Advances in Applied Mathematics 应用数学进展, **2024, 13(1), 444-452**
Published Online January 2024 in Hans. https://www.hanspub.org/journal/aam
https://doi.org/10.12677/aam.2024.131044

标准布朗运动驱动的 CIR 模型梯形数值方法

陈凯旋

辽宁师范大学数学学院，辽宁 大连

收稿日期：2023 年 12 月 27 日；录用日期：2024 年 1 月 21 日；发布日期：2024 年 1 月 29 日

摘 要

本文针对标准布朗运动驱动的 **Cox–Ingersoll–Ross (CIR)** 模型探讨了梯形数值方法的强收敛性。通过 **Lamperti** 变换，将 **CIR** 模型转换为具有局部 **Lipschitz** 条件的漂移项和具有全局 **Lipschitz** 条件的扩散项的新方程. 在适当的条件下，证明了新方程梯形数值方法的保正性和强收敛阶，并通过 **Lamperti** 逆变换得到了 **CIR** 模型数值解的强收敛阶。最后，利用数值模拟结果验证了理论分析。

关键词

随机微分方程，**Lamperti** 变换，梯形数值方法，**CIR** 模型，强收敛阶

The Trapezoidal Numerical Method for the CIR Model Driven by Standard Brownian Motion

Kaixuan Chen

School of Mathematics, Liaoning Normal University, Dalian Liaoning

Received: Dec. 27^{th}, 2023; accepted: Jan. 21^{st}, 2024; published: Jan. 29^{th}, 2024

文章引用: 陈凯旋. 标准布朗运动驱动的 CIR 模型梯形数值方法 [J]. 应用数学进展, 2024, 13(1): 444-452.
DOI: 10.12677/aam.2024.131044

Abstract

This paper investigates the strong convergence of the trapezoidal numerical method for the Cox–Ingersoll–Ross (CIR) model driven by standard Brownian motion. Through the Lamperti transformation, the CIR model is transformed into a new equation with a drift term satisfying a local Lipschitz condition and a diffusion term satisfying a global Lipschitz condition. Under suitable conditions, the positivity preservation and strong convergence order of the trapezoidal numerical method for the new equation are proven. Furthermore, the strong convergence order of the numerical solution for the CIR model is obtained through the Lamperti inverse transformation. Finally, the theoretical analysis is validated through numerical simulation results.

Keywords

Stochastic Differential Equations, Lamperti Transformation, Trapezoidal Numerical Method, CIR Model, Strong Convergence Order

Copyright © 2024 by author(s) and Hans Publishers Inc.

This work is licensed under the Creative Commons Attribution International License (CC BY 4.0).

http://creativecommons.org/licenses/by/4.0/

Open Access

1. 引言

Cox-Ingersoll-Ross (CIR) 模型是一个著名的短期利率模型, 首次在 1985 年的文献 [1] 中提出, 此模型在国内外得到了深入的研究, 尤其在金融数学和随机分析领域.CIR 模型的微分方程形式为

$$\begin{cases} dr_t = \kappa\left(\theta - r_t\right)dt + \sigma\sqrt{r_t}dB_t, \quad t \in (0, T], \\ r_0 > 0. \end{cases} \tag{1}$$

该模型表示了短期利率 r_t 如何随时间 t 以均值回归的方式变化, 并同时受到随机扰动的影响. 其中 $\{B_t\}_{t\in[0,T]}$ 为标准布朗运动. r_0 为初始的短期利率,r_t 为时刻 t 的短期利率.$T \in (0, +\infty)$ 为总时间, θ、σ 和 κ 分别为长期平均利率、波动率和回归速度. 且满足 $\kappa\theta > 0$, $\sigma > 0$. 有许多文章研究了方程(1) 的离散方案, 见 [2] [3] 等. 本文我们将提出一个新的数值方法去求解 CIR 模型, 并得到其保正性和收敛阶.

