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The Unified (*r*, *s*)-Relative Differential Entropy Based on Joint Distribution of Random Density Matrix

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Abstract

The unified (*r*, *s*)-relative differential entropy of the joint distribution of eigenvalues of random density matrices is studied by Laplace transform and Laplace inverse transform. On the one hand, the unified (*r*, *s*)-relative differential entropy of the joint distribution of the eigenvalues to diagonal entries of random density matrices induced by partial tracing (the diagonal entries of random density matrices induced by partial tracing to joint distribution of the eigenvalues) over Haar-distributed bipartite pure states is defined. On the other hand, the unified (*r*, *s*)-relative differential entropy in the three cases is calculated. The range of differential entropy is generalized.

Keywords

Unified (*r*, *s*)-Relative Differential Entropy, Diagonal Element, Random Density Matrix

基于随机密度矩阵特征值联合分布的统一(*r*, *s*)相对微分熵

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摘　要

采用Laplace变换和Laplace逆变换研究随机密度矩阵特征值联合分布的统一(*r*, *s*)相对微分熵。一方面，定义了在Haar分布的双体纯态上取部分迹所诱导的随机密度矩阵的特征值的联合分布相对于其对角元的联合分布(其对角元的联合分布相对于取部分迹所诱导的随机密度矩阵的特征值的联合分布)的统一(*r*, *s*)相对微分熵。另一方面，计算三种情形下的统一(*r*, *s*)相对微分熵，推广了微分熵的范围。

关键词

统一(*r*, *s*)相对微分熵，对角元，随机密度矩阵

1. 引言

熵在统计物理学和信息论中都起着重要作用。在 1991 年，Rathie 和 Taneja 引进了以下统一(*r*, *s*)熵，其中包括(*r*, *s*)熵，Renyi 有序熵[1]。在 2004 年，Furuichi 研究了量子 Tsallis 相对熵的数学性质[2]。在 2006 年，胡和叶介绍了经典统一(*r*, *s*)熵的量子版本[3]。在 2011 年，汪等人定义了相应的统一(*r*, *s*)相对熵并研究了它的性质[4]。在 2016 年，罗等人研究了随机密度矩阵特征值联合分布的微分熵[5]。在本文中，我们将定义和研究随机密度矩阵特征值联合分布的统一(*r*, *s*)相对微分熵。

2. 统一(*r*, *s*)相对微分熵的定义

由[5]中的定义，我们知道：

1) 在 $\mathrm{Re}(z)>0$ 上的伽马函数为 $\Gamma(z)=\int_0^{\infty} t^{z-1}\mathrm{e}^{-t}\mathrm{d}t$，$\Gamma(x),\psi(x)$ 分别代表 Gamma 函数和 Digamma 函数。

$$\psi(x)=\frac{\mathrm{d}\ln\Gamma(x)}{\mathrm{d}x}=\frac{\Gamma'(x)}{\Gamma(x)},\quad \psi(k+1)=H_k-\gamma,\ \text{其中}\ H_k=\sum_{j=1}^{k}\frac{1}{j},\quad \gamma\approx 0.57721\ \text{为欧拉常数。}$$

2) 若 Wishart 矩阵 W 的构成矩阵 Z 是复的均值为 0，方差为 σ^2 的独立同分布的高斯变量，ZZ^{+} 联合概率密度的相应特征值 $\left(\mu_j\in[0,\infty),j=1,2,\cdots,m\right)$ 的联合概率函数为

$$q(\mu_1,\cdots,\mu_m)=C_q\exp\left(-\sum_{j=1}^{m}\mu_j\right)\prod_{1\le i<j\le m}(\mu_i-\mu_j)^2\prod_{j=1}^{m}\mu_j^{n-m},\ \text{其中}\sum_{j=1}^{m}\mu_j=1 \tag{1}$$

$$C_{\mathrm{q}}=1\Big/\prod_{1\le j\le m}\left[j!(n-j)!\right] \tag{2}$$

3) 在 Haar 分布的双体纯态上取部分迹所诱导的随机密度矩阵的特征值 $\left(\lambda_j\in[0,1],j=1,2,\cdots,m\right)$ 的联合概率函数为

$$p(\lambda_1,\cdots,\lambda_m)=C_p\prod_{1\le i<j\le m}(\lambda_i-\lambda_j)^2\prod_{j=1}^{m}\lambda_j^{n-m},\ \text{其中}\sum_{j=1}^{m}\lambda_j=1 \tag{3}$$

$$C_p = \Gamma(mn) C_q \tag{4}$$

4) 在 Haar 分布的双体纯态上其对角元的联合概率函数为

$$\psi(\rho_{11}, \rho_{22}, \cdots, \rho_{mm}) = C_\psi \prod_{j=1}^{m} \rho_{jj}^{n-1} \text{，其中} \sum_{j=1}^{m} \rho_{jj} = 1 \tag{5}$$

$$C_\Psi = \frac{\Gamma(mn)}{\Gamma(n)^m} \tag{6}$$

5) $$I_m(\alpha, r) = \int_0^\infty \cdots \int_0^\infty \exp\left(-\sum_{j=1}^{m} \mu_j\right) \prod_{1 \le i < j \le m} |\mu_i - \mu_j|^{2r} \prod_{k=1}^{m} (\mu_k^\alpha \mathrm{d}\mu_k) = \prod_{\mathrm{k}=0}^{m-1} \frac{\Gamma(\alpha + 1 + kr)\Gamma(1 + (k+1)r)}{\Gamma(1+r)} \tag{7}$$

$$I_m(n-m, 1) = \frac{1}{C_q} \tag{8}$$

$$I_m(n-1, 0) = \Gamma^m(n) \tag{9}$$

因此，我们可以得到

$$\begin{aligned} \frac{\partial}{\partial r} \partial I_m(\alpha(r), \beta(r)) &= \frac{\partial I_m(\alpha(r), \beta(r))}{\partial \alpha} \times \frac{\mathrm{d}\alpha}{\mathrm{d}r} + \frac{\partial I_m(\alpha(r), \beta(r))}{\partial \beta} \times \frac{\mathrm{d}\beta}{\mathrm{d}r} \\ &= I_m(\alpha(r), \beta(r)) \left\{ \alpha'(r) \times \sum_{k=0}^{m-1} \psi(\alpha(r) + 1 + k\beta(r)) \right. \\ &\quad + \beta'(r) \times \left[\sum_{k=1}^{m-1} k\psi(\alpha(r) + 1 + k\beta(r)) \right. \\ &\quad \left. \left. + \sum_{k=1}^{m} k\psi(1 + k\beta(r)) - m\psi(1 + \beta(r)) \right] \right\} \end{aligned} \tag{10}$$

