第一章

变换核函数由指数函数，幂函数，三角函数组成

在第 1，2 章将会用到下面定理和留数级数公式。

1) 高斯留数基本定理[13]：如果函数 $f(z)$ 在扩充复平面内只有有限个奇点，那么 $f(z)$ 在所有各奇点（包括 ∞ 点）的留数总和必等于零；

2) 在文[15]关于 2 重极点留数计算公式：若 $F(z)=p(z)/q(z)$，$p(z),q(z)$ 在 $z=a$ 处解析。$p(a)\neq 0$，若 $F(z)$ 在 $z=a$ 处有 2 重极点，则留数计算公式：$\mathrm{Res}\left[F(z);a\right]=2\dfrac{p'(a)}{q''(a)}-\dfrac{2p(a)q'''(a)}{3\left[q''(a)\right]^2}$；

3) 在文[16]给出 3 重极点留数公式：设 $F(z)=p(z)/q(z)$，其中 p,q 在 a 点解析，$p(z)\neq 0$，q 在点 z 有 3 重极点，则留数公式：

$$\mathrm{Res}\left[F(z);z\right]=\frac{3p''(z)}{q'''(z)}-\frac{3p'(z)q^{(4)}(z)}{2\left[q'''(z)\right]^2}-\frac{3p(z)q^{(5)}(z)}{10\left[q'''(z)\right]^2}+\frac{3p(z)\left[q^{(4)}(z)\right]^2}{8\left[q'''(z)\right]^3}\text{。}$$

1. 变换核函数由指数函数，幂函数，三角函数组成

命题 1 无穷级数恒等式

$$\sum_{n=1}^{\infty}-\frac{1}{n\sinh(2n\pi)}+\sum_{n=1}^{\infty}\frac{2(-1)^n}{n\left(e^{n\pi}-1\right)}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n}=-\frac{\pi}{4} \tag{1}$$

$$\sum_{n=1}^{\infty}\frac{1}{4n^3\sinh(2n\pi)}+\sum_{n=1}^{\infty}\frac{2(-1)^n}{n^3\left(e^{n\pi}-1\right)}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}=-\frac{23\pi^3}{720} \tag{2}$$

$$\sum_{n=1}^{\infty}-\frac{1}{16n^5\sinh(2n\pi)}+\sum_{n=1}^{\infty}\frac{2(-1)^n}{n^5\left(e^{n\pi}-1\right)}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^5}=-\frac{\pi^5}{288} \tag{3}$$

证明 变换核函数，$f(z)=\dfrac{\pi}{z^{2k+1}\left(e^{\pi z}-1\right)\sin(\pi z)}$ 有单极点 $z=\pm n$， $z=\pm 2ni$。

1）函数展开成幂级数

$$\begin{aligned}
f(z)&=\frac{\pi\csc(\pi z)}{z^{2k+1}\left(e^{\pi z}-1\right)}=\frac{\pi\left[1/\pi z+(\pi z)/6+7(\pi z)^3/360+\cdots\right]}{z^{2k+1}\left[\pi z+(\pi z)^2/2+(\pi z)^3/6+\cdots\right]}\\
&=\frac{\pi\left[1+(\pi z)^2/6+7(\pi z)^4/360+\cdots\right]}{\pi^2 z^{2k+3}\left[1+\pi z/2+(\pi z)^2/6+(\pi z)^3/24+\cdots\right]}\\
&=\frac{1}{\pi z^{2k+3}}\left(1+\frac{1}{6}(\pi z)^2+\frac{7}{360}(\pi z)^4+\frac{31}{15120}(\pi z)^6+\cdots\right)\\
&\quad\cdot\left(1-\frac{1}{2}(\pi z)+\frac{1}{12}(\pi z)^2-\frac{1}{720}(\pi z)^4+\frac{1}{30240}(\pi z)^6+\cdots\right);
\end{aligned}$$

$f(z)$ 的留数 $\mathrm{Res}[f;0]=p(2k+3)$。其中 $p(3)=\dfrac{\pi}{4}$； $p(5)=\dfrac{23\pi^3}{720}$； $p(7)=\dfrac{\pi^5}{288}$；

2）在单极点 $z=\pm 2ni$，计算 $f(z)$ 的留数

$$\begin{aligned}
\mathrm{Res}[f;2ni]&=\lim_{z\to ni}\frac{(z-2ni)\pi}{\left(e^{\pi z}-1\right)}\frac{1}{z^{2k+1}\sin(\pi z)}=\frac{\pi}{\pi e^{2ni\pi}}\frac{1}{[2ni]^{2k+1}\left[\sin(\pi 2ni)\right]}\\
&=\frac{1}{(2n)^{2k+1}i^{2k+1}i\sinh(2n\pi)}=\frac{1}{(2n)^{2k+1}i^{2k+2}\sinh(2n\pi)}\\
&=-\frac{1}{(2n)^{2k+1}i^{2k}\sinh(2n\pi)}=\mathrm{Res}[f;-2ni];
\end{aligned}$$

3）在单极点 $z=\pm n$，计算 $f(z)$ 的留数，用洛必达法则可得

$$\mathrm{Res}[f;n]=\lim_{z\to n}\frac{\pi}{z^{2k+1}\left(e^{\pi z}-1\right)}\frac{[z-n]}{\sin(\pi z)}=\lim_{z\to n}\frac{\pi}{n^{2k+1}\left(e^{n\pi}-1\right)}\frac{1}{\left[\pi\cos(\pi n)\right]}=\frac{(-1)^n}{n^{2k+1}\left(e^{n\pi}-1\right)};$$

$$\operatorname{Res}[f;-n]=\lim_{z\to -n}\frac{\pi}{z^{2k+1}\left(e^{\pi z}-1\right)}\frac{[z+n]}{\sin(\pi z)}=\frac{\pi}{[-n]^{2k+1}\left(e^{-n\pi}-1\right)}\frac{1}{[\pi\cos(-\pi n)]}$$
$$=\frac{(-1)^n}{-n^{2k+1}\left(e^{-n\pi}-1\right)}=\frac{(-1)^n}{n^{2k+1}\left(e^{n\pi}-1\right)}+\frac{(-1)^n}{n^{2k+1}};$$