2. 梯形数值方法

对于 CIR 过程的逼近获得强收敛率的困难是由于其乘性噪声. 因此, 依赖全局 Lipschitz 假设的标准理论不适用 [4] . 通过 Lamperti 变换 $Y_t = \sqrt{r_t}$, 我们将 CIR 模型(1)转换为具有局部 Lipschitz 条件的漂移项和具有全局 Lipschitz 条件的扩散项的新方程

$$\begin{cases} dY_t = \frac{1}{2}\kappa\left(\frac{\theta}{Y_t} - Y_t\right)dt + \frac{1}{2}\sigma dB_t, \quad t \in (0, T], \\ Y_0 = \sqrt{r_0}, \end{cases} \tag{2}$$

设 $N \in \mathbb{N}_+, h := \frac{T}{N}, t_n := nh, n = 0, 1, \cdots, N$, 将梯形数值方法应用于方程(2), 可得

$$\begin{cases} Y^h_{t_{n+1}} = Y^h_{t_n} + \frac{1}{4}\kappa\left\{\left(\frac{\theta}{Y^h_{t_n}} + \frac{\theta}{Y^h_{t_{n+1}}}\right) - \left(Y^h_{t_n} + Y^h_{t_{n+1}}\right)\right\}h + \frac{1}{2}\sigma B_{t_n,t_{n+1}}, \\ Y^h_0 = Y_0, \end{cases} \tag{3}$$

其中 $B_{t_n,t_{n+1}} := B_{t_{n+1}} - B_{t_n}$.

由于利率实际是正值的, 为了确保方程解的唯一性和保正性, 故方程 (3) 的解为

$$Y^h_{t_{n+1}} = \frac{Y^h_{t_n}\left(1 - \frac{\kappa h}{4}\right) + \frac{\kappa h\theta}{4Y^h_{t_n}} + \frac{1}{2}\sigma B_{t_n,t_{n+1}} + \sqrt{\left(Y^h_{t_n}\left(1 - \frac{\kappa h}{4}\right) + \frac{\kappa h\theta}{4Y^h_{t_n}} + \frac{1}{2}\sigma B_{t_n,t_{n+1}}\right)^2 + \kappa h\theta(1 + \frac{\kappa h}{4})}}{2 + \frac{\kappa h}{2}}.$$

其中 $\tilde{\kappa} := \max\left\{0, -\frac{\kappa}{4}\right\}, h$ 满足 $h\tilde{\kappa} < 1$.

3. 梯形数值方法的强收敛性

定理 1

设 $\xi \in (0,1)$,. 若 $2\kappa\theta > \sigma^2$. $1 \le p < \frac{2\kappa\theta}{\sigma^2}$, 则存在常数 $C = C(T, p, Y_0, \kappa, \theta, \sigma, \xi)$, 使得对满足 $h\tilde{\kappa} < 1 - \xi$ 的任意 $h \in (0, 1]$, 有

$$\left\|\max_{n=1,\cdots,N}\left|Y_{t_n} - Y^h_{t_n}\right|\right\|_{L^p(\Omega)} \le Ch, \tag{4}$$

其中 $\tilde{\kappa} := \max\left\{0, -\frac{\kappa}{4}\right\}, Y$ 为方程 (2) 的精确解, Y^h 为方程(3) 的数值解.

证明. 记 $e_n := Y_{t_n} - Y^h_{t_n}, f(x) := \frac{1}{2}\kappa\left(\frac{\theta}{x} - x\right)$.

$$\begin{aligned} e_{n+1} &= e_n + \left(Y_{t_{n+1}} - Y^h_{t_{n+1}}\right) - \left(Y_{t_n} - Y^h_{t_n}\right) \\ &= e_n + \frac{1}{2}\left[\left(f\left(Y_{t_{n+1}}\right) + f\left(Y_{t_n}\right)\right) - \left(f\left(Y^h_{t_{n+1}}\right) + f\left(Y^h_{t_n}\right)\right)\right]h \\ &- \frac{1}{2}\left[\int_{t_n}^{t_{n+1}}\int_t^{t_{n+1}} f'\left(Y_u\right)\left(f\left(Y_u\right)du + \frac{1}{2}\sigma dB_u\right)dt + \int_{t_n}^{t_{n+1}}\int_t^{t_n} f'\left(Y_u\right)\left(f\left(Y_u\right)du + \frac{1}{2}\sigma dB_u\right)dt\right] \\ &= e_n + \frac{1}{2}\left[\left(f\left(Y_{t_{n+1}}\right) - f\left(Y^h_{t_{n+1}}\right)\right) + \left(f\left(Y_{t_n}\right) - f\left(Y^h_{t_n}\right)\right)\right]h + R_n, \end{aligned}$$