定义 2.1：对任意的 $r > 0$ 和实数 s，在 Haar 分布的双体纯态上取部分迹所诱导的随机密度矩阵的特征值的联合分布相对于其对角元的联合分布的统一(r, s)相对微分熵为

$$E_r^s(p \| \psi) = \begin{cases} h_r^s(p \| \psi), r \ne 1, s \ne 0 \\ h_r(p \| \psi), r \ne 1, s = 0 \\ h^r(p \| \psi), r \ne 1, s = 1 \\ {}_r h(p \| \psi), r \ne 1, s = r^{-1} \\ h(p \| \psi), r = 1 \end{cases}$$

其中

$$h_r^s(p \| \psi) = -[(1-r)s]^{-1} \left[\left(\int p(\lambda) (p(\lambda)/\psi(\lambda))^{r-1} \mathrm{d}\lambda \right)^s - 1 \right], r > 0, r \ne 1, s \ne 0$$

$$h_r(p \| \psi) = -(1-r)^{-1} \ln\left(\int p(\lambda) (p(\lambda)/\psi(\lambda))^{r-1} \mathrm{d}\lambda \right), r > 0, r \ne 1$$

$$h^r(p \| \psi) = -(1-r)^{-1} \left(\int p(\lambda) (p(\lambda)/\psi(\lambda))^{r-1} \mathrm{d}\lambda - 1 \right), r > 0, r \ne 1$$

$${}_r h(p \| \psi) = -(r-1)^{-1} \left[\left(\int p(\lambda) (p(\lambda)/\psi(\lambda))^{1/r-1} \mathrm{d}\lambda \right)^r - 1 \right], r > 0, r \ne 1$$

$$h(p \| \psi) = \int p(\lambda) \ln(p(\lambda)/\psi(\lambda)) \mathrm{d}\lambda$$

定义 2.2：对任意的 $r>0$ 和实数 s，在 Haar 分布的双体纯态上其对角元的联合分布相对于取部分迹所诱导的随机密度矩阵的特征值的联合分布的统一(r, s)相对微分熵为

$$E_r^s(\psi \| p)=\begin{cases} h_r^s(\psi \| p), r\neq 1, s\neq 0 \\ h_r(\psi \| p), r\neq 1, s=0 \\ h^r(\psi \| p), r\neq 1, s=1 \\ {}_r h(\psi \| p), r\neq 1, s=r^{-1} \\ h(\psi \| p), r=1 \end{cases}$$

其中

$$h_r^s(\psi \| p)=-\left[(1-r)s\right]^{-1}\left[\left(\int \psi(\lambda)\left(\psi(\lambda)/p(\lambda)\right)^{r-1}\mathrm{d}\lambda\right)^s-1\right], r>0, r\neq 1, s\neq 0$$

$$h_r(\psi \| p)=-(1-r)^{-1}\ln\left(\int \psi(\lambda)\left(\psi(\lambda)/p(\lambda)\right)^{r-1}\mathrm{d}\lambda\right), r>0, r\neq 1$$

$$h^r(\psi \| p)=-(1-r)^{-1}\left(\int \psi(\lambda)\left(\psi(\lambda)/p(\lambda)\right)^{r-1}\mathrm{d}\lambda-1\right), r>0, r\neq 1$$

$${}_r h(\psi \| p)=-(r-1)^{-1}\left[\left(\int \psi(\lambda)\left(\psi(\lambda)/p(\lambda)\right)^{1/r-1}\mathrm{d}\lambda\right)^r-1\right], r>0, r\neq 1$$

$$h(\psi \| p)=\int \psi(\lambda)\ln\left(\psi(\lambda)/p(\lambda)\right)\mathrm{d}\lambda$$

3. 统一(r, s)相对微分熵的计算

定理 3.1：当 $r>0, r\neq 1, s=0$ 时，在 Haar 分布的双体纯态上取部分迹所诱导的随机密度矩阵的特征值的联合分布相对于其对角元的联合分布的统一(r, s)相对微分熵为

$$h_r(p \| \psi)=(r-1)^{-1}\left[r\ln C_p+(1-r)\ln C_\psi+\ln I_m\left((1-m)r+n-1, r\right)-\ln\Gamma(mn)\right]$$

证明：

$$\begin{aligned} h_r(p \| \psi)&=-(1-r)^{-1}\ln\left(\int_0^\infty\cdots\int_0^\infty p(\lambda_1,\cdots,\lambda_m)\left(p(\lambda_1,\cdots,\lambda_m)/\psi(\lambda_1,\cdots,\lambda_m)\right)^{r-1}\prod_{k=1}^m \mathrm{d}\lambda_k\right) \\ &=(r-1)^{-1}\ln\left(\int_0^\infty\cdots\int_0^\infty p^r(\lambda_1,\cdots,\lambda_m)\psi^{1-r}(\lambda_1,\cdots,\lambda_m)\prod_{k=1}^m \mathrm{d}\lambda_k\right) \end{aligned}$$

令

$$F_r(t)=(r-1)^{-1}\ln\left(\int_0^\infty\cdots\int_0^\infty \delta\left(t-\sum_{j=1}^m \lambda_j\right)p^r(\lambda_1,\cdots,\lambda_m)\psi^{1-r}(\lambda_1,\cdots,\lambda_m)\prod_{k=1}^m \mathrm{d}\lambda_k\right)$$

对 $F_r(t)$ 应用 Laplace 变换 $(t\to s)$，则有

$$\tilde{F}_r(s)=(r-1)^{-1}\ln\left(\int_0^\infty\cdots\int_0^\infty \exp\left(-s\sum_{j=1}^m \lambda_j\right)p^r(\lambda_1,\cdots,\lambda_m)\psi^{1-r}(\lambda_1,\cdots,\lambda_m)\prod_{k=1}^m \mathrm{d}\lambda_k\right)$$

令 $\lambda_k=\dfrac{\mu_k}{s}, k=1,2,\cdots,m$

$$\frac{C_p}{C_q}\exp\left(\sum_{i=1}^m \mu_m\right)q(\mu_1,\cdots\mu_m)=p(\mu_1,\cdots\mu_m),\quad p\left(\frac{\mu_1}{s},\cdots,\frac{\mu_m}{s}\right)=s^{m-nm}p(\mu_1,\cdots,\mu_m)$$

再由式(1)和式(3)联立得到

$$p\left(\lambda_1,\cdots,\lambda_m\right)=s^{m-mn}C_p\prod_{1\le i<j\le m}\left(\mu_i-\mu_j\right)^2\prod_{k=1}^{m}\mu_k^{n-m} \tag{11}$$

$$\psi\left(\lambda_1,\cdots,\lambda_m\right)=s^{m-mn}C_\psi\prod_{j=1}^{m}\mu_j^{n-1} \tag{12}$$