根据高斯留数基本定理，从而有

$$0=p(2k+3)+\sum_{n=-\infty}^{\infty}-\frac{1}{(2n)^{2k+1}i^{2k}\sinh(2n\pi)}+\sum_{k=-\infty}^{\infty}\frac{(-1)^n}{n^{2k+1}\left(e^{n\pi}-1\right)}+\sum_{n=0}^{\infty}\frac{(-1)^n}{n^{2k+1}};$$

$$\sum_{n=1}^{\infty}-\frac{2}{(2n)^{2k+1}i^{2k}\sinh(2n\pi)}+\sum_{n=1}^{\infty}\frac{2(-1)^n}{n^{2k+1}\left(e^{n\pi}-1\right)}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{2k+1}}=-p(2k+3);$$

令 $k=0,1,2$ 整理得到如下级数恒等式

$$\sum_{n=1}^{\infty}-\frac{1}{n\sinh(2n\pi)}+\sum_{n=1}^{\infty}\frac{2(-1)^n}{n\left(e^{n\pi}-1\right)}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n}=-\frac{\pi}{4};$$

$$\sum_{n=1}^{\infty}\frac{1}{4n^3\sinh(2n\pi)}+\sum_{n=1}^{\infty}\frac{2(-1)^n}{n^3\left(e^{n\pi}-1\right)}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}=-\frac{23\pi^3}{720};$$

$$\sum_{n=1}^{\infty}-\frac{1}{16n^5\sinh(2n\pi)}+\sum_{n=1}^{\infty}\frac{2(-1)^n}{n^5\left(e^{n\pi}-1\right)}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^5}=-\frac{\pi^5}{288}。$$

命题 2 无穷级数恒等式

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2\cosh(\pi(2n+1/2))}-\sum_{n=0}^{\infty}\frac{2(-1)^n}{(2n+1)^2\left(e^{\pi(2n+1)}+1\right)}+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=\frac{\pi^2}{8}\tag{1}$$

$$\sum_{n=0}^{\infty}-\frac{1}{(2n+1)^4\cosh(\pi(2n+1/2))}-\sum_{n=0}^{\infty}\frac{2(-1)^n}{(2n+1)^4\left(e^{\pi(2n+1)}+1\right)}+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^4}=\frac{1}{192}\pi^4\tag{2}$$

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^6\cosh(\pi(2n+1/2))}-\sum_{n=0}^{\infty}\frac{2(-1)^n}{(2n+1)^6\left(e^{\pi(2n+1)}+1\right)}+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^6}=\frac{7\pi^6}{5120}\tag{3}$$

$$\sum_{n=0}^{\infty}-\frac{1}{(2n+1)^8\cosh(\pi(2n+1/2))}-\sum_{n=0}^{\infty}\frac{2(-1)^n}{(2n+1)^8\left(e^{\pi(2n+1)}+1\right)}+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^8}=\frac{47}{860160}\pi^8 \tag{4}$$

证明 变换核函数 $f(z)=\dfrac{\pi}{z^{2k+2}\left(e^{2\pi z}+1\right)\cos(\pi z)}$ 有单极点 $z=\pm(2n+1)i/2$， $z=\pm(2n+1)/2$。

1) 展开成幂级数

$$\begin{aligned}
f(z)&=\frac{\pi\sec(\pi z)}{z^{2k+2}\left(e^{2\pi z}+1\right)}=\frac{\pi\left[1+(\pi z)^2/2+5(\pi z)^4/24+\cdots\right]}{z^{2k+2}\left[2+(2\pi z)+(2\pi z)^2/2+(2\pi z)^3/6+\cdots\right]}\\
&=\frac{\pi\left[1+(\pi z)^2/2+5(\pi z)^4/24+\cdots\right]}{2z^{2k+2}\left[1+\pi z+(2\pi z)^2/4+(2\pi z)^3/12+\cdots\right]}\\
&=\frac{\pi}{2z^{2k+2}}\left(1+\frac{1}{2}(\pi z)^2+\frac{5}{24}(\pi z)^4+\frac{61}{720}(\pi z)^6+\cdots\right)\\
&\quad\cdot\left(1-\pi z+\frac{1}{3}(\pi z)^3-\frac{2}{15}(\pi z)^5+\frac{17}{315}(\pi z)^7+\cdots\right);
\end{aligned}$$

$f(z)$ 的留数 $\mathrm{Res}[f;0]=q(2k+2)$；其中 $q(2)=-\dfrac{\pi^2}{2}$； $q(4)=-\dfrac{1}{12}\pi^4$； $q(6)=-\dfrac{7\pi^6}{80}$； $q(8)=-\dfrac{47}{3360}\pi^8$；

2) 在单极点 $z=\pm(2n+1)i/2$，计算 $f(z)$ 的留数，用洛必达法则可得

$$\begin{aligned}
\mathrm{Res}\left[f;(2n+1)i/2\right]&=\lim_{z\to(2n+1)i/2}\frac{z-(2n+1)i/2}{\left(e^{2\pi z}+1\right)}\frac{\pi}{z^{2k+2}\cos(\pi z)}\\
&=\frac{1}{2\pi e^{\pi(2n+1)i}}\frac{\pi}{\left[(2n+1)i/2\right]^{2k+2}\cos(\pi(2n+1)i/2)}\\
&=-\frac{2^{2k+1}}{(2n+1)^{2k+2}i^{2k+2}\cosh(\pi(2n+1/2))}\\
&=\frac{2^{2k+1}}{(2n+1)^{2k+2}i^{2k}\cosh(\pi(2n+1/2))}\\
&=\mathrm{Res}\left[f;-(2n+1)i/2\right];
\end{aligned}$$

3) 在单极点 $z=\pm(2n+1)/2$，计算 $f(z)$ 的留数

$$\operatorname{Res}\left[f;(2n+1)/2\right]=\lim_{z\to(2n+1)/2}\frac{\pi}{z^{2k+2}\left(e^{2\pi z}+1\right)}\frac{\left(z-(2n+1)/2\right)}{\cos(\pi z)}$$
$$=\frac{2^{2k+2}\pi}{(2n+1)^{2k+2}\left(e^{\pi(2n+1)}+1\right)}\frac{1}{\left[-\pi\sin\pi(2n+1)/2\right]}$$
$$=-\frac{2^{2k+2}(-1)^n}{(2n+1)^{2k+2}\left(e^{\pi(2n+1)}+1\right)};$$

$$\operatorname{Res}\left[f;-(2n+1)/2\right]=\lim_{z\to-(2n+1)/2}\frac{1}{z^{2k+2}\left(e^{2\pi z}+1\right)}\frac{\left(z+(2n+1)/2\right)\pi}{\cos(\pi z)}$$
$$=\frac{2^{2k+2}}{(2n+1)^{2k+2}\left(e^{-\pi(2n+1)}+1\right)}\frac{\pi}{\left[-\pi\sin\left(-\pi(2n+1)/2\right)\right]}$$
$$=\frac{2^{2k+2}(-1)^n}{(2n+1)^{2k+2}\left(e^{-\pi(2n+1)}+1\right)}=\frac{2^{2k+2}(-1)^n}{(2n+1)^{2k+2}}-\frac{2^{2k+2}(-1)^n}{(2n+1)^{2k+2}\left(e^{\pi(2n+1)}+1\right)};$$

