_2+\theta_{12}(x_1,y_1)\theta_{13}(z_1,b_1)a_2\\&\quad+\lambda\{x_2,y_2,\theta_{13}(z_1,b_1)a_2\}_2+\theta_{12}(x_1,y_1)\theta_{12}(b_1,a_1)z_2+\lambda\{x_2,y_2,\theta_{12}(b_1,a_1)z_2\}_2\\&\quad+\theta_{12}(x_1,y_1)\theta_{12}(z_1,a_1)b_2+\lambda\{x_2,y_2,\theta_{12}(z_1,a_1)b_2\}_2+\lambda\theta_{12}(x_1,y_1)\{z_2,a_2,b_2\}_2\\&\quad+\lambda\{x_2,y_2,\lambda\{z_2,a_2,b_2\}_2\}_2-\{z_1,a_1,\{x_1,y_1,b_1\}_1\}_1-\theta_{13}(z_1,\{x_1,y_1,b_1\}_1)a_2\\&\quad-\theta_{12}(\{x_1,y_1,b_1\}_1,a_1)z_2-\theta_{12}(z_1,a_1)\theta_{13}(x_1,b_1)y_2-\lambda\{z_2,a_2,\theta_{13}(x_1,b_1)y_2\}_2\\&\quad-\theta_{12}(z_1,a_1)\theta_{12}(b_1,y_1)x_2-\lambda\{z_2,a_2,\theta_{12}(b_1,y_1)x_2\}_2-\theta_{12}(z_1,a_1)\theta_{12}(x_1,y_1)b_2\\&\quad-\lambda\{z_2,a_2,\theta_{12}(x_1,y_1)b_2\}_2-\lambda\theta_{12}(z_1,a_1)\{x_2,y_2,b_2\}_2-\lambda\{z_2,a_2,\lambda\{x_2,y_2,b_2\}_2\}_2\\&\quad-\{\{x_1,y_1,z_1\}_1,a_1,b_1\}_1-\theta_{13}(\{x_1,y_1,z_1\}_1,b_1)a_2-\theta_{12}(\{x_1,y_1,z_1\}_1,a_1)b_2\\&\quad-\theta_{12}(b_1,a_1)\theta_{13}(x_1,z_1)y_2-\lambda\{\theta_{13}(x_1,z_1)y_2,a_2,b_2\}_2-\theta_{12}(b_1,a_1)\theta_{12}(z_1,y_1)x_2\\&\quad-\lambda\{\theta_{12}(z_1,y_1)x_2,a_2,b_2\}_2-\theta_{12}(b_1,a_1)\theta_{12}(x_1,y_1)z_2-\lambda\{\theta_{12}(x_1,y_1)z_2,a_2,b_2\}_2\\&\quad-\lambda\theta_{12}(b_1,a_1)\{x_2,y_2,z_2\}_2-\lambda\{\lambda\{x_2,y_2,z_2\}_2,a_2,b_2\}_2+\{z_1,\{y_1,x_1,a_1\}_1,b_1\}_1\\&\quad+\theta_{12}(b_1,\{y_1,x_1,a_1\}_1)z_2+\theta_{12}(z_1,\{y_1,x_1,a_1\}_1)b_2+\theta_{13}(z_1,b_1)\theta_{13}(y_1,a_1)x_2\\&\quad+\lambda\{z_2,\theta_{13}(y_1,a_1)x_2,b_2\}_2+\theta_{13}(z_1,b_1)\theta_{12}(a_1,x_1)y_2+\lambda\{z_2,\theta_{12}(a_1,x_1),b_2\}_2\\&\quad+\theta_{13}(z_1,b_1)\theta_{12}(y_1,x_1)a_2+\lambda\{z_2,\theta_{12}(y_1,x_1)a_2,b_2\}_2+\lambda\theta_{13}(z_1,b_1)\{y_2,x_2,a_2\}_2+\lambda\{z_2,\lambda\{y_2,x_2,a_2\}_2,b_2\}_2\\&=\left(\{x_1,y_1,\{z_1,a_1,b_1\}_1\}_1-\{z_1,a_1,\{x_1,y_1,b_1\}_1\}_1-\{\{x_1,y_1,z_1\}_1,a_1,b_1\}_1+\{z_1,\{y_1,x_1,a_1\}_1,b_1\}_1\right)\\&\quad+\lambda^2\left(\{x_2,y_2,\{z_2,a_2,b_2\}_2\}_2-\{z_2,a_2,\{x_2,y_2,b_2\}_2\}_2-\{\{x_2,y_2,z_2\}_2,a_2,b_2\}_2\right.\\&\quad\left.+\{z_2,\{y_2,x_2,a_2\}_2,b_2\}_2\right)+A_1x_2+A_2y_2+A_3z_2+A_4a_2+A_5b_2+\lambda A_6\end{aligned}$$

其中，

$$A_1=\theta_{12}\left(\{z_1,a_1,b_1\}_1,y_1\right)-\theta_{12}(z_1,a_1)\theta_{12}(b_1,y_1)-\theta_{12}(b_1,a_1)\theta_{12}(z_1,y_1)+\theta_{13}(z_1,b_1)\theta_{13}(y_1,a_1),$$

$$A_2=\theta_{13}\left(x_1,\{z_1,a_1,b_1\}_1\right)-\theta_{12}(z_1,a_1)\theta_{13}(x_1,b_1)-\theta_{12}(b_1,a_1)\theta_{13}(x_1,z_1)+\theta_{13}(z_1,b_1)\theta_{12}(a_1,x_1),$$

$$A_3=\theta_{12}(x_1,y_1)\theta_{12}(b_1,a_1)-\theta_{12}\left(\{x_1,y_1,b_1\}_1,a_1\right)-\theta_{12}(b_1,a_1)\theta_{12}(x_1,y_1)+\theta_{12}\left(b_1,\{y_1,x_1,a_1\}_1\right),$$

$$A_4=\theta_{12}(x_1,y_1)\theta_{13}(z_1,b_1)-\theta_{13}\left(z_1,\{x_1,y_1,b_1\}_1\right)-\theta_{13}\left(\{x_1,y_1,z_1\}_1,b_1\right)+\theta_{13}(z_1,b_1)\theta_{12}(y_1,x_1),$$

$$A_5=\theta_{12}(x_1,y_1)\theta_{12}(z_1,a_1)-\theta_{12}(z_1,a_1)\theta_{12}(x_1,y_1)-\theta_{12}\left(\{x_1,y_1,z_1\}_1,a_1\right)+\theta_{12}\left(z_1,\{y_1,x_1,a_1\}_1\right),$$

$$\begin{aligned}A_6=&\{x_2,y_2,\theta_{13}(z_1,b_1)a_2\}_2+\{x_2,y_2,\theta_{12}(b_1,a_1)z_2\}_2+\{x_2,y_2,\theta_{12}(z_1,a_1)b_2\}_2+\theta_{12}(x_1,y_1)\{z_2,a_2,b_2\}_2\\&-\{z_2,a_2,\theta_{13}(x_1,b_1)y_2\}_2-\{z_2,a_2,\theta_{12}(b_1,y_1)x_2\}_2-\{z_2,a_2,\theta_{12}(x_1,y_1)b_2\}_2-\theta_{12}(z_1,a_1)\{x_2,y_2,b_2\}_2\\&-\{\theta_{13}(x_1,z_1)y_2,a_2,b_2\}_2-\{\theta_{12}(z_1,y_1)x_2,a_2,b_2\}_2-\{\theta_{12}(x_1,y_1)z_2,a_2,b_2\}_2-\theta_{12}(b_1,a_1)\{x_2,y_2,z_2\}_2\\&+\{z_2,\theta_{13}(y_1,a_1)x_2,b_2\}_2+\{z_2,\theta_{12}(a_1,x_1)y_2,b_2\}_2+\{z_2,\theta_{12}(y_1,x_1)a_2,b_2\}_2+\theta_{13}(z_1,b_1)\{y_2,x_2,a_2\}_2\end{aligned},$$

由$\left(J_1,\{\cdot,\cdot,\cdot\}_1\right),\left(J_2,\{\cdot,\cdot,\cdot\}_2\right)$是约当三系知(1.2)式成立，由(2.4)式成立知 A_1 为零，由(2.5)式成立知 A_2 为零，由(2.2)式成立知 A_3 为零，由(2.3)式成立知 A_4 为零，由(2.2)式成立知 A_5 为零。由于θ_{12},θ_{13}是$\left(J_1,\{\cdot,\cdot,\cdot\}_1\right)$在$\left(J_2,\{\cdot,\cdot,\cdot\}_2\right)$上的作用，故有$\{x_2,y_2,\theta_{13}(z_1,b_1)a_2\}_2=0$，$\theta_{12}(y_1,x_1)\{z_2,a_2,b_2\}_2=0$，类似可知 A_6 为零。所以在$J_1\oplus J_2$上(1.2)式成立。综上，$\left(J_1\oplus J_2,\{\cdot,\cdot,\cdot\}_\theta\right)$是一个约当三系。