其中

$$R_n := -\frac{1}{2}\int_{t_n}^{t_{n+1}} (2u - t_{n+1} - t_n) f'(Y_u) f(Y_u)\, du - \frac{1}{4}\sigma \int_{t_n}^{t_{n+1}} (2u - t_{n+1} - t_n) f'(Y_u)\, dB_u.$$

$$f(Y_{t_{n+1}}) - f\left(Y_{t_{n+1}}^h\right) = \left(-\frac{\kappa\theta}{2Y_{t_{n+1}}Y_{t_{n+1}}^h} - \frac{\kappa}{2}\right) e_{n+1} =: \gamma_{n+1}e_{n+1}$$

因为 $\kappa\theta, Y_{t_{n+1}}, Y_{t_{n+1}}^h$ 非负, 可得 $\frac{1}{2}\gamma_{n+1} \le \tilde{\kappa}$. 因此,

$$e_{n+1} = e_n + \frac{1}{2}(\gamma_{n+1}e_{n+1} + \gamma_n e_n) h + R_n.$$

定义 $\zeta_0 := 1$,$\zeta_n := \prod_{j=1}^{n}\left(1 - \frac{1}{2}\gamma_j h\right)$, $\tilde{e}_n := \zeta_n e_n$, 得

$$\tilde{e}_{n+1} = \left(1 + \frac{1}{2}\gamma_n h\right)\tilde{e}_n + \zeta_n R_n.$$

定义 $\eta_0 := 1$,$\eta_n := \prod_{j=1}^{n} \frac{1}{\left(1+\frac{1}{2}\gamma_{j-1}h\right)}$, $\hat{e}_n := \eta_n\tilde{e}_n$, 得

$$\hat{e}_{n+1} = \hat{e}_n + \eta_{n+1}\zeta_n R_n.$$

定义 $\tilde{u} := \left[\frac{u}{h}\right]$ 表示 $\frac{u}{h}$ 的整数部分,

$$\tilde{\zeta}_n := \frac{\eta_{n+1}\zeta_n}{(1-\tilde{\kappa}h)^{2n}}, \quad \tilde{R}_t := -\int_0^t \frac{1}{4}\sigma(1-\tilde{\kappa}h)^{2\tilde{u}} (2u - t_{\tilde{u}+1} - t_{\tilde{u}}) f'(Y_u)\, dB_u,$$

可得

$$\tilde{\zeta}_n\left(\tilde{R}_{t_{n+1}} - \tilde{R}_{t_n}\right) = -\eta_{n+1}\zeta_n \int_{t_n}^{t_{n+1}} \frac{1}{4}\sigma (2u - t_{n+1} - t_n) f'(Y_u)\, dB_u$$

由于

$$R_n = -\frac{1}{2}\int_{t_n}^{t_{n+1}} (2u - t_{n+1} - t_n) f'(Y_u) f(Y_u)\, du - \frac{1}{4}\sigma \int_{t_n}^{t_{n+1}} (2u - t_{n+1} - t_n) f'(Y_u)\, dB_u.$$

结合上式, 可得

$$\begin{aligned}\hat{e}_n =& \hat{e}_{n-1} - \frac{1}{2}\int_{t_{n-1}}^{t_n} \eta_n\zeta_{n-1} (2u - t_{n-1} - t_n) f'(Y_u) f(Y_u)\, du + \tilde{\zeta}_{n-1}\left(\tilde{R}_{t_n} - \tilde{R}_{t_{n-1}}\right) \\ =& -\frac{1}{2}\int_0^{t_n} \eta_{\tilde{u}+1}\zeta_{\tilde{u}} (2u - t_{\tilde{u}} - t_{\tilde{u}+1}) f'(Y_u) f(Y_u)\, du - \sum_{j=0}^{n-1}\tilde{\zeta}_j\left(\tilde{R}_{t_{j+1}} - \tilde{R}_{t_j}\right).\end{aligned}$$

由于 $\hat{e}_n = \eta_n\zeta_n e_n$, 可得

$$e_n = -\frac{1}{2}\int_0^{t_n} \frac{\eta_{\tilde{u}+1}\zeta_{\tilde{u}}}{\eta_n\zeta_n} (2u - t_{\tilde{u}} - t_{\tilde{u}+1}) f'(Y_u) f(Y_u)\, du - \frac{1}{2}\int_0^{t_n} \sum_{j=0}^{n-1} \frac{\tilde{\zeta}_j}{\eta_n\zeta_n}\left(\tilde{R}_{t_{j+1}} - \tilde{R}_{t_j}\right).$$