根据高斯留数基本定理，从而有

$$0=q(2k+2)+\sum_{n=-\infty}^{\infty}\frac{2^{2k+1}}{(2n+1)^{2k+2}i^{2k}\cosh\left(\pi(2n+1/2)\right)}$$
$$+\sum_{n=-\infty}^{\infty}-\frac{2^{2k+2}(-1)^n}{(2n+1)^{2k+2}\left(e^{\pi(2n+1)}+1\right)}+\sum_{n=0}^{\infty}\frac{2^{2k+2}(-1)^n}{(2n+1)^{2k+2}};$$

$$\sum_{n=0}^{\infty}\frac{2^{2k+2}}{(2n+1)^{2k+2}i^{2k}\cosh\left(\pi(2n+1/2)\right)}+\sum_{n=0}^{\infty}-\frac{2^{2k+3}(-1)^n}{(2n+1)^{2k+2}\left(e^{\pi(2n+1)}+1\right)}$$
$$+\sum_{n=0}^{\infty}\frac{2^{2k+2}(-1)^n}{(2n+1)^{2k+2}}=-q(2k+2);$$

令 $k=0,1,2,3$ 整理得到如下级数恒等式

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2\cosh\left(\pi(2n+1/2)\right)}-\sum_{n=0}^{\infty}\frac{2(-1)^n}{(2n+1)^2\left(e^{\pi(2n+1)}+1\right)}+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=\frac{\pi^2}{8};$$

$$\sum_{n=0}^{\infty}-\frac{1}{(2n+1)^4\cosh\left(\pi(2n+1/2)\right)}-\sum_{n=0}^{\infty}\frac{2(-1)^n}{(2n+1)^4\left(e^{\pi(2n+1)}+1\right)}+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^4}=\frac{1}{192}\pi^4;$$

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^6\cosh(\pi(2n+1/2))}-\sum_{n=0}^{\infty}\frac{2(-1)^n}{(2n+1)^6\left(e^{\pi(2n+1)}+1\right)}+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^6}=\frac{7\pi^6}{5120};$$

$$\sum_{n=0}^{\infty}-\frac{1}{(2n+1)^8\cosh(\pi(2n+1/2))}-\sum_{n=0}^{\infty}\frac{2(-1)^n}{(2n+1)^8\left(e^{\pi(2n+1)}+1\right)}+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^8}=\frac{47}{860160}\pi^8。$$

命题 3 无穷级数恒等式

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2\cosh((2n+1)\pi)}-\sum_{n=0}^{\infty}\frac{4(-1)^n}{(2n+1)^2\left(e^{(2n+1)\pi/2}+1\right)}+\sum_{n=0}^{\infty}\frac{2(-1)^n}{(2n+1)^2}=\frac{\pi^2}{8} \tag{1}$$

$$\sum_{n=0}^{\infty}-\frac{1}{(2n+1)^4\cosh((2n+1)\pi)}-\sum_{n=0}^{\infty}\frac{16(-1)^n}{(2n+1)^4\left(e^{(2n+1)\pi/2}+1\right)}+\sum_{n=0}^{\infty}\frac{8(-1)^n}{(2n+1)^4}=\frac{5}{96}\pi^4 \tag{2}$$

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^6\cosh((2n+1)\pi)}-\sum_{n=0}^{\infty}\frac{64(-1)^n}{(2n+1)^6\left(e^{(2n+1)\pi/2}+1\right)}+\sum_{n=0}^{\infty}\frac{32(-1)^n}{(2n+1)^6}=\frac{7\pi^6}{320} \tag{3}$$

$$\sum_{n=0}^{\infty}-\frac{1}{(2n+1)^8\cosh((2n+1)\pi)}-\sum_{n=0}^{\infty}\frac{256(-1)^n}{(2n+1)^8\left(e^{(2n+1)\pi/2}+1\right)}+\sum_{n=0}^{\infty}\frac{128(-1)^n}{(2n+1)^8}=\frac{95}{10752}\pi^8 \tag{4}$$

证明 选择变换核函数 $f(z)=\dfrac{\pi}{z^{2k+2}\left(e^{\pi z}+1\right)\cos(\pi z)}$ 有单极点 $z=\pm(2n+1)i$， $z=\pm(2n+1)/2$。

1) 展开成幂级数

$$\begin{aligned}
f(z)&=\frac{\pi\sec(\pi z)}{z^{2k+2}\left(e^{\pi z}+1\right)}=\frac{\pi\left[1+(\pi z)^2/2+5(\pi z)^4/24+\cdots\right]}{z^{2k+2}\left[2+(\pi z)+(\pi z)^2/2+(\pi z)^3/6+\cdots\right]}\\
&=\frac{\pi\left[1+(\pi z)^2/2+5(\pi z)^4/24+\cdots\right]}{2z^{2k+2}\left[1+\pi z/2+(\pi z)^2/4+(\pi z)^3/12+\cdots\right]}\\
&=\frac{\pi}{2z^{2k+2}}\left(1+\frac{1}{2}(\pi z)^2+\frac{5}{24}(\pi z)^4+\frac{61}{720}(\pi z)^6+\cdots\right)\\
&\quad\cdot\left(1-\frac{1}{2}\pi z+\frac{1}{24}(\pi z)^3-\frac{1}{240}(\pi z)^5+\frac{17}{40320}(\pi z)^7+\cdots\right);
\end{aligned}$$

$f(z)$ 的留数 $\mathrm{Res}[f;0]=q(2k+2)$ 。其中 $q(2)=-\dfrac{\pi^2}{4}$ ； $q(4)=-\dfrac{5}{48}\pi^4$ ； $q(6)=-\dfrac{7\pi^6}{160}$ ； $q(8)=-\dfrac{95}{5376}\pi^8$；