则

$$\begin{aligned}\tilde{F}_r\left(s\right)&=\left(r-1\right)^{-1}\ln\left(\int_0^\infty\cdots\int_0^\infty\exp\left(-s\sum_{j=1}^{m}\lambda_j\right)\left(s^{m-mn}C_p\prod_{1\le i<j\le m}\left(\mu_i-\mu_j\right)^2\prod_{k=1}^{m}\mu_k^{n-m}\right)^r\left(s^{m-mn}C_\psi\prod_{j=1}^{m}\mu_j^{n-1}\right)^{1-r}\prod_{k=1}^{m}\frac{\mathrm{d}\mu_k}{s}\right)\\&=\left(r-1\right)^{-1}\ln\left(\int_0^\infty\cdots\int_0^\infty s^{-mn}C_p^rC_\psi^{1-r}\prod_{1\le i<j\le m}\left(\mu_i-\mu_j\right)^{2r}\exp\left(-\sum_{j=1}^{m}\mu_j\right)\prod_{k=1}^{m}\mu_k^{(1-m)r+n-1}\mathrm{d}\mu_k\right)\\&=\left(r-1\right)^{-1}\ln\left(s^{-mn}C_p^rC_\psi^{1-r}I_m\left(\left(1-m\right)r+n-1,r\right)\right)\end{aligned}$$

由于

$$L^{-1}\left\{s^{-m}\right\}\left(t\right)=\frac{t^{m-1}}{\Gamma\left(m\right)} \tag{13}$$

所以

$$L^{-1}\left\{s^{-mn}\right\}\left(t\right)=\frac{t^{mn-1}}{\Gamma\left(mn\right)} \tag{14}$$

$$F_r\left(t\right)=\left(r-1\right)^{-1}\ln\left(\frac{t^{mn-1}}{\Gamma\left(mn\right)}C_p^rC_\psi^{1-r}I_m\left(\left(1-m\right)r+n-1,r\right)\right)$$

那么得到

$$\begin{aligned}h_r\left(p\,\|\,\psi\right)=F_r\left(1\right)&=\left(r-1\right)^{-1}\ln\left(\frac{C_p^rC_\psi^{1-r}}{\Gamma\left(mn\right)}I_m\left(\left(1-m\right)r+n-1,r\right)\right)\\&=\left(r-1\right)^{-1}\left[r\ln C_p+\left(1-r\right)\ln C_\psi+\ln I_m\left(\left(1-m\right)r+n-1,r\right)-\ln\Gamma\left(mn\right)\right]\end{aligned} \tag{15}$$

注 1：当 $r\to1$ 时，

$$\begin{aligned}\lim_{r\to1}h_r\left(p\,\|\,\psi\right)=&-\ln\left(\prod_{k=1}^{m}k!\left(n-k\right)!\right)-m\psi\left(2\right)+m\ln\Gamma\left(n\right)\\&-\sum_{k=n-m}^{n-1}\left(n-k-1\right)\psi\left(k+1\right)+\sum_{k=1}^{m}k\psi\left(k+1\right)\end{aligned}$$

证明：

由式(10)得到

$$\begin{aligned}\left.\frac{\partial\ln I_m\left(\left(1-m\right)r+n-1,r\right)}{\partial r}\right|_{r=1}&=\left.\frac{1}{I_m\left(\left(1-m\right)r+n-1,r\right)}\frac{\partial I_m\left(\left(1-m\right)r+n-1,r\right)}{\partial r}\right|_{r=1}\\&=\left(1-m\right)\times\sum_{k=0}^{m-1}\psi\left(\left(1-m\right)r+n+kr\right)-m\psi\left(1+r\right)\\&\quad\left.+\sum_{k=1}^{m-1}k\psi\left(\left(1-m\right)r+n+kr\right)+\sum_{k=1}^{m}k\psi\left(1+kr\right)\right|_{r=1}\\&=\sum_{k=0}^{m-1}\left(1-m+n+k\right)\times\psi\left(1-m+n+k\right)-m\psi\left(2\right)+\sum_{k=1}^{m}k\psi\left(1+k\right)\end{aligned} \tag{16}$$

把式(2)、式(4)、式(6)、式(8)、式(16)代入式(15)得到

$$\begin{aligned}\lim_{r\to 1} h_r(p\|\psi) &= (r-1)^{-1}\left[r\ln C_p+(1-r)\ln C_\psi+\ln I_m\left((1-m)r+n-1,r\right)-\ln\Gamma(mn)\right]\\ &=\ln C_p-\ln C_\psi+\left.\frac{\partial \ln I_m\left((1-m)r+n-1,r\right)}{\partial r}\right|_{r=1}\\ &=-\ln\left(\prod_{k=1}^{m}k!(n-k)!\right)-m\psi(2)+m\ln\Gamma(n)\\ &\quad-\sum_{k=n-m}^{n-1}(n-k-1)\psi(k+1)+\sum_{k=1}^{m}k\psi(k+1)\end{aligned}$$

定理 3.2：当 $r>0, r\neq 1, s=1$ 时，在 Haar 分布的双体纯态上取部分迹所诱导的随机密度矩阵的特征值的联合分布相对于其对角元的联合分布的统一(r, s)相对微分熵为

$$h^r(p\|\psi)=(r-1)^{-1}\left(\frac{C_p^rC_\psi^{1-r}}{\Gamma(mn)}I_m\left((1-m)r+n-1,r\right)-1\right)$$

证明：

$$\begin{aligned}h^r(p\|\psi) &= -(1-r)^{-1}\left(\int_0^\infty\cdots\int_0^\infty p(\lambda_1,\cdots,\lambda_m)\left(p(\lambda_1,\cdots,\lambda_m)/\psi(\lambda_1,\cdots,\lambda_m)\right)^{r-1}\prod_{k=1}^{m}\mathrm{d}\lambda_k-1\right)\\ &=(r-1)^{-1}\left(\int_0^\infty\cdots\int_0^\infty p^r(\lambda_1,\cdots,\lambda_m)\psi^{1-r}(\lambda_1,\cdots,\lambda_m)\prod_{k=1}^{m}\mathrm{d}\lambda_k-1\right)\end{aligned}$$

令

$$F^r(t)=(r-1)^{-1}\left(\int_0^\infty\cdots\int_0^\infty\delta\left(t-\sum_{j=1}^{m}\lambda_j\right)p^r(\lambda_1,\cdots,\lambda_m)\psi^{1-r}(\lambda_1,\cdots,\lambda_m)\prod_{k=1}^{m}\mathrm{d}\lambda_k-1\right)$$

对 $F^r(t)$ 应用 Laplace 变换 $(t\to s)$，则有

$$\tilde{F}^r(s)=(r-1)^{-1}\left(\int_0^\infty\cdots\int_0^\infty\exp\left(-s\sum_{j=1}^{m}\lambda_j\right)p^r(\lambda_1,\cdots,\lambda_m)\psi^{1-r}(\lambda_1,\cdots,\lambda_m)\prod_{k=1}^{m}\mathrm{d}\lambda_k-1\right)$$