定理 2.1 $\left(J_1,\{\cdot,\cdot,\cdot\}_1\right),\left(J_2,\{\cdot,\cdot,\cdot\}_2\right)$是约当三系，$\theta_{12},\theta_{13}$是$\left(J_1,\{\cdot,\cdot,\cdot\}_1\right)$在$\left(J_2,\{\cdot,\cdot,\cdot\}_2\right)$上的作用，则线性映射$T:J_2\to J_1$是关于作用$\theta_{12},\theta_{13}$的权为$\lambda$的相对 Rota-Baxter 算子当且仅当图$Gr(T)=\{Tu+u\mid u\in J_2\}$是$\left(J_1\oplus J_2,\{\cdot,\cdot,\cdot\}_\theta\right)$的子系。

证明：由于$T:J_2\to J_1$是线性映射，$\forall u,v,w\in J_2$，由命题 2.1 有

$$\{Tu+u,Tv+v,Tw+w\}_\theta=\{Tu,Tv,Tw\}_1+\theta_{13}(Tu,Tw)v+\theta_{12}(Tw,Tv)u+\theta_{12}(Tu,Tv)w+\lambda\{u,v,w\}_2,$$

这表明，图$Gr(T)$是$\left(J_1\oplus J_2,\{\cdot,\cdot,\cdot\}_\theta\right)$的子系当且仅当 T 满足(2.6)式，所以结论成立。

4. 相对 Rota-Baxter 算子的应用

命题 4.1 $\left(J_1,\{\cdot,\cdot,\cdot\}_1\right),\left(J_2,\{\cdot,\cdot,\cdot\}_2\right)$是约当三系，线性映射$T:J_2\to J_1$是关于作用$\theta_{12},\theta_{13}$的权为$\lambda$的相对 Rota-Baxter 算子，在 J_2 上定义新运算$\{\cdot,\cdot,\cdot\}_T$，其中

$$\{u,v,w\}_T=\theta_{12}(Tw,Tv)u+\theta_{13}(Tu,Tw)v+\theta_{12}(Tu,Tv)w+\lambda\{u,v,w\}_2,$$

$\forall u,v,w\in J_2$，则$\left(J_2,\{\cdot,\cdot,\cdot\}_T\right)$也是约当三系，并且$T$是从约当三系$\left(J_2,\{\cdot,\cdot,\cdot\}_T\right)$到约当三系$\left(J_1,\{\cdot,\cdot,\cdot\}_1\right)$的同态。

证明：显然，在 J_2 上定义的新代数运算$\{\cdot,\cdot,\cdot\}_T$对三个变量都是线性的，若要证明$\left(J_2,\{\cdot,\cdot,\cdot\}_T\right)$是一个约当三系，只需在$\left(J_2,\{\cdot,\cdot,\cdot\}_T\right)$上验证(1.1)~(1.2)式成立。$\forall u_1,u_2,u_3,u_4,u_5\in J_2$，

$$\begin{aligned}&\{u_1,u_2,u_3\}_T-\{u_3,u_2,u_1\}_T\\&=\theta_{12}(Tu_3,Tu_2)u_1+\theta_{13}(Tu_1,Tu_3)u_2+\theta_{12}(Tu_1,Tu_2)u_3+\lambda\{u_1,u_2,u_3\}_2\\&\quad-\theta_{12}(Tu_1,Tu_2)u_3-\theta_{13}(Tu_3,Tu_1)u_2-\theta_{12}(Tu_3,Tu_2)u_1-\lambda\{u_3,u_2,u_1\}_2\end{aligned},$$

由于$\left(J_2,\{\cdot,\cdot,\cdot\}_2\right)$是约当三系且$\theta_{13}$是对称的，所以在$\left(J_2,\{\cdot,\cdot,\cdot\}_T\right)$上(1.1)式成立。

$$
\begin{aligned}
&\{u_1,u_2,\{u_3,u_4,u_5\}_T\}_T-\{u_3,u_4,\{u_1,u_2,u_5\}_T\}_T-\{\{u_1,u_2,u_3\}_T,u_4,u_5\}_T+\{u_3,\{u_2,u_1,u_4\}_T,u_5\}_T\\
&=\{u_1,u_2,\theta_{12}(Tu_5,Tu_4)u_3+\theta_{13}(Tu_3,Tu_5)u_4+\theta_{12}(Tu_3,Tu_4)u_5+\lambda\{u_3,u_4,u_5\}_2\}_T\\
&\quad-\{u_3,u_4,\theta_{12}(Tu_5,Tu_2)u_1+\theta_{13}(Tu_1,Tu_5)u_2+\theta_{12}(Tu_1,Tu_2)u_5+\lambda\{u_1,u_2,u_5\}_2\}_T\\
&\quad-\{\theta_{12}(Tu_3,Tu_2)u_1+\theta_{13}(Tu_1,Tu_3)u_2+\theta_{12}(Tu_1,Tu_2)u_3+\lambda\{u_1,u_2,u_3\}_2,u_4,u_5\}_T\\
&\quad+\{u_3,\theta_{12}(Tu_4,Tu_1)u_2+\theta_{13}(Tu_2,Tu_4)u_1+\theta_{12}(Tu_2,Tu_1)u_4+\lambda\{u_2,u_1,u_4\}_2,u_5\}_T\\
&=\theta_{12}\left(T\left(\theta_{12}(Tu_5,Tu_4)u_3\right),Tu_2\right)u_1+\theta_{13}\left(Tu_1,T\left(\theta_{12}(Tu_5,Tu_4)u_3\right)\right)u_2+\theta_{12}(Tu_1,Tu_2)\theta_{12}(Tu_5,Tu_4)u_3\\
&\quad+\lambda\{u_1,u_2,\theta_{12}(Tu_5,Tu_4)u_3\}_2+\theta_{12}\left(T\left(\theta_{13}(Tu_3,Tu_5)u_4\right),Tu_2\right)u_1+\theta_{13}\left(Tu_1,T\left(\theta_{13}(Tu_3,Tu_5)u_4\right)\right)u_2\\
&\quad+\theta_{12}(Tu_1,Tu_2)\theta_{13}(Tu_3,Tu_5)u_4+\lambda\{u_1,u_2,\theta_{13}(Tu_3,Tu_5)u_4\}_2+\theta_{12}\left(T\left(\theta_{12}(Tu_3,Tu_4)u_5\right),Tu_2\right)u_1\\
&\quad+\theta_{13}\left(Tu_1,T\left(\theta_{12}(Tu_3,Tu_4)u_5\right)\right)u_2+\theta_{12}(Tu_1,Tu_2)\theta_{12}(Tu_3,Tu_4)u_5+\lambda\{u_1,u_2,\theta_{12}(Tu_3,Tu_4)u_5\}_2\\
&\quad+\theta_{12}\left(T\left(\lambda\{u_3,u_4,u_5\}_2\right),Tu_2\right)u_1+\theta_{13}\left(Tu_1,T\left(\lambda\{u_3,u_4,u_5\}_2\right)\right)u_2+\lambda\theta_{12}(Tu_1,Tu_2)\{u_3,u_4,u_5\}_2\\
&\quad+\lambda\{u_1,u_2,\lambda\{u_3,u_4,u_5\}_2\}_2-\theta_{12}\left(T\left(\theta_{12}(Tu_5,Tu_2)u_1\right),Tu_4\right)u_3-\theta_{13}\left(Tu_3,T\left(\theta_{12}(Tu_5,Tu_2)u_1\right)\right)u_4\\
&\quad-\theta_{12}(Tu_3,Tu_4)\theta_{12}(Tu_5,Tu_2)u_1-\lambda\{u_3,u_4,\theta_{12}(Tu_5,Tu_2)u_1\}_2-\theta_{12}\left(T\left(\theta_{13}(Tu_1,Tu_5)u_2\right),Tu_4\right)u_3\\
&\quad-\theta_{13}\left(Tu_3,T\left(\theta_{13}(Tu_1,Tu_5)u_2\right)\right)u_4-\theta_{12}(Tu_3,Tu_4)\theta_{13}(Tu_1,Tu_5)u_2-\lambda\{u_3,u_4,\theta_{13}(Tu_1,Tu_5)u_2\}_2\\
&\quad-\theta_{12}\left(T\left(\theta_{12}(Tu_1,Tu_2)u_5\right),Tu_4\right)u_3-\theta_{13}\left(Tu_3,T\left(\theta_{12}(Tu_1,Tu_2)u_5\right)\right)u_4-\theta_{12}(Tu_3,Tu_4)\theta_{12}(Tu_1,Tu_2)u_5\\
&\quad-\lambda\{u_3,u_4,\theta_{12}(Tu_1,Tu_2)u_5\}_2-\theta_{12}\left(T\left(\lambda\{u_1,u_2,u_5\}_2\right),Tu_4\right)u_3-\theta_{13}\left(Tu_3,T\left(\lambda\{u_1,u_2,u_5\}_2\right)\right)u_4\\
&\quad-\lambda\theta_{12}(Tu_3,Tu_4)\{u_1,u_2,u_5\}_2-\lambda\{u_3,u_4,\lambda\{u_1,u_2,u_5\}_2\}_2-\theta_{12}(Tu_5,Tu_4)\theta_{12}(Tu_3,Tu_2)u_1\\
&\quad-\theta_{13}\left(T\left(\theta_{12}(Tu_3,Tu_2)u_1\right),Tu_5\right)u_4-\theta_{12}\left(T\left(\theta_{12}(Tu_3,Tu_2)u_1\right),Tu_4\right)u_5-\lambda\{\theta_{12}(Tu_3,Tu_2)u_1,u_4,u_5\}_2\\
&\quad-\theta_{12}(Tu_5,Tu_4)\theta_{13}(Tu_1,Tu_3)u_2-\theta_{13}\left(T\left(\theta_{13}(Tu_1,Tu_3)u_2\right),Tu_5\right)u_4-\theta_{12}\left(T\left(\theta_{13}(Tu_1,Tu_3)u_2\right),Tu_4\right)u_5\\
&\quad-\lambda\{\theta_{13}(Tu_1,Tu_3)u_2,u_4,u_5\}_2-\theta_{12}(Tu_5,Tu_4)\theta_{12}(Tu_1,Tu_2)u_3-\theta_{13}\left(T\left(\theta_{12}(Tu_1,Tu_2)u_3\right),Tu_5\right)u_4\\
&\quad-\theta_{12}\left(T\left(\theta_{12}(Tu_1,Tu_2)u_3\right),Tu_4\right)u_5-\lambda\{\theta_{12}(Tu_1,Tu_2)u_3,u_4,u_5\}_2-\lambda\theta_{12}(Tu_5,Tu_4)\{u_1,u_2,u_3\}_2\\
&\quad-\theta_{13}\left(T\left(\lambda\{u_1,u_2,u_3\}_2\right),Tu_5\right)u_4-\theta_{12}\left(T\left(\lambda\{u_1,u_2,u_3\}_2\right),Tu_4\right)u_5-\lambda\{\lambda\{u_1,u_2,u_3\}_2,u_4,u_5\}_2\\
&\quad+\theta_{12}\left(Tu_5,T\left(\theta_{12}(Tu_4,Tu_1)u_2\right)\right)u_3+\theta_{13}(Tu_3,Tu_5)\theta_{12}(Tu_4,Tu_1)u_2+\theta_{12}\left(Tu_3,T\left(\theta_{12}(Tu_4,Tu_1)u_2\right)\right)u_5\\
&\quad+\lambda\{u_3,\theta_{12}(Tu_4,Tu_1)u_2,u_5\}_2+\theta_{12}\left(Tu_5,T\left(\theta_{13}(Tu_2,Tu_4)u_1\right)\right)u_3+\theta_{13}(Tu_3,Tu_5)\theta_{13}(Tu_2,Tu_4)u_1\\
&\quad+\theta_{12}\left(Tu_3,T\left(\theta_{13}(Tu_2,Tu_4)u_1\right)\right)u_5+\lambda\{u_3,\theta_{13}(Tu_2,Tu_4)u_1,u_5\}_2+\theta_{12}\left(Tu_5,T\left(\theta_{12}(Tu_2,Tu_1)u_4\right)\right)u_3\\
&\quad+\theta_{13}(Tu_3,Tu_5)\theta_{12}(Tu_2,Tu_1)u_4+\theta_{12}\left(Tu_3,T\left(\theta_{12}(Tu_2,Tu_1)u_4\right)\right)u_5+\lambda\{u_3,\theta_{12}(Tu_2,Tu_1)u_4,u_5\}_2\\
&\quad+\theta_{12}\left(Tu_5,T\left(\lambda\{u_2,u_1,u_4\}_2\right)\right)u_3+\lambda\theta_{13}(Tu_3,Tu_5)\{u_2,u_1,u_4\}_2+\theta_{12}\left(Tu_3,T\left(\lambda\{u_2,u_1,u_4\}_2\right)\right)u_5\\
&\quad+\lambda\{u_3,\lambda\{u_2,u_1,u_4\}_2,u_5\}_2\\
&=\lambda^2\left(\{u_1,u_2,\{u_3,u_4,u_5\}_2\}_2-\{u_3,u_4,\{u_1,u_2,u_5\}_2\}_2-\{\{u_1,u_2,u_3\}_2,u_4,u_5\}_2+\{u_3,\{u_2,u_1,u_4\}_2,u_5\}_2\right)\\
&\quad+B_1u_1+B_2u_2+B_3u_3+B_4u_4+B_5u_5+\lambda B_6
\end{aligned}
$$