2) 在单极点 $z=\pm(2n+1)i$，计算 $f(z)$ 的留数，用洛必达法则可得

$$\begin{aligned}\mathrm{Res}\left[f;(2n+1)i\right]&=\lim_{z\to(2n+1)i}\frac{\left[z-(2n+1)i\right]}{\left(e^{\pi z}+1\right)}\frac{\pi}{z^{2k+2}\cos(\pi z)}\\&=\lim_{z\to(2n+1)i}\frac{1}{\pi e^{(2n+1)\pi i}}\frac{\pi}{\left[(2n+1)i\right]^{2k+2}\cos\left(\pi(2n+1)i\right)}\\&=-\frac{1}{(2n+1)^{2k+2}i^{2k+2}\cosh\left((2n+1)\pi\right)}\\&=\frac{1}{(2n+1)^{2k+2}i^{2k}\cosh\left((2n+1)\pi\right)}=\mathrm{Res}\left[f;-(2n+1)i\right];\end{aligned}$$

3) 在单极点 $z=\pm(2n+1)/2$，计算 $f(z)$ 的留数

$$\begin{aligned}\mathrm{Res}\left[f;(2n+1)/2\right]&=\lim_{z\to(2n+1)/2}\frac{1}{z^{2k+2}\left(e^{\pi z}+1\right)}\frac{\left(z-(2n+1)/2\right)\pi}{\cos(\pi z)}\\&=\frac{1}{\left[(2n+1)/2\right]^{2k+2}\left(e^{(2n+1)\pi/2}+1\right)}\frac{\pi}{\left[-\pi\sin\left(\pi(2n+1)/2\right)\right]}\\&=-\frac{2^{2k+2}(-1)^n}{(2n+1)^{2k+2}\left(e^{(2n+1)\pi/2}+1\right)};\end{aligned}$$

$$\begin{aligned}\mathrm{Res}\left[f;-(2n+1)/2\right]&=\lim_{z\to-(2n+1)/2}\frac{1}{z^{2k+2}\left(e^{\pi z}+1\right)}\frac{(z+(2n+1)/2)\pi}{\cos(\pi z)}\\&=\frac{1}{\left[-(2n+1)/2\right]^{2k+2}\left(e^{-(2n+1)\pi/2}+1\right)}\frac{\pi}{\left[-\pi\sin\left(-\pi(2n+1)/2\right)\right]}\\&=-\frac{2^{2k+2}(-1)^n}{(2n+1)^{2k+2}\left(e^{-(2n+1)\pi/2}+1\right)}\\&=-\frac{2^{2k+2}(-1)^n}{(2n+1)^{2k+2}\left(e^{(2n+1)\pi/2}+1\right)}+\frac{2^{2k+2}(-1)^n}{(2n+1)^{2k+2}};\end{aligned}$$

根据高斯留数基本定理，从而有

$$0=q(2k+2)+\sum_{n=-\infty}^{\infty}\frac{1}{(2n+1)^{2k+2}i^{2k}\cosh((2n+1)\pi)}$$
$$+\sum_{n=-\infty}^{\infty}-\frac{2^{2k+2}(-1)^n}{(2n+1)^{2k+2}\left(e^{(2n+1)\pi/2}+1\right)}+\sum_{n=0}^{\infty}\frac{2^{2k+2}(-1)^n}{(2n+1)^{2k+2}};$$

$$\sum_{n=0}^{\infty}\frac{2}{(2n+1)^{2k+2}i^{2k}\cosh((2n+1)\pi)}-\sum_{n=0}^{\infty}\frac{2^{2k+3}(-1)^n}{(2n+1)^{2k+2}\left(e^{(2n+1)\pi/2}+1\right)}$$
$$+\sum_{n=0}^{\infty}\frac{2^{2k+2}(-1)^n}{(2n+1)^{2k+2}}=-q(2k+2);$$

令 $k=0,1,2,3$ 整理得到如下级数恒等式

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2\cosh((2n+1)\pi)}-\sum_{n=0}^{\infty}\frac{4(-1)^n}{(2n+1)^2\left(e^{(2n+1)\pi/2}+1\right)}+\sum_{n=0}^{\infty}\frac{2(-1)^n}{(2n+1)^2}=\frac{\pi^2}{8};$$

$$\sum_{n=0}^{\infty}-\frac{1}{(2n+1)^4\cosh((2n+1)\pi)}-\sum_{n=0}^{\infty}\frac{16(-1)^n}{(2n+1)^4\left(e^{(2n+1)\pi/2}+1\right)}+\sum_{n=0}^{\infty}\frac{8(-1)^n}{(2n+1)^4}=\frac{5}{96}\pi^4;$$

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^6\cosh((2n+1)\pi)}-\sum_{n=0}^{\infty}\frac{64(-1)^n}{(2n+1)^6\left(e^{(2n+1)\pi/2}+1\right)}+\sum_{n=0}^{\infty}\frac{32(-1)^n}{(2n+1)^6}=\frac{7\pi^6}{320};$$

$$\sum_{n=0}^{\infty}-\frac{1}{(2n+1)^8\cosh((2n+1)\pi)}-\sum_{n=0}^{\infty}\frac{256(-1)^n}{(2n+1)^8\left(e^{(2n+1)\pi/2}+1\right)}+\sum_{n=0}^{\infty}\frac{128(-1)^n}{(2n+1)^8}=\frac{95}{10752}\pi^8。$$

命题 4 无穷级数恒等式

$$\sum_{n=1}^{\infty}\frac{1}{(2n)^2\cosh(2n\pi)}+\sum_{n=0}^{\infty}\frac{4(-1)^n}{(2n+1)^2\left(e^{(2n+1)\pi/2}-1\right)}+\sum_{n=0}^{\infty}\frac{2(-1)^n}{(2n+1)^2}=\frac{7\pi^2}{24}\quad(1)$$

$$\sum_{n=1}^{\infty}-\frac{1}{(2n)^4\cosh(2n\pi)}+\sum_{n=0}^{\infty}\frac{16(-1)^n}{(2n+1)^4\left(e^{(2n+1)\pi/2}-1\right)}+\sum_{n=0}^{\infty}\frac{8(-1)^n}{(2n+1)^4}=\frac{179}{1440}\pi^4\quad(2)$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n)^6\cosh(2n\pi)}+\sum_{n=0}^{\infty}\frac{64(-1)^n}{(2n+1)^6\left(e^{(2n+1)\pi/2}-1\right)}+\sum_{n=0}^{\infty}\frac{32(-1)^n}{(2n+1)^6}=\frac{3067\pi^6}{60480}\quad(3)$$

证明 变换核函数$f(z)=\dfrac{\pi}{z^{2k+2}\left(e^{\pi z}-1\right)\cos(\pi z)}$有单极点$z=\pm 2ni$，$z=\pm(2n+1)/2$。