把式(7)、式(11)、式(12)代入上式得到

$$\begin{aligned}\tilde{F}^r(s) &= (r-1)^{-1}\left(\int_0^\infty\cdots\int_0^\infty\exp\left(-s\sum_{j=1}^{m}\lambda_j\right)\left(s^{m-mn}C_p\prod_{1\le i<j\le m}(\mu_i-\mu_j)^2\prod_{k=1}^{m}\mu_k^{n-m}\right)^r\left(s^{m-mn}C_\psi\prod_{j=1}^{m}\mu_j^{n-1}\right)^{1-r}\prod_{k=1}^{m}\frac{\mathrm{d}\mu_k}{s}-1\right)\\ &=(r-1)^{-1}\left(\int_0^\infty\cdots\int_0^\infty s^{-mn}C_p^rC_\psi^{1-r}\prod_{1\le i<j\le m}(\mu_i-\mu_j)^{2r}\exp\left(-\sum_{j=1}^{m}\mu_j\right)\prod_{k=1}^{m}\mu_k^{(1-m)r+n-1}\mathrm{d}\mu_k-1\right)\\ &=(r-1)^{-1}\left(s^{-mn}C_p^rC_\psi^{1-r}I_m\left((1-m)r+n-1,r\right)-1\right)\end{aligned}$$

把式(14)代入上式得到

$$F^r(t)=(r-1)^{-1}\left(\frac{t^{mn-1}}{\Gamma(mn)}C_p^rC_\psi^{1-r}I_m\left((1-m)r+n-1,r\right)-1\right)$$

那么

$$h^r(p\|\psi)=F^r(1)=(r-1)^{-1}\left(\frac{C_p^rC_\psi^{1-r}}{\Gamma(mn)}I_m\left((1-m)r+n-1,r\right)-1\right)\tag{17}$$

注 2：当 $r \to 1$ 时，

$$\lim_{r\to 1} h^r\left(p \,\|\, \psi\right) = -\ln\left(\prod_{k=1}^{m} k!(n-k)!\right) - m\psi(2) + m\ln\Gamma(n) - \sum_{k=n-m}^{n-1}(n-k-1)\psi(k+1) + \sum_{k=1}^{m} k\psi(k+1)$$

证明：

由式(10)，我们得到

$$\begin{aligned} &\left.\frac{\partial I_m\left((1-m)r+n-1,r\right)}{\partial r}\right|_{r=1} \\ &= I_m\left((1-m)r+n-1,r\right)\Bigg[(1-m)\times\sum_{k=0}^{m-1}\psi\left((1-m)r+n+kr\right) - m\psi(1+r) \\ &\quad \left.+\sum_{k=1}^{m-1}k\psi\left((1-m)r+n+kr\right) + \sum_{k=1}^{m}k\psi(1+kr)\Bigg]\right|_{r=1} \\ &= I_m(n-m,1)\left[\sum_{k=0}^{m-1}(1-m+n+k)\times\psi(1-m+n+k) - m\psi(2) + \sum_{k=1}^{m}k\psi(1+k)\right] \end{aligned} \tag{18}$$

把式(2)、式(4)、式(6)、式(8)、式(18)代入式(17)得到

$$\begin{aligned} \lim_{r\to 1} h^r\left(p \,\|\, \psi\right) &= \Bigg[\ln C_p C_p^r C_\psi^{1-r}\frac{I_m\left((1-m)r+n-1,r\right)}{\Gamma(mn)} - \ln C_\psi C_p^r C_\psi^{1-r}\frac{I_m\left((1-m)r+n-1,r\right)}{\Gamma(mn)} \\ &\quad \left.+\frac{C_p^r C_\psi^{1-r}}{\Gamma(mn)}\frac{\partial I_m\left((1-m)r+n-1,r\right)}{\partial r}\Bigg]\right|_{r=1} \\ &= \ln C_p - \ln C_\psi + \left.\frac{C_p}{\Gamma(mn)}\frac{\partial I_m\left((1-m)r+n-1,r\right)}{\partial r}\right|_{r=1} \\ &= -\ln\left(\prod_{k=1}^{m}k!(n-k)!\right) - m\psi(2) + m\ln\Gamma(n) - \sum_{k=n-m}^{n-1}(n-k-1)\psi(k+1) + \sum_{k=1}^{m}k\psi(k+1) \end{aligned}$$

定理 3.3：当 $r > 0, r \neq 1, s = 1/r$ 时，在 Haar 分布的双体纯态上取部分迹所诱导的随机密度矩阵的特征值的联合分布相对于其对角元的联合分布的统一(r, s)相对微分熵为

$${}_r h\left(p \,\|\, \psi\right) = (1-r)^{-1}\left[\frac{C_p C_\psi^{r-1}}{\Gamma(mnr)}\left(I_m\left(n-1-\frac{m-1}{r},\frac{1}{r}\right)\right)^r - 1\right]$$

证明：

$$\begin{aligned} {}_r h\left(p \,\|\, \psi\right) &= -(r-1)^{-1}\left[\left(\int_0^\infty\cdots\int_0^\infty p(\lambda_1,\cdots,\lambda_m)\left(p(\lambda_1,\cdots,\lambda_m)/\psi(\lambda_1,\cdots,\lambda_m)\right)^{1/r-1}\prod_{k=1}^{m}\mathrm{d}\lambda_k\right)^r - 1\right] \\ &= (1-r)^{-1}\left[\left(\int_0^\infty\cdots\int_0^\infty p^{1/r}(\lambda_1,\cdots,\lambda_m)\psi^{1-1/r}(\lambda_1,\cdots,\lambda_m)\prod_{k=1}^{m}\mathrm{d}\lambda_k\right)^r - 1\right] \end{aligned}$$

令

$${}_r F(t) = (1-r)^{-1}\left[\left(\int_0^\infty\cdots\int_0^\infty \delta\left(t-\sum_{j=1}^{m}\lambda_j\right)p^{1/r}(\lambda_1,\cdots,\lambda_m)\psi^{1-1/r}(\lambda_1,\cdots,\lambda_m)\prod_{k=1}^{m}\mathrm{d}\lambda_k\right)^r - 1\right]$$

对 ${}_rF(t)$ 应用 Laplace 变换 $(t\to s)$，则有

$${}_r\tilde{F}(s)=(1-r)^{-1}\left[\left(\int_0^\infty\cdots\int_0^\infty\exp\left(-s\sum_{j=1}^m\lambda_j\right)p^{1/r}(\lambda_1,\cdots,\lambda_m)\psi^{1-1/r}(\lambda_1,\cdots,\lambda_m)\prod_{k=1}^m\mathrm{d}\lambda_k\right)^r-1\right]$$