由于 h 满足 $h\tilde{\kappa} < 1-\xi$, 可知 $0<\xi<1-\tilde{\kappa}h \le 1-\frac{1}{2}\gamma_j h$,
从而

$$0<\frac{\tilde{\zeta}_{j-1}}{\tilde{\zeta}_j}=\frac{1+\frac{1}{2}\gamma_j h}{1-\frac{1}{2}\gamma_j h}\left(1-\tilde{\kappa}h\right)^2 \le \left(1+\tilde{\kappa}h\right)\left(1-\tilde{\kappa}h\right)\le 1.$$

所以

$$\begin{aligned}\left|\sum_{j=0}^{n-1}\tilde{\zeta}_j\left(\tilde{R}_{t_{j+1}}-\tilde{R}_{t_j}\right)\right| &= \left|\tilde{\zeta}_{n-1}\tilde{R}_{t_n}+\sum_{j=1}^{n-1}\left(\tilde{\zeta}_{j-1}-\tilde{\zeta}_j\right)\tilde{R}_{t_j}\right| \\ &\le \tilde{\zeta}_{n-1}\left|\tilde{R}_{t_n}\right|+\sum_{j=1}^{n-1}\left(\tilde{\zeta}_j-\tilde{\zeta}_{j-1}\right)\left|\tilde{R}_{t_j}\right| \\ &\le 2\tilde{\zeta}_{n-1}\max_{j=1,\cdots,n}\left|\tilde{R}_{t_j}\right|.\end{aligned}$$

根据 $\xi<1-\tilde{\kappa}h\le 1$, 有

$$\frac{\tilde{\zeta}_{n-1}}{\eta_n\zeta_n}=\frac{1}{\left(1-\frac{1}{2}\gamma_n h\right)}(1-\tilde{\kappa}h)^{-2n+2}\le \exp\left\{\frac{2\tilde{\kappa}T}{\xi}\right\},\quad n=0,\cdots,N,$$

可得

$$\max_{n=1,\cdots,N}\left|\sum_{j=0}^{n-1}\frac{\tilde{\zeta}_j}{\eta_n\zeta_n}\left(\tilde{R}_{t_{j+1}}-\tilde{R}_{t_j}\right)\right|\le 2\exp\left\{\frac{2\tilde{\kappa}T}{\xi}\right\}\max_{j=1,\cdots,N}\left|\tilde{R}_{t_j}\right|$$

结合

$$\frac{\eta_{\tilde{u}+1}\zeta_{\tilde{u}}}{\eta_n\zeta_n}\le\frac{(1+\tilde{\kappa}h)^{n-\tilde{u}-1}}{(1-\tilde{\kappa}h)^{n-\tilde{u}}}\le\left(1+\frac{2\tilde{\kappa}h}{1-\tilde{\kappa}h}\right)^{n-\tilde{u}}\le\exp\left\{\frac{2\tilde{\kappa}T}{\xi}\right\},\quad 0<u<t_n,$$

可得

$$\begin{aligned}&\mathbb{E}\left[\max_{n=1,\cdots,N}|e_n|^p\right] \\ &\le C\mathbb{E}\left[\max_{n=1,\cdots,N}\left|\int_0^{t_n}\frac{\eta_{\tilde{u}+1}\zeta_{\tilde{u}}}{\eta_n\zeta_n}\left(2u-t_{\tilde{u}}-t_{\tilde{u}+1}\right)f'\left(Y_u\right)f\left(Y_u\right)du\right|^p\right] \\ &+C\mathbb{E}\left[\max_{n=1,\cdots,N}\left|\sum_{j=0}^{n-1}\frac{\tilde{\zeta}_j}{\eta_n\zeta_n}\left(\tilde{R}_{t_{j+1}}-\tilde{R}_{t_j}\right)\right|^p\right] \\ &\le C\mathbb{E}\left[\left(\int_0^T\left|\left(2u-t_{\tilde{u}}-t_{\tilde{u}+1}\right)f'\left(Y_u\right)f\left(Y_u\right)\right|du\right)^p\right] \\ &+C\mathbb{E}\left[\max_{j=1,\cdots,N}\left|\tilde{R}_{t_j}\right|^p\right] \\ &=: CI_1+CI_2.\end{aligned}$$