1) 函数展开成幂级数

$$f(z)=\frac{\pi\sec(\pi z)}{z^{2k+2}(e^{\pi z}-1)}=\frac{\pi\left[1+(\pi z)^2/2+5(\pi z)^4/24+\cdots\right]}{z^{2k+2}\left[(\pi z)+(\pi z)^2/2+(\pi z)^3/6+\cdots\right]}$$

$$=\frac{\left[1+(\pi z)^2/2+5(\pi z)^4/24+\cdots\right]}{z^{2k+3}\left[1+\pi z/2+(\pi z)^2/6+(\pi z)^3/24+\cdots\right]}$$

$$=\frac{1}{z^{2k+3}}\left(1+\frac{1}{2}(\pi z)^2+\frac{5}{24}(\pi z)^4+\frac{61}{720}(\pi z)^6+\cdots\right)$$

$$\cdot\left[1-\frac{1}{2}\pi z+\frac{1}{12}(\pi z)^2-\frac{1}{720}(\pi z)^4+\frac{1}{30240}(\pi z)^6+\cdots\right];$$

$f(z)$的留数$\mathrm{Res}[f;0]=q(2k+3)$；其中$q(3)=\dfrac{7\pi^2}{12}$；$q(5)=\dfrac{179}{720}\pi^4$；$q(7)=\dfrac{3067\pi^6}{30240}$；

2) 在单极点$z=\pm 2ni$，计算$f(z)$的留数，用洛必达法则可得

$$\mathrm{Res}[f;2ni]=\lim_{z\to 2ni}\frac{[z-2ni]}{\left(e^{\pi z}-1\right)}\frac{\pi}{z^{2k+2}\cos(\pi z)}$$

$$=\lim_{z\to 2ni}\frac{1}{\pi e^{2n\pi i}}\frac{\pi}{[2ni]^{2k+2}\cos(\pi 2ni)}$$

$$=\frac{1}{(2n)^{2k+2}i^{2k+2}\cosh(2n\pi)}=-\frac{1}{(2n)^{2k+2}i^{2k}\cosh(2n\pi)}=\mathrm{Res}[f;-2ni];$$

3) 在单极点$z=\pm(2n+1)/2$，计算$f(z)$的留数

$$\mathrm{Res}[f;(2n+1)/2]=\lim_{z\to(2n+1)/2}\frac{\pi}{z^{2k+2}\left(e^{\pi z}-1\right)}\frac{\left(z-(2n+1)/2\right)}{\cos(\pi z)}$$

$$=\frac{\pi}{\left[(2n+1)/2\right]^{2k+2}\left(e^{(2n+1)\pi/2}-1\right)}\frac{1}{\left[-\pi\sin\left(\pi(2n+1)/2\right)\right]}$$

$$=-\frac{2^{2k+2}(-1)^n}{(2n+1)^{2k+2}\left(e^{(2n+1)\pi/2}-1\right)};$$

$$\operatorname{Res}\left[f;-(2n+1)/2\right]=\lim_{z\to-(2n+1)/2}\frac{\pi}{z^{2k+2}\left(e^{\pi z}-1\right)}\frac{\left(z+(2n+1)/2\right)}{\cos(\pi z)}$$
$$=\frac{\pi}{\left[-(2n+1)/2\right]^{2k+2}\left(e^{-(2n+1)\pi/2}-1\right)}\frac{1}{\left[-\pi\sin\left(-\pi(2n+1)/2\right)\right]}$$
$$=\frac{2^{2k+2}(-1)^n}{(2n+1)^{2k+2}\left(e^{-(2n+1)\pi/2}-1\right)}$$
$$=-\frac{2^{2k+2}(-1)^n}{(2n+1)^{2k+2}}-\frac{2^{2k+2}(-1)^n}{(2n+1)^{2k+2}\left(e^{(2n+1)\pi/2}-1\right)};$$

根据高斯留数基本定理，从而有

$$0=q(2k+3)+\sum_{n=-\infty}^{\infty}-\frac{1}{(2n)^{2k+2}i^{2k}\cosh(2n\pi)}+\sum_{k=-\infty}^{\infty}-\frac{2^{2k+2}(-1)^n}{(2n+1)^{2k+2}\left(e^{(2n+1)\pi/2}-1\right)}+\sum_{n=0}^{\infty}-\frac{2^{2k+2}(-1)^n}{(2n+1)^{2k+2}};$$

$$\sum_{n=1}^{\infty}\frac{2}{(2n)^{2k+2}i^{2k}\cosh(2n\pi)}+\sum_{n=0}^{\infty}\frac{2^{2k+3}(-1)^n}{(2n+1)^{2k+2}\left(e^{(2n+1)\pi/2}-1\right)}+\sum_{n=0}^{\infty}\frac{2^{2k+2}(-1)^n}{(2n+1)^{2k+2}}=q(2k+3)\text{。}$$

令 $k=0,1,2$ 整理得到如下级数恒等式

$$\sum_{n=1}^{\infty}\frac{1}{(2n)^2\cosh(2n\pi)}+\sum_{n=0}^{\infty}\frac{4(-1)^n}{(2n+1)^2\left(e^{(2n+1)\pi/2}-1\right)}+\sum_{n=0}^{\infty}\frac{2(-1)^n}{(2n+1)^2}=\frac{7\pi^2}{24};$$

$$\sum_{n=1}^{\infty}-\frac{1}{(2n)^4\cosh(2n\pi)}+\sum_{n=0}^{\infty}\frac{16(-1)^n}{(2n+1)^4\left(e^{(2n+1)\pi/2}-1\right)}+\sum_{n=0}^{\infty}\frac{8(-1)^n}{(2n+1)^4}=\frac{179}{1440}\pi^4;$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n)^6\cosh(2n\pi)}+\sum_{n=0}^{\infty}\frac{64(-1)^n}{(2n+1)^6\left(e^{(2n+1)\pi/2}-1\right)}+\sum_{n=0}^{\infty}\frac{32(-1)^n}{(2n+1)^6}=\frac{3067\pi^6}{60480}\text{。}$$