把式(7)、式(11)、式(12) 代入上式得到

$$\begin{aligned}{}_r\tilde{F}(s)&=(1-r)^{-1}\left[\left(\int_0^\infty\cdots\int_0^\infty\exp\left(-s\sum_{j=1}^m\lambda_j\right)\left(s^{m-mn}C_p\prod_{1\le i<j\le m}(\mu_i-\mu_j)^2\prod_{k=1}^m\mu_k^{n-m}\right)^{1/r}\left(s^{m-mn}C_\psi\prod_{j=1}^m\mu_j^{n-1}\right)^{1-1/r}\prod_{k=1}^m\frac{\mathrm{d}\mu_k}{s}\right)^r-1\right]\\&=(1-r)^{-1}\left[\left(\int_0^\infty\cdots\int_0^\infty s^{-mn}C_p^{1/r}C_\psi^{1-1/r}\prod_{1\le i<j\le m}(\mu_i-\mu_j)^{2/r}\exp\left(-\sum_{j=1}^m\mu_j\right)\prod_{k=1}^m\mu_k^{n-1+(1-m)/r}\mathrm{d}\mu_k\right)^r-1\right]\\&=(1-r)^{-1}\left[s^{-mnr}C_pC_\psi^{r-1}\left(I_m\left(n-1+(1-m)/r,1/r\right)\right)^r-1\right]\end{aligned}$$

由式(13)得到

$$L^{-1}\left\{s^{-mnr}\right\}(t)=\frac{t^{mnr-1}}{\Gamma(mnr)}\tag{19}$$

$${}_rF(t)=(1-r)^{-1}\left[\frac{t^{mnr-1}}{\Gamma(mnr)}C_pC_\psi^{r-1}\left(I_m\left(n-1+(1-m)/r,1/r\right)\right)^r-1\right]$$

那么

$${}_rh(p\,\|\,\psi)={}_rF(1)=(1-r)^{-1}\left[\frac{C_pC_\psi^{r-1}}{\Gamma(mnr)}\left(I_m\left(n-1+(1-m)/r,1/r\right)\right)^r-1\right]\tag{20}$$

注 3：当 $r\to1$ 时，

$$\begin{aligned}\lim_{r\to1}{}_rh(p\,\|\,\psi)=&-\ln\left(\prod_{k=1}^m k!(n-k)!\right)-m\psi(2)+m\ln\Gamma(n)-\ln\Gamma(mn)-mn\psi(mn)\\&-\sum_{k=n-m}^{n-1}(n-k-1)\psi(k+1)+\sum_{k=1}^m k\psi(k+1)\end{aligned}$$

证明：

令

$$I_m\left(\alpha(r),\beta(r)\right)^r=y$$

那么对等式两边取以 e 为底的对数得到

$$r\ln I_m\left(\alpha(r),\beta(r)\right)=\ln y$$

再对 r 求导得到

$$\begin{aligned}&\ln I_m\left(\alpha(r),\beta(r)\right)+\frac{r}{I_m\left(\alpha(r),\beta(r)\right)}\frac{\partial I_m\left(\alpha(r),\beta(r)\right)}{\partial r}=\frac{y'}{y}\\&y'=y\left[\ln I_m\left(\alpha(r),\beta(r)\right)+\frac{r}{I_m\left(\alpha(r),\beta(r)\right)}\frac{\partial I_m\left(\alpha(r),\beta(r)\right)}{\partial r}\right]\end{aligned}\tag{21}$$

由式(21)得到

$$\frac{\partial\left(I_m\left(n-1+(1-m)/r,1/r\right)\right)^r}{\partial r}=\left(I_m\left(n-1+(1-m)/r,1/r\right)\right)^r \left[\ln I_m\left(n-1+(1-m)/r,1/r\right)+\frac{r}{I_m\left(n-1+(1-m)/r,1/r\right)}\times\frac{\partial I_m\left(n-1+(1-m)/r,1/r\right)}{\partial r}\right] \tag{22}$$

由式(10)得到

$$\begin{aligned}&\left.\frac{\partial I_m\left(n-1+(1-m)/r,1/r\right)}{\partial r}\right|_{r=1}\\&=I_m\left(n-1+(1-m)/r,1/r\right)\left[\frac{m-1}{r^2}\sum_{k=0}^{m-1}\psi\left(n+(1-m)/r+k/r\right)\right.\\&\left.\left.-\frac{1}{r^2}\sum_{k=1}^{m-1}k\psi\left(n+(1-m)/r+k/r\right)-\frac{1}{r^2}\sum_{k=1}^{m}k\psi\left(1+k/r\right)+\frac{1}{r^2}m\psi\left(1+1/r\right)\right]\right|_{r=1}\\&=I_m\left(n-m,1\right)\left[\sum_{k=n-m}^{n-1}\left(1+k-n\right)\psi\left(k+1\right)-\sum_{k=1}^{m}k\psi\left(1+k\right)+m\psi\left(2\right)\right]\end{aligned} \tag{23}$$

把式(2)、式(4)、式(6)、式(8)、式(22)、式(23)代入式(20)得到

$$\begin{aligned}\lim_{r\to1}{}_r h\left(p\,\|\,\psi\right)&=-\left[\frac{C_pC_\psi^{r-1}\ln C_\psi}{\Gamma\left(mnr\right)}\left(I_m\left(n-1+(1-m)/r,1/r\right)\right)^r\right.\\&\quad+\frac{C_pC_\psi^{r-1}\psi\left(mnr\right)mn}{\Gamma\left(mnr\right)}\left(I_m\left(n-1+(1-m)/r,1/r\right)\right)^r\\&\quad\left.\left.+\frac{C_pC_\psi^{r-1}}{\Gamma\left(mnr\right)}\frac{\partial\left(I_m\left(n-1+(1-m)/r,1/r\right)\right)^r}{\partial r}\right]\right|_{r=1}\\&=-\left.\left[\ln C_p-\ln C_\psi+\frac{C_p}{\Gamma\left(mn\right)}\frac{\partial\left(I_m\left(n-1+(1-m)/r,1/r\right)\right)^r}{\partial r}\right]\right|_{r=1}\\&=-\ln\left(\prod_{k=1}^{m}k!\left(n-k\right)!\right)-m\psi\left(2\right)+m\ln\Gamma\left(n\right)-\ln\Gamma\left(mn\right)-mn\psi\left(mn\right)\\&\quad-\sum_{k=n-m}^{n-1}\left(n-k-1\right)\psi\left(k+1\right)+\sum_{k=1}^{m}k\psi\left(k+1\right)\end{aligned}$$

定理 3.4：当 $r>0, r\neq1, s=0$ 时，在 Haar 分布的双体纯态上其对角元的联合分布相对于取部分迹所诱导的随机密度矩阵的联合分布的统一(r, s)相对微分熵为

$$h_r\left(\psi\,\|\,p\right)=\left(r-1\right)^{-1}\left[r\ln C_\psi+\left(1-r\right)\ln C_p+\ln I_m\left(\left(m-1\right)r-m+n,1-r\right)-\ln\Gamma\left(mn\right)\right]$$