根据 [3] 可知, 当 $2\kappa\theta>\sigma^2$. $1\le p<\frac{2\kappa\theta}{\sigma^2}$ 有

$$\sup_{t\in[0,T]}\mathbb{E}\left[\left|Y_t^{-1}\right|^p\right]+\sup_{t\in[0,T]}\mathbb{E}\left[\left|Y_t^{-1}\right|^{2p}\right]\le C.$$

也有

$$\sup_{t\in[0,T]} E[Y(t)^p] \le C \tag{5}$$

由于 $f'(x) = -\frac{\kappa\theta}{2x^2} - \frac{\kappa}{2}$, 对于 I_1, 根据 Minkowski 不等式得到

$$\begin{aligned} I_1 &= \left\| \int_0^T \left| (2u - t_{\tilde{u}} - t_{\tilde{u}+1}) f'(Y_u) f(Y_u) \right| du \right\|_{L^p(\Omega)}^p \\ &\le (2h)^p \left(\int_0^T \| f'(Y_u) f(Y_u) \|_{L^p(\Omega)} du \right)^p \\ &\le Ch^p. \end{aligned}$$

对于 I_2, 根据 Burkholder-Davis-Gundy 不等式得到

$$\begin{aligned} I_2 =& \mathbb{E}\left[\max_{j=1,\cdots,N} \left| \tilde{R}_{t_j} \right|^p \right] \\ \le& C\mathbb{E}\left[\left| \int_0^T \frac{1}{4}\sigma(1-\tilde{\kappa}h)^{2\tilde{u}} (2u - t_{\tilde{u}+1} - t_{\tilde{u}}) f'(Y_u) dB_u \right|^p \right] \\ \le& C\mathbb{E}\left[\left(\int_0^T \left(\frac{1}{4}\sigma(1-\tilde{\kappa}h)^{2\tilde{u}} (2u - t_{\tilde{u}+1} - t_{\tilde{u}}) f'(Y_u) \right)^2 du \right)^{\frac{p}{2}} \right] \\ \le& Ch^p. \end{aligned}$$

综上所述,

$$\mathbb{E}\left[\max_{n=1,\cdots,N} |e_n|^p \right] \le Ch^p.$$

定理 2

设 $\xi \in (0,1)$, 若 $2\kappa\theta > \sigma^2$. $1 \le p < \frac{2\kappa\theta}{\sigma^2}$, 则存在常数 $C = C(T, p, Y_0, \kappa, \theta, \sigma, \xi)$, 使得对满足 $h\tilde{\kappa} < 1 - \xi$ 的任意 $h \in (0,1]$, 有

$$\left\| \sup_{t\in[0,T]} \left| Y_t - Y_t^h \right| \right\|_{LP(\Omega)} \le Ch^{\frac{1}{2}} \sqrt{\ln \frac{T}{h}}$$

其中 $\tilde{\kappa} := \max\left\{0, -\frac{\kappa}{4}\right\}$, Y 为方程(2) 的精确解, Y^h 为方程 (3) 的数值解.

证明. 对 Y^h 取分段线性插值, 当 $t \in (t_n, t_{n+1}]$, 有

$$\begin{aligned} Y_t - Y_t^h =& Y_t - \frac{t - t_n}{h} Y_{t_{n+1}}^h - \frac{t_{n+1} - t}{h} Y_{t_n}^h \\ =& \frac{t_n - t}{h} \left(\int_t^{t_{n+1}} f(Y_s) ds + \frac{1}{2}\sigma \left(B_{t_{n+1}} - B_t \right) \right) \\ &+ \frac{t_{n+1} - t}{h} \left(\int_{t_n}^t f(Y_s) ds + \frac{1}{2}\sigma (B_t - B_{t_n}) \right) \\ &+ \frac{t - t_n}{h} \left(Y_{t_{n+1}} - Y_{t_{n+1}}^h \right) + \frac{t_{n+1} - t}{h} \left(Y_{t_n} - Y_{t_n}^h \right). \end{aligned}$$

特别地,$Y_0^h = Y_0$, 根据 [5], 可得

$$\left\|\sup_{t\in[0,T]}\left|\frac{t_n - t}{h}\left(B_{t_{n+1}} - B_t\right) + \frac{t_{n+1} - t}{h}\left(B_t - B_{t_n}\right)\right|\right\|_{L^p(\Omega)} \leq Ch^{\frac{1}{2}}\sqrt{\ln\frac{T}{h}}.$$

联立上式并结合定理 1, 得证.