命题 5 无穷级数恒等式

$$\sum_{n=1}^{\infty}-\frac{\tanh(n\pi)}{n}+\sum_{n=0}^{\infty}\frac{4}{(2n+1)\left(e^{(2n+1)\pi}-1\right)}+\sum_{n=0}^{\infty}\frac{2}{(2n+1)}=\frac{1}{2}\pi \tag{1}$$

$$\sum_{n=1}^{\infty}\frac{\tanh(n\pi)}{n^3}+\sum_{n=0}^{\infty}\frac{16}{(2n+1)^3\left(e^{(2n+1)\pi}-1\right)}+\sum_{n=0}^{\infty}\frac{8}{(2n+1)^3}=\frac{\pi^3}{3} \tag{2}$$

$$\sum_{n=1}^{\infty}-\frac{\tanh(n\pi)}{n^5}+\sum_{n=0}^{\infty}\frac{64}{(2n+1)^5\left(e^{(2n+1)\pi}-1\right)}+\sum_{n=0}^{\infty}\frac{32}{(2n+1)^5}=\frac{\pi^5}{9} \tag{3}$$

$$\sum_{n=1}^{\infty}\frac{\tanh(n\pi)}{n^7}+\sum_{n=0}^{\infty}\frac{256}{(2n+1)^7\left(e^{(2n+1)\pi}-1\right)}+\sum_{n=0}^{\infty}\frac{128}{(2n+1)^7}=\frac{44\pi^7}{945} \tag{4}$$

证明 变换核函数 $f(z)=\dfrac{\pi\sin(\pi z)}{z^{2k+1}\left(e^{2\pi z}-1\right)\cos(\pi z)}$ 有单极点 $z=\pm ni$， $z=\pm(2n+1)/2$ 。

1) 函数展开成幂级数

$$\begin{aligned}
f(z)&=\frac{\pi\tan(\pi z)}{z^{2k+1}\left(e^{2\pi z}-1\right)}=\frac{\pi\left[\pi z+(\pi z)^3/3+2(\pi z)^5/15+\cdots\right]}{z^{2k+1}\left[(2\pi z)+(2\pi z)^2/2+(2\pi z)^3/6+\cdots\right]}\\
&=\frac{\pi^2\left[1+(\pi z)^2/3+2(\pi z)^4/15+\cdots\right]}{2\pi z^{2k+1}\left[1+\pi z+2(\pi z)^2/3+(\pi z)^3/3+\cdots\right]}\\
&=\frac{\pi}{2z^{2k+1}}\left(1+\frac{1}{3}(\pi z)^2+\frac{2}{15}(\pi z)^4+\frac{17}{315}(\pi z)^6+\cdots\right)\\
&\quad\cdot\left(1-\pi z+\frac{1}{3}(\pi z)^2-\frac{1}{45}(\pi z)^4+\frac{2}{945}(\pi z)^6+\cdots\right);
\end{aligned}$$

$f(z)$的留数 $\mathrm{Res}[f;0]=p(2k+1)$；其中 $p(1)=\dfrac{1}{2}\pi$； $p(3)=\dfrac{\pi^3}{3}$； $p(5)=\dfrac{\pi^5}{9}$； $p(7)=\dfrac{44\pi^7}{945}$；

2) 在单极点 $z=\pm ni$，计算 $f(z)$的留数，用洛必达法则可得

$$\begin{aligned}
\mathrm{Res}[f;ni]&=\lim_{z\to ni}\frac{[z-ni]}{\left(e^{2\pi z}-1\right)}\frac{\pi\tan(\pi z)}{z^{2k+1}}=\lim_{z\to ni}\frac{1}{2\pi e^{2n\pi i}}\frac{\pi\tan(\pi ni)}{(ni)^{2k+1}}\\
&=\frac{\tanh(n\pi)}{2n^{2k+1}i^{2k}}=\mathrm{Res}[f;-ni];
\end{aligned}$$

3) 在单极点 $z=\pm(2n+1)/2$，计算 $f(z)$的留数

$$\mathrm{Res}\left[f;(2n+1)/2\right]=\lim_{z\to(2n+1)/2}\frac{\pi}{z^{2k+1}\left(e^{2\pi z}-1\right)}\frac{\left(z-(2n+1)/2\right)\sin(\pi z)}{\cos(\pi z)}$$

$$=\frac{\pi}{\left[(2n+1)/2\right]^{2k+1}\left(e^{(2n+1)\pi}-1\right)}\frac{\sin\left(\pi(2n+1)/2\right)}{\left[-\pi\sin\left(\pi(2n+1)/2\right)\right]}$$

$$=-\frac{2^{2k+1}}{(2n+1)^{2k+1}\left(e^{(2n+1)\pi}-1\right)};$$

$$\mathrm{Res}\left[f;-(2n+1)/2\right]=\lim_{z\to-(2n+1)/2}\frac{\pi}{z^{2k+1}\left(e^{2\pi z}-1\right)}\frac{\left(z+(2n+1)/2\right)\sin(\pi z)}{\cos(\pi z)}$$

$$=\frac{\pi}{\left[-(2n+1)/2\right]^{2k+1}\left(e^{-(2n+1)\pi}-1\right)}\frac{\sin\left(-\pi(2n+1)/2\right)}{\left[-\pi\sin\left(-\pi(2n+1)/2\right)\right]}$$

$$=\frac{2^{2k+1}}{(2n+1)^{2k+1}\left(e^{-(2n+1)\pi/2}-1\right)}$$

$$=-\frac{2^{2k+1}}{(2n+1)^{2k+1}\left(e^{(2n+1)\pi}-1\right)}-\frac{2^{2k+1}}{(2n+1)^{2k+1}};$$

根据高斯留数基本定理，从而有

$$0=p(2k+1)+\sum_{n=-\infty}^{\infty}\frac{\tanh(n\pi)}{2n^{2k+1}i^{2k}}+\sum_{n=-\infty}^{\infty}-\frac{2^{2k+1}}{(2n+1)^{2k+1}\left(e^{(2n+1)\pi}-1\right)}$$

$$+\sum_{n=0}^{\infty}-\frac{2^{2k+1}}{(2n+1)^{2k+1}};$$

$$\sum_{n=1}^{\infty}-\frac{\tanh(n\pi)}{n^{2k+1}i^{2k}}+\sum_{n=0}^{\infty}\frac{2^{2k+2}}{(2n+1)^{2k+1}\left(e^{(2n+1)\pi}-1\right)}+\sum_{n=0}^{\infty}\frac{2^{2k+1}}{(2n+1)^{2k+1}}=p(2k+1);$$