其中，

$$
\begin{aligned}
B_1 = {} & \theta_{12}\left(T\left(\theta_{12}\left(Tu_5,Tu_4\right)u_3\right),Tu_2\right)+\theta_{12}\left(T\left(\theta_{13}\left(Tu_3,Tu_5\right)u_4\right),Tu_2\right)+\theta_{12}\left(T\left(\theta_{12}\left(Tu_3,Tu_4\right)u_5\right),Tu_2\right) \\
& +\theta_{12}\left(T\left(\lambda\left\{u_3,u_4,u_5\right\}_2\right),Tu_2\right)-\theta_{12}\left(Tu_3,Tu_4\right)\theta_{12}\left(Tu_5,Tu_2\right)-\theta_{12}\left(Tu_5,Tu_4\right)\theta_{12}\left(Tu_3,Tu_2\right) \\
& +\theta_{13}\left(Tu_3,Tu_5\right)\theta_{13}\left(Tu_2,Tu_4\right)
\end{aligned},
$$

$$
\begin{aligned}
B_2 = {} & \theta_{13}\left(Tu_1,T\left(\theta_{12}\left(Tu_5,Tu_4\right)u_3\right)\right)+\theta_{13}\left(Tu_1,T\left(\theta_{13}\left(Tu_3,Tu_5\right)u_4\right)\right)+\theta_{13}\left(Tu_1,T\left(\theta_{12}\left(Tu_3,Tu_4\right)u_5\right)\right) \\
& +\theta_{13}\left(Tu_1,T\left(\lambda\left\{u_3,u_4,u_5\right\}_2\right)\right)-\theta_{12}\left(Tu_3,Tu_4\right)\theta_{13}\left(Tu_1,Tu_5\right)-\theta_{12}\left(Tu_5,Tu_4\right)\theta_{13}\left(Tu_1,Tu_3\right) \\
& +\theta_{13}\left(Tu_3,Tu_5\right)\theta_{12}\left(Tu_4,Tu_1\right)
\end{aligned},
$$

$$
\begin{aligned}
B_3 = {} & \theta_{12}\left(Tu_1,Tu_2\right)\theta_{12}\left(Tu_5,Tu_4\right)-\theta_{12}\left(T\left(\theta_{12}\left(Tu_5,Tu_2\right)u_1\right),Tu_4\right)-\theta_{12}\left(T\left(\theta_{13}\left(Tu_1,Tu_5\right)u_2\right),Tu_4\right) \\
& -\theta_{12}\left(T\left(\theta_{12}\left(Tu_1,Tu_2\right)u_5\right),Tu_4\right)-\theta_{12}\left(T\left(\lambda\left\{u_1,u_2,u_5\right\}_2\right),Tu_4\right)-\theta_{12}\left(Tu_5,Tu_4\right)\theta_{12}\left(Tu_1,Tu_2\right) \\
& +\theta_{12}\left(Tu_5,T\left(\theta_{12}\left(Tu_4,Tu_1\right)u_2\right)\right)+\theta_{12}\left(Tu_5,T\left(\theta_{13}\left(Tu_2,Tu_4\right)u_1\right)\right)+\theta_{12}\left(Tu_5,T\left(\theta_{12}\left(Tu_2,Tu_1\right)u_4\right)\right) \\
& +\theta_{12}\left(Tu_5,T\left(\lambda\left\{u_2,u_1,u_4\right\}_2\right)\right)
\end{aligned},
$$

$$
\begin{aligned}
B_4 = {} & \theta_{12}\left(Tu_1,Tu_2\right)\theta_{13}\left(Tu_3,Tu_5\right)-\theta_{13}\left(Tu_3,T\left(\theta_{12}\left(Tu_5,Tu_2\right)u_1\right)\right)-\theta_{13}\left(Tu_3,T\left(\theta_{13}\left(Tu_1,Tu_5\right)u_2\right)\right) \\
& -\theta_{13}\left(Tu_3,T\left(\theta_{12}\left(Tu_1,Tu_2\right)u_5\right)\right)-\theta_{13}\left(Tu_3,T\left(\lambda\left\{u_1,u_2,u_5\right\}_2\right)\right)-\theta_{13}\left(T\left(\theta_{12}\left(Tu_3,Tu_2\right)u_1\right),Tu_5\right) \\
& -\theta_{13}\left(T\left(\theta_{13}\left(Tu_1,Tu_3\right)u_2\right),Tu_5\right)-\theta_{13}(T(\theta_{12}(Tu_1,Tu_2)u_3),Tu_5)-\theta_{13}\left(T\left(\lambda\left\{u_1,u_2,u_3\right\}_2\right),Tu_5\right) \\
& +\theta_{13}\left(Tu_3,Tu_5\right)\theta_{12}\left(Tu_2,Tu_1\right)
\end{aligned},
$$