证明：

$$\begin{aligned}h_r\left(\psi\,\|\,p\right)&=-\left(1-r\right)^{-1}\ln\left(\int_0^\infty\cdots\int_0^\infty\psi\left(\lambda_1,\cdots,\lambda_m\right)\left(\psi\left(\lambda_1,\cdots,\lambda_m\right)/p\left(\lambda_1,\cdots,\lambda_m\right)\right)^{r-1}\prod_{k=1}^{m}\mathrm{d}\lambda_k\right)\\&=\left(r-1\right)^{-1}\ln\left(\int_0^\infty\cdots\int_0^\infty\psi^r\left(\lambda_1,\cdots,\lambda_m\right)p^{1-r}\left(\lambda_1,\cdots,\lambda_m\right)\prod_{k=1}^{m}\mathrm{d}\lambda_k\right)\end{aligned}$$

令

$$F_r'(t)=(r-1)^{-1}\ln\left(\int_0^{\infty}\cdots\int_0^{\infty}\delta\left(t-\sum_{j=1}^{m}\lambda_j\right)\psi^r(\lambda_1,\cdots,\lambda_m)p^{1-r}(\lambda_1,\cdots,\lambda_m)\prod_{k=1}^{m}\mathrm{d}\lambda_k\right)$$

对 $F_r'(t)$ 应用 Laplace 变换 $(t\to s)$，则有

$$\tilde{F}_r'(s)=(r-1)^{-1}\ln\left(\int_0^{\infty}\cdots\int_0^{\infty}\exp\left(-s\sum_{j=1}^{m}\lambda_j\right)\psi^r(\lambda_1,\cdots,\lambda_m)p^{1-r}(\lambda_1,\cdots,\lambda_m)\prod_{k=1}^{m}\mathrm{d}\lambda_k\right)$$

把式(7)、式(11)、式(12)代入上式得到

$$\begin{aligned}\tilde{F}_r'(s)&=(r-1)^{-1}\ln\left(\int_0^{\infty}\cdots\int_0^{\infty}\exp\left(-s\sum_{j=1}^{m}\lambda_j\right)\left(s^{m-mn}C_p\prod_{1\le i<j\le m}(\mu_i-\mu_j)^2\prod_{k=1}^{m}\mu_k^{n-m}\right)^{1-r}\left(s^{m-mn}C_\psi\prod_{j=1}^{m}\mu_j^{n-1}\right)^r\prod_{k=1}^{m}\frac{\mathrm{d}\mu_k}{s}\right)\\&=(r-1)^{-1}\ln\left(\int_0^{\infty}\cdots\int_0^{\infty}s^{-mn}C_\psi^rC_p^{1-r}\prod_{1\le i<j\le m}(\mu_i-\mu_j)^{2(1-r)}\exp\left(-\sum_{j=1}^{m}\mu_j\right)\prod_{k=1}^{m}\mu_k^{(m-1)r-m+n}\mathrm{d}\mu_k\right)\\&=(r-1)^{-1}\ln\left(s^{-mn}C_\psi^rC_p^{1-r}I_m\left((m-1)r-m+n,1-r\right)\right)\end{aligned}$$

把式(14)代入上式得到

$$F_r'(t)=(r-1)^{-1}\ln\left(\frac{t^{mn-1}}{\Gamma(mn)}C_\psi^rC_p^{1-r}I_m\left((m-1)r+n-m,1-r\right)\right)$$

那么

$$\begin{aligned}h_r(\psi\parallel p)=F_r'(1)&=(r-1)^{-1}\ln\left(\frac{C_\psi^rC_p^{1-r}}{\Gamma(mn)}I_m\left((m-1)r+n-m,1-r\right)\right)\\&=(r-1)^{-1}\left[r\ln C_\psi+(1-r)\ln C_p+\ln I_m\left((m-1)r+n-m,1-r\right)-\ln\Gamma(mn)\right]\end{aligned}\tag{24}$$

注 4：当 $r\to1$ 时，

$$\lim_{r\to1}h_r(\psi\parallel p)=\ln\left(\prod_{k=1}^{m}k!(n-k)!\right)-m\ln\Gamma(n)+\frac{m^2-m}{2}\psi(n)+\frac{m-m^2}{2}\psi(1)$$

证明：

由式(10)，我们得到

$$\begin{aligned}\left.\frac{\partial\ln I_m\left((m-1)r+n-m,1-r\right)}{\partial r}\right|_{r=1}&=\left.\frac{1}{I_m\left((m-1)r+n-m,1-r\right)}\frac{\partial I_m\left((m-1)r+n-m,1-r\right)}{\partial r}\right|_{r=1}\\&=(m-1)\times\sum_{k=0}^{m-1}\psi\left((m-1)r+n-m+1+k(1-r)\right)+m\psi\left(1+(1-r)\right)\\&\quad\left.-\sum_{k=1}^{m-1}k\psi\left((m-1)r+n-m+1+k(1-r)\right)-\sum_{k=1}^{m}k\psi\left(1+k(1-r)\right)\right|_{r=1}\\&=(m-1)\times m\psi(n)-\frac{m^2+m}{2}\psi(1)+m\psi(1)\end{aligned}\tag{25}$$

把式(2)、式(4)、式(6)、式(8)、式(25)代入式(24)得到

$$\begin{aligned}\lim_{r\to1}h_r(\psi\parallel p)&=(r-1)^{-1}\left[r\ln C_p+(1-r)\ln C_\psi+\ln I_m\left((m-1)r+n-m,1-r\right)-\ln\Gamma(mn)\right]\\&=\ln C_\psi-\ln C_p+\left.\frac{\partial\ln I_m\left((m-1)r+n-m,1-r\right)}{\partial r}\right|_{r=1}\\&=\ln\left(\prod_{k=1}^{m}k!(n-k)!\right)-m\ln\Gamma(n)+\frac{m^2-m}{2}\psi(n)+\frac{m-m^2}{2}\psi(1)\end{aligned}$$

定理 3.5：当 $r>0, r\neq 1, s=1$ 时，在 Haar 分布的双体纯态上其对角元的联合分布相对于取部分迹所诱导的随机密度矩阵的联合分布的统一(r, s)相对微分熵为

$$h^{r}(\psi \| p)=(r-1)^{-1}\left[\frac{C_{\psi}^{r} C_{p}^{1-r}}{\Gamma(mn)} I_{m}((m-1)r+n-m, 1-r)-1\right]$$