令 $k=0,1,2,3$ 整理得到如下级数恒等式

$$\sum_{n=1}^{\infty}-\frac{\tanh(n\pi)}{n}+\sum_{n=0}^{\infty}\frac{4}{(2n+1)\left(e^{(2n+1)\pi}-1\right)}+\sum_{n=0}^{\infty}\frac{2}{(2n+1)}=\frac{1}{2}\pi;$$

$$\sum_{n=1}^{\infty}\frac{\tanh(n\pi)}{n^3}+\sum_{n=0}^{\infty}\frac{16}{(2n+1)^3\left(e^{(2n+1)\pi}-1\right)}+\sum_{n=0}^{\infty}\frac{8}{(2n+1)^3}=\frac{\pi^3}{3};$$

$$\sum_{n=1}^{\infty}-\frac{\tanh(n\pi)}{n^5}+\sum_{n=0}^{\infty}\frac{64}{(2n+1)^5\left(e^{(2n+1)\pi}-1\right)}+\sum_{n=0}^{\infty}\frac{32}{(2n+1)^5}=\frac{\pi^5}{9};$$

$$\sum_{n=1}^{\infty}\frac{\tanh(n\pi)}{n^7}+\sum_{n=0}^{\infty}\frac{256}{(2n+1)^7\left(e^{(2n+1)\pi}-1\right)}+\sum_{n=0}^{\infty}\frac{128}{(2n+1)^7}=\frac{44\pi^7}{945}。$$

命题 6 无穷级数恒等式

$$\sum_{n=0}^{\infty}-\frac{\tanh(\pi(2n+1)/2)}{(2n+1)^3}+\sum_{n=0}^{\infty}\frac{2}{(2n+1)^3\left(e^{(2n+1)\pi}+1\right)}-\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}=-\frac{1}{16}\pi^3 \tag{1}$$

$$\sum_{n=0}^{\infty}\frac{\tanh(\pi(2n+1)/2)}{(2n+1)^5}+\sum_{n=0}^{\infty}\frac{2}{(2n+1)^5\left(e^{(2n+1)\pi}+1\right)}-\sum_{n=0}^{\infty}\frac{1}{(2n+1)^5}=0 \tag{2}$$

$$\sum_{n=0}^{\infty}-\frac{\tanh(\pi(2n+1)/2)}{(2n+1)^7}+\sum_{n=0}^{\infty}\frac{2}{(2n+1)^7\left(e^{(2n+1)\pi}+1\right)}-\sum_{n=0}^{\infty}\frac{1}{(2n+1)^7}=-\frac{7\pi^7}{11520} \tag{3}$$

$$\sum_{n=0}^{\infty}\frac{\tanh(\pi(2n+1)/2)}{(2n+1)^9}+\sum_{n=0}^{\infty}\frac{2}{(2n+1)^9\left(e^{(2n+1)\pi}+1\right)}-\sum_{n=0}^{\infty}\frac{1}{(2n+1)^9}=0 \tag{4}$$

证明 变换核函数 $f(z)=\dfrac{\pi\sin(\pi z)}{z^{2k+1}\left(e^{2\pi z}+1\right)\cos(\pi z)}$ 有单极点 $z=\pm(2n+1)i/2$，$z=\pm(2n+1)/2$。

1) 函数展开成幂级数

$$\begin{aligned}
f(z)&=\frac{\pi\tan(\pi z)}{z^{2k+1}\left(e^{2\pi z}+1\right)}=\frac{\pi\left[(\pi z)+(\pi z)^3/3+2(\pi z)^5/15+\cdots\right]}{z^{2k+1}\left[2+(2\pi z)+(2\pi z)^2/2+(2\pi z)^3/6+\cdots\right]}\\
&=\frac{\pi^2\left[1+(\pi z)^3/3+2(\pi z)^4/15+\cdots\right]}{2z^{2k}\left[1+\pi z+(\pi z)^2+(\pi z)^3/3+\cdots\right]}\\
&=\frac{\pi^2}{2z^{2k}}\left(1+\frac{1}{3}(\pi z)^2+\frac{2}{15}(\pi z)^4+\frac{17}{315}(\pi z)^6+\cdots\right)\\
&\cdot\left(1-\pi z+\frac{1}{3}(\pi z)^3-\frac{2}{15}(\pi z)^5+0+\frac{17}{315}(\pi z)^7+\cdots\right);
\end{aligned}$$

$f(z)$ 的留数 $\mathrm{Res}[f;0]=p(2k)$。其中 $p(2)=-\dfrac{1}{2}\pi^3$；$p(4)=0$；$p(6)=-\dfrac{7\pi^7}{90}$；$p(8)=0$；

2) 在单极点 $z=\pm(2n+1)i/2$，计算 $f(z)$ 的留数，用洛必达法则

$$\operatorname{Res}\left[f;i(2n+1)/2\right]=\lim_{z\to(2n+1)i/2}\frac{\left[z-(2n+1)i/2\right]}{\left(e^{2\pi z}+1\right)}\frac{\pi\tan(\pi z)}{z^{2k+1}}=\frac{1}{2\pi e^{\pi(2n+1)i}}\frac{\pi\tan\left(\pi(2n+1)i/2\right)}{\left[(2n+1)i/2\right]^{2k+1}}$$

$$=\frac{2^{2k}i\tanh\left(\pi(2n+1)/2\right)}{-(2n+1)^{2k+1}i^{2k+1}}=-\frac{2^{2k}\tanh\left(\pi(2n+1)/2\right)}{(2n+1)^{2k+1}i^{2k}}=\operatorname{Res}\left[f;-i(2n+1)/2\right];$$

3) 在单极点 $z=\pm(2n+1)/2$，计算 $f(z)$ 的留数

$$\operatorname{Res}\left[f;(2n+1)/2\right]=\lim_{z\to(2n+1)/2}\frac{\pi}{z^{2k+1}\left(e^{2\pi z}+1\right)}\frac{\left(z-(2n+1)/2\right)\sin(\pi z)}{\cos(\pi z)}$$

$$=\frac{\pi}{\left[(2n+1)/2\right]^{2k+1}\left(e^{(2n+1)\pi}+1\right)}\frac{\sin\left(\pi(2n+1)/2\right)}{\left[-\pi\sin\left(\pi(2n+1)/2\right)\right]}=-\frac{2^{2k+1}}{(2n+1)^{2k+1}\left(e^{(2n+1)\pi}+1\right)};$$

$$\operatorname{Res}\left[f;-(2n+1)/2\right]=\lim_{z\to-(2n+1)/2}\frac{\pi}{z^{2k+1}\left(e^{2\pi z}+1\right)}\frac{\left(z+(2n+1)/2\right)\sin(\pi z)}{\cos(\pi z)}$$

$$=\frac{\pi}{\left[-(2n+1)/2\right]^{2k+1}\left(e^{-(2n+1)\pi}+1\right)}\frac{\sin\left(-\pi(2n+1)/2\right)}{\left[-\pi\sin\left(-\pi(2n+1)/2\right)\right]}$$

$$=\frac{2^{2k+1}}{(2n+1)^{2k+1}\left(e^{-(2n+1)\pi}+1\right)}=-\frac{2^{2k+1}}{(2n+1)^{2k+1}\left(e^{(2n+1)\pi}+1\right)}+\frac{2^{2k+1}}{(2n+1)^{2k+1}};$$