$$
\begin{aligned}
B_5 = {} & \theta_{12}\left(Tu_1,Tu_2\right)\theta_{12}\left(Tu_3,Tu_4\right)-\theta_{12}\left(Tu_3,Tu_4\right)\theta_{12}\left(Tu_1,Tu_2\right)-\theta_{12}\left(T\left(\theta_{12}\left(Tu_3,Tu_2\right)u_1\right),Tu_4\right) \\
& -\theta_{12}\left(T\left(\theta_{13}\left(Tu_1,Tu_3\right)u_2\right),Tu_4\right)-\theta_{12}\left(T\left(\theta_{12}\left(Tu_1,Tu_2\right)u_3\right),Tu_4\right)-\theta_{12}\left(T\left(\lambda\left\{u_1,u_2,u_3\right\}_2\right),Tu_4\right) \\
& +\theta_{12}\left(Tu_3,T\left(\theta_{12}\left(Tu_4,Tu_1\right)u_2\right)\right)+\theta_{12}\left(Tu_3,T\left(\theta_{13}\left(Tu_2,Tu_4\right)u_1\right)\right)+\theta_{12}\left(Tu_3,T\left(\theta_{12}\left(Tu_2,Tu_1\right)u_4\right)\right) \\
& +\theta_{12}\left(Tu_3,T\left(\lambda\left\{u_2,u_1,u_4\right\}_2\right)\right)
\end{aligned},
$$

$$
\begin{aligned}
B_6 = {} & \left\{u_1,u_2,\theta_{12}\left(Tu_5,Tu_4\right)u_3\right\}_2+\left\{u_1,u_2,\theta_{13}\left(Tu_3,Tu_5\right)u_4\right\}_2+\left\{u_1,u_2,\theta_{12}\left(Tu_3,Tu_4\right)u_5\right\}_2 \\
& +\theta_{12}\left(Tu_1,Tu_2\right)\left\{u_3,u_4,u_5\right\}_2-\left\{u_3,u_4,\theta_{12}\left(Tu_5,Tu_2\right)u_1\right\}_2-\left\{u_3,u_4,\theta_{13}\left(Tu_1,Tu_5\right)u_2\right\}_2 \\
& -\left\{u_3,u_4,\theta_{12}\left(Tu_1,Tu_2\right)u_5\right\}_2-\theta_{12}\left(Tu_3,Tu_4\right)\left\{u_1,u_2,u_5\right\}_2-\left\{\theta_{12}\left(Tu_3,Tu_2\right)u_1,u_4,u_5\right\}_2 \\
& -\left\{\theta_{13}\left(Tu_1,Tu_3\right)u_2,u_4,u_5\right\}_2-\left\{\theta_{12}\left(Tu_1,Tu_2\right)u_3,u_4,u_5\right\}_2-\theta_{12}\left(Tu_5,Tu_4\right)\left\{u_1,u_2,u_3\right\}_2 \\
& +\left\{u_3,\theta_{12}\left(Tu_4,Tu_1\right)u_2,u_5\right\}_2+\left\{u_3,\theta_{13}\left(Tu_2,Tu_4\right)u_1,u_5\right\}_2+\left\{u_3,\theta_{12}\left(Tu_2,Tu_1\right)u_4,u_5\right\}_2 \\
& +\theta_{13}\left(Tu_3,Tu_5\right)\left\{u_2,u_1,u_4\right\}_2
\end{aligned},
$$

由$\left(J_2,\{\cdot,\cdot,\cdot\}_2\right)$是约当三系知(1.2)式成立，由(2.4)式成立知 B_1 为零，由(2.5)式成立知 B_2 为零，由(2.2)式成立知 B_3 为零，由(2.3)式成立知 B_4 为零，由(2.2)式成立知 B_5 为零，由于 θ_{12},θ_{13} 是$\left(J_1,\{\cdot,\cdot,\cdot\}_1\right)$在$\left(J_2,\{\cdot,\cdot,\cdot\}_2\right)$上的作用，故有$\left\{u_1,u_2,\theta_{12}\left(Tu_5,Tu_4\right)u_3\right\}_2=0$，$\theta_{12}\left(Tu_1,Tu_2\right)\left\{u_3,u_4,u_5\right\}_2=0$，类似可知 B_6 为零。故在$\left(J_2,\{\cdot,\cdot,\cdot\}_T\right)$上(1.2)式成立。综上，$\left(J_2,\{\cdot,\cdot,\cdot\}_T\right)$也是约当三系。

下证 T 是从$\left(J_2,\{\cdot,\cdot,\cdot\}_T\right)$到$\left(J_1,\{\cdot,\cdot,\cdot\}_1\right)$的同态。由于线性映射 $T:J_2\to J_1$ 是关于作用 θ_{12},θ_{13} 的权为 λ 的相对 Rota-Baxter 算子，所以，

$$
T\left(\{u,v,w\}_T\right)=T\left(\theta_{12}\left(Tw,Tv\right)u+\theta_{13}\left(Tu,Tw\right)v+\theta_{12}\left(Tu,Tv\right)w+\lambda\{u,v,w\}_2\right)=\{Tu,Tv,Tw\}_1,
$$

$\forall u,v,w\in J_2$，因此，T 是从 $\left(J_2,\{\cdot,\cdot,\cdot\}_T\right)$ 到 $\left(J_1,\{\cdot,\cdot,\cdot\}_1\right)$ 的代数同态。

命题 3.2 $\left(J_1,\{\cdot,\cdot,\cdot\}_1\right),\left(J_2,\{\cdot,\cdot,\cdot\}_2\right)$ 是约当三系，线性映射 $T:J_2\to J_1$ 是关于作用 θ_{12},θ_{13} 的权为 λ 的相对 Rota-Baxter 算子，定义双线性映射 $L_T,M_T:J_2\times J_2\to End\left(J_1\right)$，其中

$$L_T\left(u,v\right)x=\{Tu,Tv,x\}_1-T\left(\theta_{13}\left(Tu,x\right)v+\theta_{12}\left(x,Tv\right)u\right),$$

$$M_T\left(u,v\right)x=\{Tu,x,Tv\}_1-T\left(\theta_{12}\left(Tv,x\right)u+\theta_{12}\left(Tu,x\right)v\right),$$

$x\in J_1,u,v\in J_2$，则 $\left(L_T,M_T,J_1\right)$ 是 $\left(J_2,\{\cdot,\cdot,\cdot\}_T\right)$ 的表示。

证：显然，L_T,M_T 是线性映射，$\forall u_1,u_2,u_3,u_4\in J_2,x\in J_1$，计算(2.1)式左边可得

$$\begin{aligned}&\left(M_T\left(u_1,u_2\right)-M_T\left(u_2,u_1\right)\right)x\\&=\{Tu_1,x,Tu_2\}_1-T\left(\theta_{12}\left(Tu_2,x\right)u_1+\theta_{12}\left(Tu_1,x\right)u_2\right)-\{Tu_2,x,Tu_1\}_1+T\left(\theta_{12}\left(Tu_1,x\right)u_2+\theta_{12}\left(Tu_2,x\right)u_1\right),\end{aligned}$$