证明：

$$\begin{aligned}h^{r}(\psi \| p)&=-(1-r)^{-1}\left[\int_{0}^{\infty}\cdots\int_{0}^{\infty}\psi(\lambda_{1},\cdots,\lambda_{m})\left(\psi(\lambda_{1},\cdots,\lambda_{m})/p(\lambda_{1},\cdots,\lambda_{m})\right)^{r-1}\prod_{k=1}^{m}\mathrm{d}\lambda_{k}-1\right]\\&=(r-1)^{-1}\left[\int_{0}^{\infty}\cdots\int_{0}^{\infty}\psi^{r}(\lambda_{1},\cdots,\lambda_{m})p^{1-r}(\lambda_{1},\cdots,\lambda_{m})\prod_{k=1}^{m}\mathrm{d}\lambda_{k}-1\right]\end{aligned}$$

令

$$F^{r'}(t)=(r-1)^{-1}\left[\int_{0}^{\infty}\cdots\int_{0}^{\infty}\delta\left(t-\sum_{j=1}^{m}\lambda_{j}\right)\psi^{r}(\lambda_{1},\cdots,\lambda_{m})p^{1-r}(\lambda_{1},\cdots,\lambda_{m})\prod_{k=1}^{m}\mathrm{d}\lambda_{k}-1\right]$$

对 $F^{r'}(t)$ 应用 Laplace 变换 $(t\to s)$，则有

$$\tilde{F}^{r'}(s)=(r-1)^{-1}\left[\int_{0}^{\infty}\cdots\int_{0}^{\infty}\exp\left(-s\sum_{j=1}^{m}\lambda_{j}\right)\psi^{r}(\lambda_{1},\cdots,\lambda_{m})p^{1-r}(\lambda_{1},\cdots,\lambda_{m})\prod_{k=1}^{m}\mathrm{d}\lambda_{k}-1\right]$$

把式(7)、式(11)、式(12) 代入上式得到

$$\begin{aligned}\tilde{F}^{r'}(s)&=(r-1)^{-1}\left[\int_{0}^{\infty}\cdots\int_{0}^{\infty}\exp\left(-s\sum_{j=1}^{m}\lambda_{j}\right)\left(s^{m-mn}C_{p}\prod_{1\le i<j\le m}(\mu_{i}-\mu_{j})^{2}\prod_{k=1}^{m}\mu_{k}^{n-m}\right)^{1-r}\left(s^{m-mn}C_{\psi}\prod_{j=1}^{m}\mu_{j}^{n-1}\right)^{r}\prod_{k=1}^{m}\frac{\mathrm{d}\mu_{k}}{s}-1\right]\\&=(r-1)^{-1}\left[\int_{0}^{\infty}\cdots\int_{0}^{\infty}s^{-mn}C_{\psi}^{r}C_{p}^{1-r}\prod_{1\le i<j\le m}(\mu_{i}-\mu_{j})^{2(1-r)}\exp\left(-\sum_{j=1}^{m}\mu_{j}\right)\prod_{k=1}^{m}\mu_{k}^{(m-1)r+n-m}\mathrm{d}\mu_{k}-1\right]\\&=(r-1)^{-1}\left[s^{-mn}C_{\psi}^{r}C_{p}^{1-r}I_{m}((m-1)r+n-m, 1-r)-1\right]\end{aligned}$$

把式(14)代入上式得到

$$F^{r'}(t)=(r-1)^{-1}\left[\frac{t^{mn-1}}{\Gamma(mn)}C_{\psi}^{r}C_{p}^{1-r}I_{m}((m-1)r+n-m, 1-r)-1\right]$$

那么

$$h^{r'}(\psi \| p)=F^{r'}(1)=(r-1)^{-1}\left[\frac{C_{\psi}^{r}C_{p}^{1-r}}{\Gamma(mn)}I_{m}((m-1)r+n-m, 1-r)-1\right]$$

注 5：当 $r\to 1$ 时，

$$\lim_{r\to 1}h^{r}(\psi \| p)=\ln\left(\prod_{k=1}^{m}k!(n-k)!\right)-m\ln\Gamma(n)+\frac{m^{2}-m}{2}\psi(n)+\frac{m-m^{2}}{2}\psi(1) \tag{26}$$

证明：

由式(10)，我们得到

$$\begin{aligned}&\left.\frac{\partial I_m\left((m-1)r+n-m,1-r\right)}{\partial r}\right|_{r=1}\\&=I_m\left((m-1)r+n-m,1-r\right)\left[(m-1)\times\sum_{k=0}^{m-1}\psi\left((m-1)r+n-m+1+k(1-r)\right)+m\psi(2-r)\right.\\&\quad\left.\left.-\sum_{k=1}^{m-1}k\psi\left((m-1)r+n-m+1+k(1-r)\right)-\sum_{k=1}^{m}k\psi\left(1+k(1-r)\right)\right]\right|_{r=1}\\&=I_m(n-1,0)\left[\sum_{k=0}^{m-1}(m-1-k)\times\psi(n)+m\psi(1)-\sum_{k=1}^{m}k(1)\right]\end{aligned}\tag{27}$$

把式(2)、式(4)、式(6)、式(9)、式(27)代入式(26)得到

$$\begin{aligned}\lim_{r\to1}h^r(\psi\parallel p)&=\left[\ln C_\psi C_\psi^r C_p^{1-r}\frac{I_m\left((m-1)r+n-m,1-r\right)}{\Gamma(mn)}-\ln C_p C_\psi^r C_p^{1-r}\frac{I_m\left((m-1)r+n-m,1-r\right)}{\Gamma(mn)}\right.\\&\quad\left.\left.+\frac{C_\psi^r C_p^{1-r}}{\Gamma(mn)}\frac{\partial I_m\left((m-1)r+n-m,1-r\right)}{\partial r}\right]\right|_{r=1}\\&=\frac{\ln C_\psi I_m(n-1,0)}{\Gamma^m(n)}-\frac{\ln C_p I_m(n-1,0)}{\Gamma^m(n)}+\frac{I_m(n-1,0)}{\Gamma^m(n)}\left.\frac{\partial I_m\left((m-1)r+n-m,1-r\right)}{\partial r}\right|_{r=1}\\&=\ln\left(\prod_{k=1}^m k!(n-k)!\right)-m\ln\Gamma(n)+\frac{m^2-m}{2}\psi(n)+\frac{m-m^2}{2}\psi(1)\end{aligned}$$

定理 3.6：当 $r>0,r\neq1,s=1/r$ 时，在 Haar 分布的双体纯态上取部分迹所诱导的随机密度矩阵的特征值的联合分布相对于其对角元的联合分布的统一(r,s)相对微分熵为

$$_r h(\psi\parallel p)=(1-r)^{-1}\left[\frac{C_\psi C_p^{r-1}}{\Gamma(mnr)}\left(I_m\left(n-m+\frac{m-1}{r},1-\frac{1}{r}\right)\right)^r-1\right]$$