接下来, 利用 Lamperti 逆变换 $r^h := \left(Y^h\right)^2$, 即可求出原 CIR 方程数值解的强收敛阶.

定理 3

设 $\xi \in (0,1)$. 若 $2\kappa\theta > \sigma^2$. $1 \leq p < \frac{2\kappa\theta}{\sigma^2}$, 则存在常数 $C = C\left(T, p, Y_0, \kappa, \theta, \sigma, \xi\right)$, 使得对满足 $h\tilde{\kappa} < 1 - \xi$ 的任意 $h \in (0,1]$, 有

$$\left\|\max_{n=1,\cdots,N}\left|r_{t_n} - r_{t_n}^h\right|\right\|_{L^{\frac{p}{2}}(\Omega)} \leq Ch, \quad \left\|\sup_{t\in[0,T]}\left|r_t - r_t^h\right|\right\|_{L^{\frac{p}{2}}(\Omega)} \leq Ch^{\frac{1}{2}}\sqrt{\ln\frac{T}{h}},$$

其中 $\tilde{\kappa} := \max\left\{0, -\frac{\kappa}{4}\right\}$, r 为 (1) 的精确解, r^h 为数值解.

证明. 结合 (5) 和定理 1, 可得

$$\left\|\max_{n=1,\cdots,N}\left|Y_{t_n} + Y_{t_n}^h\right|\right\|_{L^p(\Omega)} \leq 2\left\|\max_{n=1,\cdots,N}\left|Y_{t_n}\right|\right\|_{L^p(\Omega)} + \left\|\max_{n=1,\cdots,N}\left|Y_{t_n}^h - Y_{t_n}\right|\right\|_{L^p(\Omega)} \leq C.$$

由于

$$r_t - r_t^h = \left(Y_t - Y_t^h\right)\left(Y_t + Y_t^h\right),$$

利用 Hölder 不等式和定理 1, 可得

$$\left\|\max_{n=1,\cdots,N}\left|r_{t_n} - r_{t_n}^h\right|\right\|_{L^{\frac{p}{2}}(\Omega)} \leq \left\|\max_{n=1,\cdots,N}\left|Y_{t_n} - Y_{t_n}^h\right|\right\|_{L^p(\Omega)}\left\|\max_{n=1,\cdots,N}\left|Y_{t_n} + Y_{t_n}^h\right|\right\|_{L^p(\Omega)} \leq Ch.$$

同理, 利用 Hölder 不等式和定理 2, 可得

$$\left\|\sup_{t\in[0,T]}\left|r_t - r_t^h\right|\right\|_{L^{\frac{p}{2}}(\Omega)} \leq Ch^{\frac{1}{2}}\sqrt{\ln\frac{T}{h}}.$$

4. 数值实验

下面通过数值模拟验证 CIR 模型梯形数值方法的强收敛阶. 设初值为 $r_0 = 1$, 模型参数为 $\kappa = 2, \theta = 0.5, \sigma = 0.5$, 令 $p = 2$ 和 $T = 1$ 以满足条件.

我们选择步长 $h^* = \frac{T}{N^*}$, 其中 $N^* = 2^{15}$ 的数值解来模拟精确解. 选择步长 $h = \frac{T}{N}$, 其中 $N = 2^i$,$i = 6,7,8,9,10$ 来得到数值解. 基于 500 条轨迹得到了均方误差

$$\left(\mathbb{E}\left[\max_{n=1,\cdots,N}\left|Y_{t_n} - Y_{t_n}^h\right|^2\right]\right)^{\frac{1}{2}}.$$

由图 1展示的数值结果, 可以看出强收敛阶为 1, 与定理 1 的结论相符.