由于 $\left(J_1,\{\cdot,\cdot,\cdot\}_1\right)$ 是约当三系，故(2.1)式成立。

计算(2.2)式左边可得

$$\begin{aligned}&\left(L_T\left(u_1,u_2\right)L_T\left(u_3,u_4\right)-L_T\left(u_3,u_4\right)L_T\left(u_1,u_2\right)-L_T\left(\{u_1,u_2,u_3\}_T,u_4\right)+L_T\left(u_3,\{u_2,u_1,u_4\}_T\right)\right)x\\
&=L_T\left(u_1,u_2\right)\{Tu_3,Tu_4,x\}_1-L_T\left(u_1,u_2\right)T\theta_{12}\left(x,Tu_4\right)u_3-L_T\left(u_1,u_2\right)T\theta_{13}\left(Tu_3,x\right)u_4\\
&\quad-L_T\left(u_3,u_4\right)\{Tu_1,Tu_2,x\}_1+L_T\left(u_3,u_4\right)T\theta_{12}\left(x,Tu_2\right)u_1+L_T\left(u_3,u_4\right)T\theta_{13}\left(Tu_1,x\right)u_2\\
&\quad-\left\{\{Tu_1,Tu_2,Tu_3\}_1,Tu_4,x\right\}_1+T\theta_{12}\left(x,Tu_4\right)\{u_1,u_2,u_3\}_T+T\theta_{13}\left(\{Tu_1,Tu_2,Tu_3\}_1,x\right)u_4\\
&\quad+\left\{Tu_3,\{Tu_2,Tu_1,Tu_4\}_1,x\right\}_1-T\theta_{12}\left(x,\{Tu_2,Tu_1,Tu_4\}_1\right)u_3-T\theta_{13}\left(Tu_3,x\right)\{u_2,u_1,u_4\}_T\\
&=\left\{Tu_1,Tu_2,\{Tu_3,Tu_4,x\}_1\right\}_1-T\theta_{12}\left(\{Tu_3,Tu_4,x\}_1,Tu_2\right)u_1-T\theta_{13}\left(Tu_1,\{Tu_3,Tu_4,x\}_1\right)u_2\\
&\quad-\left\{Tu_1,Tu_2,T\theta_{12}\left(x,Tu_4\right)u_3\right\}_1+T\theta_{12}\left(T\theta_{12}\left(x,Tu_4\right)u_3,Tu_2\right)u_1+T\theta_{13}\left(Tu_1,T\theta_{12}\left(x,Tu_4\right)u_3\right)u_2\\
&\quad-\left\{Tu_1,Tu_2,T\theta_{13}\left(Tu_3,x\right)u_4\right\}_1+T\theta_{12}\left(T\theta_{13}\left(Tu_3,x\right)u_4,Tu_2\right)u_1+T\theta_{13}\left(Tu_1,T\theta_{13}\left(Tu_3,x\right)u_4\right)u_2\\
&\quad-\left\{Tu_3,Tu_4,\{Tu_1,Tu_2,x\}_1\right\}_1+T\theta_{12}\left(\{Tu_1,Tu_2,x\}_1,Tu_4\right)u_3+T\theta_{13}\left(Tu_3,\{Tu_1,Tu_2,x\}_1\right)u_4\\
&\quad+\left\{Tu_3,Tu_4,T\theta_{12}\left(x,Tu_2\right)u_1\right\}_1-T\theta_{12}\left(T\theta_{12}\left(x,Tu_2\right)u_1,Tu_4\right)u_3-T\theta_{13}\left(Tu_3,T\theta_{12}\left(x,Tu_2\right)u_1\right)u_4\\
&\quad+\left\{Tu_3,Tu_4,T\theta_{13}\left(Tu_1,x\right)u_2\right\}_1-T\theta_{12}\left(T\theta_{13}\left(Tu_1,x\right)u_2,Tu_4\right)u_3-T\theta_{13}\left(Tu_3,T\theta_{13}\left(Tu_1,x\right)u_2\right)u_4\\
&\quad-\left\{\{Tu_1,Tu_2,Tu_3\}_1,Tu_4,x\right\}_1+T\theta_{13}\left(\{Tu_1,Tu_2,Tu_3\}_1,x\right)u_4+T\theta_{12}\left(x,Tu_4\right)\theta_{12}\left(Tu_3,Tu_2\right)u_1\\
&\quad+T\theta_{12}\left(x,Tu_4\right)\theta_{12}\left(Tu_1,Tu_2\right)u_3+T\theta_{12}\left(x,Tu_4\right)\theta_{13}\left(Tu_1,Tu_3\right)u_2+\lambda T\theta_{12}\left(x,Tu_4\right)\{u_1,u_2,u_3\}_2\\
&\quad+\left\{Tu_3,\{Tu_2,Tu_1,Tu_4\}_1,x\right\}_1-T\theta_{12}\left(x,\{Tu_2,Tu_1,Tu_4\}_1\right)u_3-T\theta_{13}\left(Tu_3,x\right)\theta_{12}\left(Tu_4,Tu_1\right)u_2\\
&\quad-T\theta_{13}\left(Tu_3,x\right)\theta_{12}\left(Tu_2,Tu_1\right)u_4-T\theta_{13}\left(Tu_3,x\right)\theta_{13}\left(Tu_2,Tu_4\right)u_1-\lambda T\theta_{13}\left(Tu_3,x\right)\{u_2,u_1,u_4\}_2\\
&=\left(\left\{Tu_1,Tu_2,\{Tu_3,Tu_4,x\}_1\right\}_1-\left\{Tu_3,Tu_4,\{Tu_1,Tu_2,x\}_1\right\}_1-\left\{\{Tu_1,Tu_2,Tu_3\}_1,Tu_4,x\right\}_1\right.\\
&\quad\left.+\left\{Tu_3,\{Tu_2,Tu_1,Tu_4\}_1,x\right\}_1\right)-T\left(C_{11}u_1+C_{12}u_2+C_{13}u_3+C_{14}u_4+\lambda C_{15}\right)\end{aligned}$$

其中，

$$C_{11}=\theta_{12}\left(\{Tu_3,Tu_4,x\}_1,Tu_2\right)-\theta_{12}\left(Tu_3,Tu_4\right)\theta_{12}\left(x,Tu_2\right)-\theta_{12}\left(x,Tu_4\right)\theta_{12}\left(Tu_3,Tu_2\right)+\theta_{13}\left(Tu_3,x\right)\theta_{13}\left(Tu_2,Tu_4\right),$$

$$C_{12}=\theta_{13}\left(Tu_1,\{Tu_3,Tu_4,x\}_1\right)-\theta_{12}\left(Tu_3,Tu_4\right)\theta_{13}\left(Tu_1,x\right)-\theta_{12}\left(x,Tu_4\right)\theta_{13}\left(Tu_1,Tu_3\right)+\theta_{13}\left(Tu_3,x\right)\theta_{12}\left(Tu_4,Tu_1\right),$$

$$C_{13}=\theta_{12}\left(Tu_1,Tu_2\right)\theta_{12}\left(x,Tu_4\right)-\theta_{12}\left(\{Tu_1,Tu_2,x\}_1,Tu_4\right)-\theta_{12}\left(x,Tu_4\right)\theta_{12}\left(Tu_1,Tu_2\right)+\theta_{12}\left(x,\{Tu_2,Tu_1,Tu_4\}_1\right),$$

$$C_{14}=\theta_{12}(Tu_1,Tu_2)\theta_{13}(Tu_3,x)-\theta_{13}(Tu_3,\{Tu_1,Tu_2,x\}_1)-\theta_{13}(\{Tu_1,Tu_2,Tu_3\}_1,x)+\theta_{13}(Tu_3,x)\theta_{12}(Tu_2,Tu_1),$$

$$\begin{aligned}C_{15}=&\{u_1,u_2,\theta_{12}(x,Tu_4)u_3\}_2+\{u_1,u_2,\theta_{13}(Tu_3,x)\}_2-\{u_3,u_4,\theta_{12}(x,Tu_2)u_1\}_2-\{u_3,u_4,\theta_{13}(Tu_1,x)\}_2\\&-\theta_{12}(x,Tu_4)\{u_1,u_2,u_3\}_2+\theta_{13}(Tu_3,x)\{u_2,u_1,u_4\}_2\end{aligned},$$

由于 $(J_1,\{\cdot,\cdot,\cdot\}_1)$ 是约当三系，故(1.2)式成立，由(2.4)式成立知 C_{11} 为零，由(2.5)式成立知 C_{12} 为零，由(2.2)式成立知 C_{13} 为零，由(2.3)式成立知 C_{14} 为零，由于 θ_{12},θ_{13} 是 $(J_1,\{\cdot,\cdot,\cdot\}_1)$ 在 $(J_2,\{\cdot,\cdot,\cdot\}_2)$ 上的作用，故有 $\{u_1,u_2,\theta_{12}(x,Tu_4)u_3\}_2=0$， $\theta_{12}(x,Tu_4)\{u_1,u_2,u_3\}_2=0$，类似可知 C_{15} 为零，故(2.2)式成立。

计算(2.3)式左边可得

$$\begin{aligned}&\left(L_T(u_1,u_2)M_T(u_3,u_4)+M_T(u_3,u_4)L_T(u_2,u_1)-M_T(u_3,\{u_1,u_2,u_4\}_T)-M_T(\{u_1,u_2,u_3\}_T,u_4)\right)x\\
&=L_T(u_1,u_2)\{Tu_3,x,Tu_4\}_1-L_T(u_1,u_2)T\theta_{12}(Tu_4,x)u_3-L_T(u_1,u_2)T\theta_{12}(Tu_3,x)u_4\\
&\quad+M_T(u_3,u_4)\{Tu_2,Tu_1,x\}_1-M_T(u_3,u_4)T\theta_{13}(Tu_2,x)u_1-M_T(u_3,u_4)T\theta_{12}(x,Tu_1)u_2\\
&\quad-\{Tu_3,x,\{Tu_1,Tu_2,Tu_4\}_1\}_1+T\theta_{12}(Tu_3,x)\{u_1,u_2,u_4\}_T+T\theta_{12}(\{Tu_1,Tu_2,Tu_4\}_1,x)u_3\\
&\quad-\{\{Tu_1,Tu_2,Tu_3\}_1,x,Tu_4\}_1+T\theta_{12}(Tu_4,x)\{u_1,u_2,u_3\}_T+T\theta_{12}(\{Tu_1,Tu_2,Tu_3\}_1,x)u_4\\
&=\{Tu_1,Tu_2,\{Tu_3,x,Tu_4\}_1\}_1-T\theta_{13}(Tu_1,\{Tu_3,x,Tu_4\}_1)u_2-T\theta_{12}(\{Tu_3,x,Tu_4\}_1,Tu_2)u_1\\
&\quad-\{Tu_1,Tu_2,T\theta_{12}(Tu_4,x)u_3\}_1+T\theta_{13}(Tu_1,T\theta_{12}(Tu_4,x)u_3)u_2+T\theta_{12}(T\theta_{12}(Tu_4,x)u_3,Tu_2)u_1\\
&\quad-\{Tu_1,Tu_2,T\theta_{12}(Tu_3,x)u_4\}_1+T\theta_{13}(Tu_1,T\theta_{12}(Tu_3,x)u_4)u_2+T\theta_{12}(T\theta_{12}(Tu_3,x)u_4,Tu_2)u_1\\
&\quad+\{Tu_3,\{Tu_2,Tu_1,x\}_1,Tu_4\}_1-T\theta_{12}(Tu_4,\{Tu_2,Tu_1,x\}_1)u_3-T\theta_{12}(Tu_3,\{Tu_2,Tu_1,x\}_1)u_4\\
&\quad-\{Tu_3,T\theta_{13}(Tu_2,x)u_1,Tu_4\}_1+T\theta_{12}(Tu_4,T\theta_{13}(Tu_2,x)u_1)u_3+T\theta_{12}(Tu_3,T\theta_{13}(Tu_2,x)u_1)u_4\\
&\quad-\{Tu_3,T\theta_{12}(x,Tu_1)u_2,Tu_4\}_1+T\theta_{12}(Tu_4,T\theta_{12}(x,Tu_1)u_2)u_3+T\theta_{12}(Tu_3,T\theta_{12}(x,Tu_1)u_2)u_4\\
&\quad-\{Tu_3,x,\{Tu_1,Tu_2,Tu_4\}_1\}_1+T\theta_{12}(\{Tu_1,Tu_2,Tu_4\}_1,x)u_3+T\theta_{12}(Tu_3,x)\theta_{12}(Tu_4,Tu_2)u_1\\
&\quad+T\theta_{12}(Tu_3,x)\theta_{12}(Tu_1,Tu_2)u_4+T\theta_{12}(Tu_3,x)\theta_{13}(Tu_1,Tu_4)u_2+\lambda T\theta_{12}(Tu_3,x)\{u_1,u_2,u_4\}_2\\
&\quad-\{\{Tu_1,Tu_2,Tu_3\}_1,x,Tu_4\}_1+T\theta_{12}(\{Tu_1,Tu_2,Tu_3\}_1,x)u_4+T\theta_{12}(Tu_4,x)\theta_{12}(Tu_3,Tu_2)u_1\\
&\quad+T\theta_{12}(Tu_4,x)\theta_{12}(Tu_1,Tu_2)u_3+T\theta_{12}(Tu_4,x)\theta_{13}(Tu_1,Tu_3)u_2+\lambda T\\
&=\left(\{Tu_1,Tu_2,\{Tu_3,x,Tu_4\}_1\}_1-\{Tu_3,x,\{Tu_1,Tu_2,Tu_4\}_1\}_1-\{\{Tu_1,Tu_2,Tu_3\}_1,x,Tu_4\}_1\right.\\
&\quad\left.+\{Tu_3,\{Tu_2,Tu_1,x\}_1,Tu_4\}_1\right)-T\left(C_{21}u_1+C_{22}u_2+C_{23}u_3+C_{24}u_4+\lambda C_{25}\right)\end{aligned}$$