证明：

$$\begin{aligned}{}_r h(\psi\parallel p)&=-(r-1)^{-1}\left[\left(\int_0^\infty\cdots\int_0^\infty\psi(\lambda_1,\cdots,\lambda_m)\left(\psi(\lambda_1,\cdots,\lambda_m)/p(\lambda_1,\cdots,\lambda_m)\right)^{1/r-1}\prod_{k=1}^m\mathrm{d}\lambda_k\right)^r-1\right]\\&=(1-r)^{-1}\left[\left(\int_0^\infty\cdots\int_0^\infty\psi^{1/r}(\lambda_1,\cdots,\lambda_m)p^{1-1/r}(\lambda_1,\cdots,\lambda_m)\prod_{k=1}^m\mathrm{d}\lambda_k\right)^r-1\right]\end{aligned}$$

令

$$_r F'(t)=(1-r)^{-1}\left[\left(\int_0^\infty\cdots\int_0^\infty\delta\left(t-\sum_{j=1}^m\lambda_j\right)\psi^{1/r}(\lambda_1,\cdots,\lambda_m)p^{1-1/r}(\lambda_1,\cdots,\lambda_m)\prod_{k=1}^m\mathrm{d}\lambda_k\right)^r-1\right]$$

对 $_r F'(t)$ 应用 Laplace 变换 $(t\to s)$，则有

$$_r\tilde{F}'(s)=(1-r)^{-1}\left[\left(\int_0^\infty\cdots\int_0^\infty\exp\left(-s\sum_{j=1}^m\lambda_j\right)\psi^{1/r}(\lambda_1,\cdots,\lambda_m)p^{1-1/r}(\lambda_1,\cdots,\lambda_m)\prod_{k=1}^m\mathrm{d}\lambda_k\right)^r-1\right]$$

把式(7)、式(11)、式(12)代入上式得到

$$ {}_r\tilde{F}'(s)=(1-r)^{-1}\left[\left(\int_0^\infty\cdots\int_0^\infty \exp\left(-s\sum_{j=1}^m\lambda_j\right)\left(s^{m-mn}C_p\prod_{1\le i<j\le m}(\mu_i-\mu_j)^2\prod_{k=1}^m\mu_k^{n-m}\right)^{1-1/r}\left(s^{m-mn}C_\psi\prod_{j=1}^m\mu_j^{n-1}\right)^{1/r}\prod_{k=1}^m\frac{\mathrm{d}\mu_k}{s}\right)^r-1\right] $$

$$ =(1-r)^{-1}\left[\left(\int_0^\infty\cdots\int_0^\infty s^{-mn}C_\psi^{1/r}C_p^{1-1/r}\prod_{1\le i<j\le m}(\mu_i-\mu_j)^{2(1-1/r)}\exp\left(-\sum_{j=1}^m\mu_j\right)\prod_{k=1}^m\mu_j^{n-m+(m-1)/r}\mathrm{d}\mu_k\right)^r-1\right] $$

$$ =(1-r)^{-1}\left[s^{-mnr}C_\psi C_p^{r-1}\left(I_m\left(n-m+(m-1)/r,1-1/r\right)\right)^r-1\right] $$

把式(19)代入上式得到

$$ {}_rF'(t)=(1-r)^{-1}\left[\frac{t^{mnr-1}}{\Gamma(mnr)}C_\psi C_p^{r-1}\left(I_m\left(n-m+(m-1)/r,1-1/r\right)\right)^r-1\right] $$

那么

$$ {}_rh(\psi\,\|\,p)={}_rF'(1)=(1-r)^{-1}\left[\frac{C_\psi C_p^{r-1}}{\Gamma(mnr)}\left(I_m\left(n-m+(m-1)/r,1-1/r\right)\right)^r-1\right] \tag{28} $$

注 6：当 $r\to1$ 时，

$$ \lim_{r\to1}{}_rh(\psi\,\|\,p)=\ln\left(\prod_{k=1}^m k!(n-k)!\right)-m\ln\Gamma(n)-\ln\Gamma(mn)-mn\psi(mn)+\frac{m^2-m}{2}\psi(n)+\frac{m-m^2}{2}\psi(1) $$

证明：

由式(21)得到

$$ \begin{aligned} \frac{\partial\left(I_m\left(n-m+(m-1)/r,1-1/r\right)\right)^r}{\partial r}&=\left(I_m\left(n-m+(m-1)/r,1-1/r\right)\right)^r\\ \left[\ln I_m\left(n-m+(m-1)/r,1-1/r\right)\right.&\left.+\frac{r}{I_m\left(n-m+(m-1)/r,1-1/r\right)}\times\frac{\partial I_m\left(n-m+(m-1)/r,1-1/r\right)}{\partial r}\right] \end{aligned} \tag{29} $$

由式(10)得到

$$ \left.\frac{\partial I_m\left(n-m+(m-1)/r,1-1/r\right)}{\partial r}\right|_{r=1}=I_m(n-1,0)\left[(1-m)\sum_{k=0}^{m-1}\psi(n)+\sum_{k=1}^m k\psi(n)+\sum_{k=1}^m k\psi(1)-m\psi(1)\right] \tag{30} $$

把式(2)、式(4)、式(6)、式(9)、式(22)、式(29)、式(30)代入式(28)得到

$$ \begin{aligned} \lim_{r\to1}{}_rh(\psi\,\|\,p)=-\Bigg[&\frac{C_\psi C_p^{r-1}\ln C_p}{\Gamma(mnr)}\left(I_m\left(n-m+(m-1)/r,1-1/r\right)\right)^r\\ &+\frac{C_\psi C_p^{r-1}\psi(mnr)mn}{\Gamma(mnr)}\left(I_m\left(n-m+(m-1)/r,1-1/r\right)\right)^r\\ &\left.+\frac{C_\psi C_p^{r-1}}{\Gamma(mnr)}\frac{\partial\left(I_m\left(n-m+(m-1)/r,1-1/r\right)\right)^r}{\partial r}\Bigg]\right|_{r=1}\\ =-\ln&\left(\prod_{k=1}^m k!(n-k)!\right)-m\psi(2)+m\ln\Gamma(n)-\ln\Gamma(mn)-mn\psi(mn)\\ &-\sum_{k=n-m}^{n-1}(n-k-1)\psi(k+1)+\sum_{k=1}^m k\psi(k+1) \end{aligned} $$

4. 总结

本文定义了在Haar分布的双体纯态上取部分迹所诱导的随机密度矩阵的特征值的联合分布相对于其对角元的联合分布(其对角元的联合分布相对于取部分迹所诱导的随机密度矩阵的特征值的联合分布)统一(r, s)相对微分熵，计算了三种情形下的统一(r, s)相对微分熵，取极限后的结果表明三种情形下的结果基本相等，推广了随机密度矩阵特征值联合分布的微分熵的范围。

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