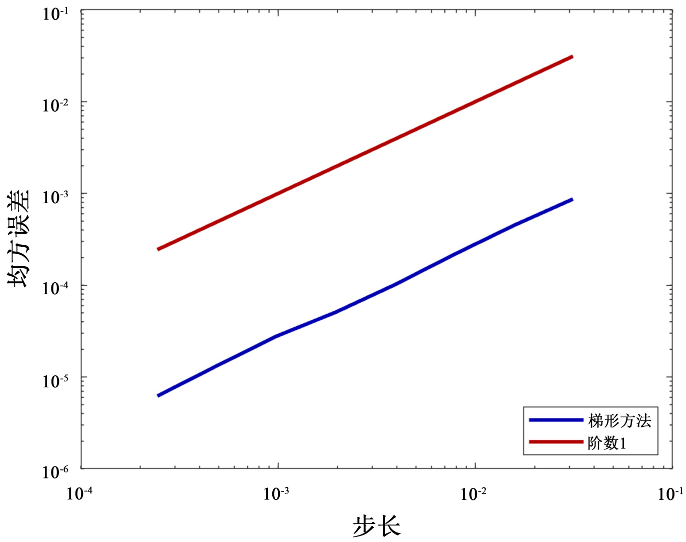

Figure 1. Mean-square convergence order of trapezoidal numerical method

图 1. 梯形数值方法均方收敛阶

对于分段线性插值的强收敛阶, 继续选择步长 $h^* = \frac{T}{N^*}$ 和 $N^* = 2^{15}$ 的数值解来模拟精确解. 当时间 $t = nh^*$, $n = 1, 2, 3, \ldots, 2^{15}$, 选择步长 $h = \frac{T}{N}$, 其中 $N = 2^i$, $i = 6, 7, 8, 9, 10$ 来得到数值解. 基于 500 条轨迹得到了均方误差

$$\left(\mathbb{E}\left[\max_{n=1,\cdots,N^*} \left|Y_{nh^*} - Y_{nh^*}^h\right|^2\right]\right)^{\frac{1}{2}}.$$

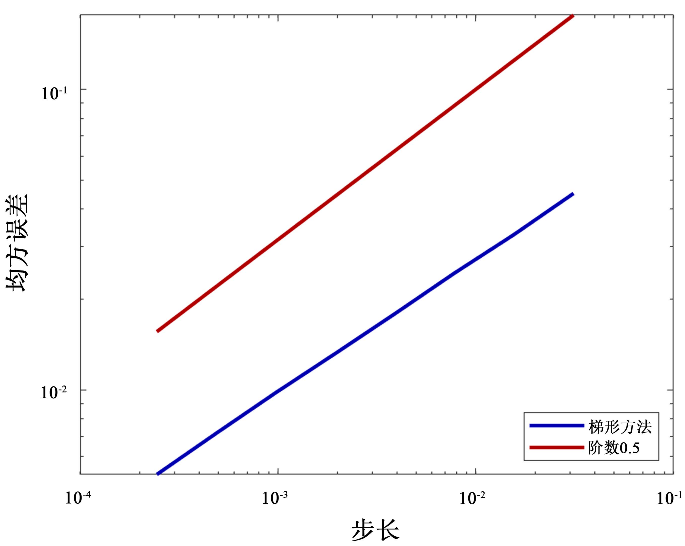

Figure 2. Mean-square convergence order of piecewise linear interpolation

图 2. 分段线性插值均方收敛阶

由图 2展示的数值结果可知, 这与定理 2 的结论相符.

参考文献

[1] Cox, J.C., Ingersoll Jr., J.E. and Ross, S.A. (1985) A Theory of the Term Structure of Interest Rates. *Econometrica*, **53**, 385-407. https://doi.org/10.2307/1911242

[2] Higham, D.J. and Mao, X. (2005) Convergence of Monte Carlo Simulations Involving the Mean-Reverting Square Root Process. *Journal of Computational Finance*, **8**, 35-61. https://doi.org/10.21314/JCF.2005.136

[3] Dereich, S., Neuenkirch, A. and Szpruch, L. (2012) An Euler-Type Method for the Strong Approximation of the Cox-Ingersoll-Ross Process. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, **468**, 1105-1115. https://doi.org/10.1098/rspa.2011.0505

[4] Milstein, G.N. (1994) Numerical Integration of Stochastic Differential Equations. Vol. 313, Springer Science Business Media, Berlin. https://doi.org/10.1007/978-94-015-8455-5

[5] Fischer, M. and Nappo, G. (2009) On the Moments of the Modulus of Continuity of Itô Processes. *Stochastic Analysis and Applications*, **28**, 103-122. https://doi.org/10.1080/07362990903415825