其中，

$$C_{21}=\theta_{12}(\{Tu_3,x,Tu_4\}_1,Tu_2)+\theta_{13}(Tu_3,Tu_4)\theta_{13}(Tu_2,x)-\theta_{12}(Tu_3,x)\theta_{12}(Tu_4,Tu_2)-\theta_{12}(Tu_4,x)\theta_{12}(Tu_3,Tu_2),$$

$$C_{22}=\theta_{13}(Tu_1,\{Tu_3,x,Tu_4\}_1)+\theta_{13}(Tu_3,Tu_4)\theta_{12}(x,Tu_1)-\theta_{12}(Tu_3,x)\theta_{13}(Tu_1,Tu_4)-\theta_{12}(Tu_4,x)\theta_{13}(Tu_1,Tu_3),$$

$$C_{23}=\theta_{12}(Tu_1,Tu_2)\theta_{12}(Tu_4,x)+\theta_{12}(Tu_4,\{Tu_2,Tu_1,x\}_1)-\theta_{12}(\{Tu_1,Tu_2,Tu_4\}_1,x)-\theta_{12}(Tu_4,x)\theta_{12}(Tu_1,Tu_2),$$

$$C_{24}=\theta_{12}(Tu_1,Tu_2)\theta_{12}(Tu_3,x)+\theta_{12}(Tu_3,\{Tu_2,Tu_1,x\}_1)-\theta_{12}(Tu_3,x)\theta_{12}(Tu_1,Tu_2)-\theta_{12}(\{Tu_1,Tu_2,Tu_3\}_1,x),$$

$$\begin{aligned}C_{25}=&\{u_1,u_2,\theta_{12}(Tu_4,x)u_3\}_2+\{u_1,u_2,\theta_{12}(Tu_3,x)u_4\}_1-\theta_{12}(Tu_4,x)\{u_1,u_2,u_3\}_2\\&+\{u_3,\theta_{13}(Tu_2,x)u_1,u_4\}_1+\{u_3,\theta_{12}(x,Tu_1)u_2,u_4\}_1-\theta_{12}(Tu_3,x)\{u_1,u_2,u_4\}_2\end{aligned},$$

由于 $(J_1,\{\cdot,\cdot,\cdot\}_1)$ 是约当三系，故(1.2)式成立，由(2.4)式成立知 C_{21} 为零，由(2.5)式成立知 C_{22} 为零，

由(2.2)式成立知 C_{23} 为零，由(2.2)式成立知 C_{24} 为零，由于 θ_{12},θ_{13} 是 $\left(J_1,\{\cdot,\cdot,\cdot\}_1\right)$ 在 $\left(J_2,\{\cdot,\cdot,\cdot\}_2\right)$ 上的作用，故有 $\left\{u_3,\theta_{13}\left(x,Tu_2\right)u_1,u_4\right\}_2=0$， $\theta_{12}\left(x,Tu_4\right)\left\{u_1,u_2,u_3\right\}_2=0$，类似可知 C_{25} 为零，故(2.3)式成立。

计算(2.4)式左边可得

$$
\begin{aligned}
&\left(L_T\left(\{u_1,u_2,u_3\}_T,u_4\right)-L_T\left(u_1,u_2\right)L_T\left(u_3,u_4\right)-L_T\left(u_3,u_2\right)L_T\left(u_1,u_4\right)+M_T\left(u_1,u_3\right)M_T\left(u_2,u_4\right)\right)x\\
&=\left\{\{Tu_1,Tu_2,Tu_3\}_1,Tu_4,x\right\}_1-T\theta_{13}\left(\{Tu_1,Tu_2,Tu_3\}_1,x\right)u_4-T\theta_{12}\left(x,Tu_4\right)\{u_1,u_2,u_3\}_T\\
&\quad-L_T\left(u_1,u_2\right)\{Tu_3,Tu_4,x\}_1+L_T\left(u_1,u_2\right)T\theta_{13}\left(Tu_3,x\right)u_4+L_T\left(u_1,u_2\right)T\theta_{12}\left(x,Tu_4\right)u_3\\
&\quad-L_T\left(u_3,u_2\right)\{Tu_1,Tu_4,x\}_1+L_T\left(u_3,u_2\right)T\theta_{13}\left(Tu_1,x\right)u_4+L_T\left(u_3,u_2\right)T\theta_{12}\left(x,Tu_4\right)u_1\\
&\quad+M_T\left(u_1,u_3\right)\{Tu_2,x,Tu_4\}_1-M_T\left(u_1,u_3\right)T\theta_{12}\left(Tu_4,x\right)u_2-M_T\left(u_1,u_3\right)T\theta_{12}\left(Tu_2,x\right)u_4\\
&=\left\{\{Tu_1,Tu_2,Tu_3\}_1,Tu_4,x\right\}_1-T\theta_{13}\left(\{Tu_1,Tu_2,Tu_3\}_1,x\right)u_4-T\theta_{12}\left(x,Tu_4\right)\theta_{12}\left(Tu_3,Tu_2\right)u_1\\
&\quad-T\theta_{12}\left(x,Tu_4\right)\theta_{12}\left(Tu_1,Tu_2\right)u_3-T\theta_{12}\left(x,Tu_4\right)\theta_{13}\left(Tu_1,Tu_3\right)u_2-\lambda T\theta_{12}\left(x,Tu_4\right)\{u_1,u_2,u_3\}_2\\
&\quad-\left\{Tu_1,Tu_2,\{Tu_3,Tu_4,x\}_1\right\}_1+T\theta_{13}\left(Tu_1,\{Tu_3,Tu_4,x\}_1\right)u_2+T\theta_{12}\left(\{Tu_3,Tu_4,x\}_1,Tu_2\right)u_1\\
&\quad+\left\{Tu_1,Tu_2,T\theta_{13}\left(Tu_3,x\right)u_4\right\}_1-T\theta_{13}\left(Tu_1,T\theta_{13}\left(Tu_3,x\right)u_4\right)u_2-T\theta_{12}\left(T\theta_{13}\left(Tu_3,x\right)u_4,Tu_2\right)u_1\\
&\quad+\left\{Tu_1,Tu_2,T\theta_{12}\left(x,Tu_4\right)u_3\right\}_1-T\theta_{13}\left(Tu_1,T\theta_{12}\left(x,Tu_4\right)u_3\right)u_2-T\theta_{12}\left(T\theta_{12}\left(x,Tu_4\right)u_3,Tu_2\right)u_1\\
&\quad-\left\{Tu_3,Tu_2,\{Tu_1,Tu_4,x\}_1\right\}_1+T\theta_{13}\left(Tu_3,\{Tu_1,Tu_4,x\}_1\right)u_2+T\theta_{12}\left(\{Tu_1,Tu_4,x\}_1,Tu_2\right)u_3\\
&\quad+\left\{Tu_3,Tu_2,T\theta_{13}\left(Tu_1,x\right)u_4\right\}_1-T\theta_{13}\left(Tu_3,T\theta_{13}\left(Tu_1,x\right)u_4\right)u_2-T\theta_{12}\left(T\theta_{13}\left(Tu_1,x\right)u_4,Tu_2\right)u_3\\
&\quad+\left\{Tu_3,Tu_2,T\theta_{12}\left(x,Tu_4\right)u_1\right\}_1-T\theta_{13}\left(Tu_3,T\theta_{12}\left(x,Tu_4\right)u_1\right)u_2-T\theta_{12}\left(T\theta_{12}\left(x,Tu_4\right)u_1,Tu_2\right)u_3\\
&\quad+\left\{Tu_1,\{Tu_2,x,Tu_4\}_1,Tu_3\right\}_1-T\theta_{12}\left(Tu_3,\{Tu_2,x,Tu_4\}_1\right)u_1-T\theta_{12}\left(Tu_1,\{Tu_2,x,Tu_4\}_1\right)u_3\\
&\quad-\left\{Tu_1,T\theta_{12}\left(Tu_4,x\right)u_2,Tu_3\right\}_1+T\theta_{12}\left(Tu_3,T\theta_{12}\left(Tu_4,x\right)u_2\right)u_1+T\theta_{12}\left(Tu_1,T\theta_{12}\left(Tu_4,x\right)u_2\right)u_3\\
&\quad-\left\{Tu_1,T\theta_{12}\left(Tu_2,x\right)u_4,Tu_3\right\}_1+T\theta_{12}\left(Tu_3,T\theta_{12}\left(Tu_2,x\right)u_4\right)u_1+T\theta_{12}\left(Tu_1,T\theta_{12}\left(Tu_2,x\right)u_4\right)u_3\\
&=\left\{\{Tu_1,Tu_2,Tu_3\}_1,Tu_4,x\right\}_1-\left\{Tu_1,Tu_2,\{Tu_3,Tu_4,x\}_1\right\}_1-\left\{Tu_3,Tu_2,\{Tu_1,Tu_4,x\}_1\right\}_1\\
&\quad+\left\{Tu_1,\{Tu_2,x,Tu_4\}_1,Tu_3\right\}_1-T\left(C_{31}u_1+C_{32}u_2+C_{33}u_3+C_{34}u_4+\lambda C_{35}\right)
\end{aligned}
$$

其中，

$$C_{31}=\theta_{12}\left(x,Tu_4\right)\theta_{12}\left(Tu_3,Tu_2\right)-\theta_{12}\left(\{Tu_3,Tu_4,x\}_1,Tu_2\right)-\theta_{12}\left(Tu_3,Tu_2\right)\theta_{12}\left(x,Tu_4\right)+\theta_{12}\left(Tu_3,\{Tu_2,x,Tu_4\}_1\right),$$

$$C_{32}=\theta_{12}\left(x,Tu_4\right)\theta_{13}\left(Tu_1,Tu_3\right)-\theta_{13}\left(Tu_1,\{Tu_3,Tu_4,x\}_1\right)-\theta_{13}\left(Tu_3,\{Tu_1,Tu_4,x\}_1\right)+\theta_{13}\left(Tu_1,Tu_3\right)\theta_{12}\left(Tu_4,x\right),$$

$$C_{33}=\theta_{12}\left(x,Tu_4\right)\theta_{12}\left(Tu_1,Tu_2\right)-\theta_{12}\left(Tu_1,Tu_2\right)\theta_{12}\left(x,Tu_4\right)-\theta_{12}\left(\{Tu_1,Tu_4,x\}_1,Tu_2\right)+\theta_{12}\left(Tu_1,\{Tu_2,x,Tu_4\}_1\right),$$

$$C_{34}=\theta_{13}\left(\{Tu_1,Tu_2,Tu_3\}_1,x\right)-\theta_{12}\left(Tu_1,Tu_2\right)\theta_{13}\left(Tu_3,x\right)-\theta_{12}\left(Tu_3,Tu_2\right)\theta_{13}\left(Tu_1,x\right)+\theta_{13}\left(Tu_1,Tu_3\right)\theta_{12}\left(Tu_2,x\right),$$

$$
\begin{aligned}
C_{35}=&\,\theta_{12}\left(x,Tu_4\right)\{u_1,u_2,u_3\}_2-\left\{u_1,u_2,\theta_{13}\left(Tu_3,x\right)u_4\right\}_2-\left\{u_1,u_2,\theta_{12}\left(x,Tu_4\right)u_3\right\}_2-\left\{u_3,u_2,\theta_{13}\left(Tu_1,x\right)u_4\right\}_2\\
&-\left\{u_3,u_2,\theta_{12}\left(x,Tu_4\right)u_1\right\}_2+\left\{u_1,\theta_{12}\left(Tu_4,x\right)u_2,u_3\right\}_2+\left\{u_1,\theta_{12}\left(Tu_2,x\right)u_4,u_3\right\}_2,
\end{aligned}
$$

由于 $\left(J_1,\{\cdot,\cdot,\cdot\}_1\right)$ 是约当三系，故(1.2)式成立，由(2.2)式成立知 C_{31} 为零，由(2.3)式成立知 C_{32} 为零，由(2.2)式成立知 C_{33} 为零，由(2.5)式成立知 C_{34} 为零，由于 θ_{12},θ_{13} 是 $\left(J_1,\{\cdot,\cdot,\cdot\}_1\right)$ 在 $\left(J_2,\{\cdot,\cdot,\cdot\}_2\right)$ 上的作用，故有 $\left\{u_1,u_2,\theta_{13}\left(Tu_3,x\right)u_4\right\}_2=0$， $\theta_{12}\left(x,Tu_4\right)\left\{u_1,u_2,u_3\right\}_2=0$，类似可知 C_{35} 为零，故(2.4)式成立。

计算(2.5)式左边可得

$$
\begin{aligned}
&\left(M_T\left(\{u_1,u_2,u_3\}_T,u_4\right)-L_T\left(u_1,u_2\right)M_T\left(u_3,u_4\right)-L_T\left(u_3,u_2\right)M_T\left(u_1,u_4\right)+M_T\left(u_1,u_3\right)L_T\left(u_2,u_4\right)\right)x\\
&=\left\{\{Tu_1,Tu_2,Tu_3\}_1,x,Tu_4\right\}_1-T\theta_{12}\left(Tu_4,x\right)\{u_1,u_2,u_3\}_T-T\theta_{12}\left(\{Tu_1,Tu_2,Tu_3\}_1,x\right)u_4\\
&\quad-L_T\left(u_1,u_2\right)\{Tu_3,x,Tu_4\}_1+L_T\left(u_1,u_2\right)T\theta_{12}\left(Tu_4,x\right)u_3+L_T\left(u_1,u_2\right)T\theta_{12}\left(Tu_3,x\right)u_4\\
&\quad-L_T\left(u_3,u_2\right)\{Tu_1,x,Tu_4\}_1+L_T\left(u_3,u_2\right)T\theta_{12}\left(Tu_4,x\right)u_1+L_T\left(u_3,u_2\right)T\theta_{12}\left(Tu_1,x\right)u_4\\
&\quad+M_T\left(u_1,u_3\right)\{Tu_2,Tu_4,x\}_1-M_T\left(u_1,u_3\right)T\theta_{13}\left(Tu_2,x\right)u_4-M_T\left(u_1,u_3\right)T\theta_{12}\left(x,Tu_4\right)u_2\\
&=\left\{\{Tu_1,Tu_2,Tu_3\}_1,x,Tu_4\right\}_1-T\theta_{12}\left(\{Tu_1,Tu_2,Tu_3\}_1,x\right)u_4-T\theta_{12}\left(Tu_4,x\right)\theta_{12}\left(Tu_3,Tu_2\right)u_1\\
&\quad-T\theta_{12}\left(Tu_4,x\right)\theta_{12}\left(Tu_1,Tu_2\right)u_3-T\theta_{12}\left(Tu_4,x\right)\theta_{13}\left(Tu_1,Tu_3\right)u_2-\lambda T\theta_{12}\left(Tu_4,x\right)\{u_1,u_2,u_3\}_2\\
&\quad-\left\{Tu_1,Tu_2,\{Tu_3,x,Tu_4\}_1\right\}_1+T\theta_{13}\left(Tu_1,\{Tu_3,x,Tu_4\}_1\right)u_2+T\theta_{12}\left(\{Tu_3,x,Tu_4\}_1,Tu_2\right)u_1\\
&\quad+\left\{Tu_1,Tu_2,T\theta_{12}\left(Tu_4,x\right)u_3\right\}_1-T\theta_{13}\left(Tu_1,T\theta_{12}\left(Tu_4,x\right)u_3\right)u_2-T\theta_{12}\left(T\theta_{12}\left(Tu_4,x\right)u_3,Tu_2\right)u_1\\
&\quad+\left\{Tu_1,Tu_2,T\theta_{12}\left(Tu_3,x\right)u_4\right\}_1-T\theta_{13}\left(Tu_1,T\theta_{12}\left(Tu_3,x\right)u_4\right)u_2-T\theta_{12}\left(T\theta_{12}\left(Tu_3,x\right)u_4,Tu_2\right)u_1\\
&\quad-\left\{Tu_3,Tu_2,\{Tu_1,x,Tu_4\}_1\right\}_1+T\theta_{13}\left(Tu_3,\{Tu_1,x,Tu_4\}_1\right)u_2+T\theta_{12}\left(\{Tu_1,x,Tu_4\}_1,Tu_2\right)u_3\\
&\quad+\left\{Tu_3,Tu_2,T\theta_{12}\left(Tu_4,x\right)u_1\right\}_1-T\theta_{13}\left(Tu_3,T\theta_{12}\left(Tu_4,x\right)u_1\right)u_2-T\theta_{12}\left(T\theta_{12}\left(Tu_4,x\right)u_1,Tu_2\right)u_3\\
&\quad+\left\{Tu_3,Tu_2,T\theta_{12}\left(Tu_1,x\right)u_4\right\}_1-T\theta_{13}\left(Tu_3,T\theta_{12}\left(Tu_1,x\right)u_4\right)u_2-T\theta_{12}\left(T\theta_{12}\left(Tu_1,x\right)u_4,Tu_2\right)u_3\\
&\quad+\left\{Tu_1,\{Tu_2,Tu_4,x\}_1,Tu_3\right\}_1-T\theta_{12}\left(Tu_3,\{Tu_2,Tu_4,x\}_1u_1\right)-T\theta_{12}\left(Tu_1,\{Tu_2,Tu_4,x\}_1\right)u_3\\
&\quad-\left\{Tu_1,T\theta_{13}\left(Tu_2,x\right)u_4,Tu_3\right\}_1+T\theta_{12}\left(Tu_3,T\theta_{13}\left(Tu_2,x\right)u_4\right)u_1+T\theta_{12}\left(Tu_1,T\theta_{13}\left(Tu_2,x\right)u_4\right)u_3\\
&\quad-\left\{Tu_1,T\theta_{12}\left(x,Tu_4\right)u_2,Tu_3\right\}_1+T\theta_{12}\left(Tu_3,x\right)T\theta_{12}\left(x,Tu_4\right)u_2u_1+T\theta_{12}\left(Tu_1,T\theta_{12}\left(x,Tu_4\right)u_2\right)u_3\\
&=\left\{\{Tu_1,Tu_2,Tu_3\}_1,x,Tu_4\right\}_1-\left\{Tu_1,Tu_2,\{Tu_3,x,Tu_4\}_1\right\}_1-\left\{Tu_3,Tu_2,\{Tu_1,x,Tu_4\}_1\right\}_1\\
&\quad+\left\{Tu_1,\{Tu_2,Tu_4,x\}_1,Tu_3\right\}-T\left(C_{41}u_1+C_{42}u_2+C_{43}u_3+C_{44}u_4+\lambda C_{45}\right)
\end{aligned}
$$

其中，

$$C_{41}=\theta_{12}\left(Tu_4,x\right)\theta_{12}\left(Tu_3,Tu_2\right)-\theta_{12}\left(\{Tu_3,x,Tu_4\}_1,Tu_2\right)-\theta_{12}\left(Tu_3,Tu_2\right)\theta_{12}\left(Tu_4,x\right)+\theta_{12}\left(Tu_3,\{Tu_2,Tu_4,x\}_1\right),$$

$$C_{42}=\theta_{12}\left(Tu_4,x\right)\theta_{13}\left(Tu_1,Tu_3\right)-\theta_{13}\left(Tu_1,\{Tu_3,x,Tu_4\}_1\right)-\theta_{13}\left(Tu_3,\{Tu_1,x,Tu_4\}_1\right)+\theta_{13}\left(Tu_1,Tu_3\right)\theta_{12}\left(x,Tu_4\right),$$

$$C_{43}=\theta_{12}\left(Tu_4,x\right)\theta_{12}\left(Tu_1,Tu_2\right)-\theta_{12}\left(Tu_1,Tu_2\right)\theta_{12}\left(Tu_4,x\right)-\theta_{12}\left(\{Tu_1,x,Tu_4\}_1,Tu_2\right)+\theta_{12}\left(Tu_1,\{Tu_2,Tu_4,x\}_1\right),$$

$$C_{44}=\theta_{12}\left(\{Tu_1,Tu_2,Tu_3\}_1,x\right)-\theta_{12}\left(Tu_1,Tu_2\right)\theta_{12}\left(Tu_3,x\right)-\theta_{12}\left(Tu_3,Tu_2\right)\theta_{12}\left(Tu_1,x\right)+\theta_{13}\left(Tu_1,Tu_3\right)\theta_{13}\left(Tu_2,x\right),$$

$$
\begin{aligned}
C_{45}=&\,\theta_{12}\left(Tu_4,x\right)\{u_1,u_2,u_3\}_2-\left\{u_1,u_2,\theta_{12}\left(Tu_4,x\right)u_3\right\}_2-\left\{u_1,u_2,\theta_{12}\left(Tu_3,x\right)u_4\right\}_2-\left\{u_3,u_2,\theta_{12}\left(Tu_4,x\right)u_1\right\}_2,\\
&-\left\{u_3,u_2,\theta_{12}\left(Tu_1,x\right)u_4\right\}_2+\left\{u_1,\theta_{12}\left(x,Tu_4\right)u_2,u_3\right\}_2+\left\{u_1,\theta_{13}\left(Tu_2,x\right)u_4,u_3\right\}_2
\end{aligned}
$$

由于$\left(J_1,\{\cdot,\cdot,\cdot\}_1\right)$是约当三系，故(1.2)式成立，由(2.2)式成立知 C_{41} 为零，由(2.3)式成立知 C_{42} 为零，由(2.2)式成立知 C_{43} 为零，由(2.4)式成立知 C_{44} 为零，由于θ_{12},θ_{13}是$\left(J_1,\{\cdot,\cdot,\cdot\}_1\right)$在$\left(J_2,\{\cdot,\cdot,\cdot\}_2\right)$上的作用，故有$\theta_{12}\left(Tu_4,x\right)\{u_1,u_2,u_3\}_2=0$，$\left\{u_1,u_2,\theta_{12}\left(Tu_4,x\right)u_3\right\}_2=0$，类似可知 C_{45} 为零，故(2.5)式成立。

综上，$\left(L_T,M_T,J_1\right)$是$\left(J_2,\{\cdot,\cdot,\cdot\}_T\right)$的表示。

5. 相对 Rota-Baxter 算子的构造

命题 5.1 $\left(J,\{\cdot,\cdot,\cdot\}\right)$是约当三系，$\left(\sigma_{12},\sigma_{13},J\right)$是$\left(J,\{\cdot,\cdot,\cdot\}\right)$的伴随表示，$J_0$ 是 J 的子系且满足 $J_0^1=\{0\}$，$J^1\cap J_0=\{0\}$，$\{J_0,J_0,J\}=\{0\}$，$\{J_0,J,J_0\}=\{0\}$。设 H 是 J_0 的补空间，使 $J=H\oplus J_0$，P 是 J 在 J_0 上的

自然投影，即

$$P(x)=x_0, x\in J, x_0\in J_0,$$

则 P 是关于作用 σ_{12},σ_{13} 的权为 λ 的相对 Rota-Baxter 算子。

证：由于 $(\sigma_{12},\sigma_{13},J)$ 是 $(J,\{\cdot,\cdot,\cdot\})$ 的伴随表示，故由例 2.2 知，σ_{12},σ_{13} 是 $(J,\{\cdot,\cdot,\cdot\})$ 在自身上的作用，$\forall x,y,z\in J$，记 x,y,z 在投影 P 下的像为 x_0,y_0,z_0，

$$\begin{aligned}&\{Px,Py,Pz\}-P\left(\sigma_{12}(Pz,Py)x+\sigma_{13}(Px,Pz)y+\sigma_{12}(Px,Py)z+\lambda\{x,y,z\}\right)\\&=\{x_0,y_0,z_0\}-P\left(\sigma_{12}(z_0,y_0)x+\sigma_{13}(x_0,z_0)y+\sigma_{12}(x_0,y_0)z+\lambda\{x,y,z\}\right)\\&=0\end{aligned},$$

所以，P 是关于作用 σ_{12},σ_{13} 的权为 λ 的相对 Rota-Baxter 算子。

例 5.1 设 $(J,\cdot)$ 是三维约当代数，n_1,n_2,n_3 是 J 的一组基，J 中运算为

$$n_1\cdot n_1=n_2, n_2\cdot n_2=0, n_3\cdot n_3=0, n_1\cdot n_2=n_2\cdot n_1=n_3, n_1\cdot n_3=n_3\cdot n_1=0, n_2\cdot n_3=n_3\cdot n_2=0。$$

由例 1.1 知，在 J 上定义运算

$$\{x,y,z\}=(x\cdot y)\cdot z+(y\cdot z)\cdot x-(x\cdot z)\cdot y,$$

则 $(J,\{\cdot,\cdot,\cdot\})$ 是约当三系，且基中元素的非零代数运算为

$$\{n_1,n_1,n_1\}=n_3,$$

$(J,\{\cdot,\cdot,\cdot\})$ 的 $C(J)$ 是由 $\{n_2,n_3\}$ 生成的子系，则伴随表示 $(\sigma_{12},\sigma_{13},J)$ 是 $(J,\{\cdot,\cdot,\cdot\})$ 在自身上的一个作用。取 J_0 为由 $\{n_2\}$ 所生成的子系，根据命题 4.1，由

$$P(n_1)=0, P(n_2)=n_2, P(n_3)=0$$

所决定的投影 $P:J\to J_0$ 是关于作用 σ_{12},σ_{13} 的权为 λ 的相对 Rota-Baxter 算子。

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