根据高斯留数基本定理，从而有

$$0=p(2k)+\sum_{n=-\infty}^{\infty}-\frac{2^{2k}\tanh\left(\pi(2n+1)/2\right)}{(2n+1)^{2k+1}i^{2k}}+\sum_{n=-\infty}^{\infty}-\frac{2^{2k+1}}{(2n+1)^{2k+1}\left(e^{(2n+1)\pi}+1\right)}+\sum_{n=0}^{\infty}\frac{2^{2k+1}}{(2n+1)^{2k+1}};$$

$$\sum_{n=0}^{\infty}\frac{2^{2k+1}\tanh\left(\pi(2n+1)/2\right)}{(2n+1)^{2k+1}i^{2k}}+\sum_{n=0}^{\infty}\frac{2^{2k+2}}{(2n+1)^{2k+1}\left(e^{(2n+1)\pi}+1\right)}-\sum_{n=0}^{\infty}\frac{2^{2k+1}}{(2n+1)^{2k+1}}=p(2k);$$

令 $k=1,2,3,4$ 整理得到如下级数恒等式。

$$\sum_{n=0}^{\infty}-\frac{\tanh\left(\pi(2n+1)/2\right)}{(2n+1)^{3}}+\sum_{n=0}^{\infty}\frac{2}{(2n+1)^{3}\left(e^{(2n+1)\pi}+1\right)}-\sum_{n=0}^{\infty}\frac{1}{(2n+1)^{3}}=-\frac{1}{16}\pi^{3};$$

$$\sum_{n=0}^{\infty}\frac{\tanh\left(\pi(2n+1)/2\right)}{(2n+1)^5}+\sum_{n=0}^{\infty}\frac{2}{(2n+1)^5\left(e^{(2n+1)\pi}+1\right)}-\sum_{n=0}^{\infty}\frac{1}{(2n+1)^5}=0\text{；}$$

$$\sum_{n=0}^{\infty}-\frac{\tanh\left(\pi(2n+1)/2\right)}{(2n+1)^7}+\sum_{n=0}^{\infty}\frac{2}{(2n+1)^7\left(e^{(2n+1)\pi}+1\right)}-\sum_{n=0}^{\infty}\frac{1}{(2n+1)^7}=-\frac{7\pi^7}{11520}\text{；}$$

$$\sum_{n=0}^{\infty}\frac{\tanh\left(\pi(2n+1)/2\right)}{(2n+1)^9}+\sum_{n=0}^{\infty}\frac{2}{(2n+1)^9\left(e^{(2n+1)\pi}+1\right)}-\sum_{n=0}^{\infty}\frac{1}{(2n+1)^9}=0\text{。}$$

命题 7 无穷级数恒等式

$$\sum_{n=0}^{\infty}-\frac{\tanh\left(\pi(2n+1)\right)}{(2n+1)^3}+\sum_{n=0}^{\infty}\frac{8}{(2n+1)^3\left(e^{(2n+1)\pi/2}+1\right)}-\sum_{n=0}^{\infty}\frac{4}{(2n+1)^3}=-\frac{1}{8}\pi^3 \tag{1}$$

$$\sum_{n=0}^{\infty}\frac{\tanh\left(\pi(2n+1)\right)}{(2n+1)^5}+\sum_{n=0}^{\infty}\frac{32}{(2n+1)^5\left(e^{(2n+1)\pi/2}+1\right)}-\sum_{n=0}^{\infty}\frac{16}{(2n+1)^5}=-\frac{1}{32}\pi^5 \tag{2}$$

$$\sum_{n=0}^{\infty}-\frac{\tanh\left(\pi(2n+1)\right)}{(2n+1)^7}+\sum_{n=0}^{\infty}\frac{128}{(2n+1)^7\left(e^{(2n+1)\pi/2}+1\right)}-\sum_{n=0}^{\infty}\frac{64}{(2n+1)^7}=-\frac{41\pi^7}{2880} \tag{3}$$

$$\sum_{n=0}^{\infty}\frac{\tanh\left(\pi(2n+1)\right)}{(2n+1)^9}+\sum_{n=0}^{\infty}\frac{512}{(2n+1)^9\left(e^{(2n+1)\pi/2}+1\right)}-\sum_{n=0}^{\infty}\frac{256}{(2n+1)^9}=-\frac{43}{7680}\pi^9 \tag{4}$$

证明 变换核函数 $f(z)=\dfrac{\pi\sin(\pi z)}{z^{2k+1}\left(e^{\pi z}+1\right)\cos(\pi z)}$ 有单极点 $z=\pm(2n+1)i$， $z=\pm(2n+1)/2$。

1) 函数展开成幂级数

$$\begin{aligned}f(z)&=\frac{\pi\tan(\pi z)}{z^{2k+1}\left(e^{\pi z}+1\right)}=\frac{\pi\left[\pi z+(\pi z)^3/3+2(\pi z)^5/15+\cdots\right]}{z^{2k+1}\left[2+(\pi z)+(\pi z)^2/2+(\pi z)^3/6+\cdots\right]}\\&=\frac{\pi^2\left[1+(\pi z)^2/3+2(\pi z)^4/15+\cdots\right]}{2z^{2k}\left[1+\pi z/2+(\pi z)^2/4+(\pi z)^3/12+\cdots\right]}\\&=\frac{\pi^2}{2z^{2k}}\left(1+\frac{1}{3}(\pi z)^2+\frac{2}{15}(\pi z)^4+\frac{17}{315}(\pi z)^6+\cdots\right)\\&\quad\cdot\left(1-\frac{1}{2}\pi z+\frac{1}{24}(\pi z)^3-\frac{1}{240}(\pi z)^5+0+\frac{17}{40320}(\pi z)^7+\cdots\right);\end{aligned}$$

$f(z)$ 的留数 $\mathrm{Res}[f;0]=p(2k)$。其中 $p(2)=-\frac{1}{4}\pi^3$； $p(4)=-\frac{1}{16}\pi^5$； $p(6)=-\frac{41\pi^7}{1440}$；$p(8)=-\frac{43}{3840}\pi^9$；

2) 在单极点 $z=\pm(2n+1)i$，计算 $f(z)$ 的留数，用洛必达法则可得

$$\begin{aligned}\mathrm{Res}\left[f;(2n+1)i\right]&=\lim_{z\to(2n+1)i}\frac{\left[z-(2n+1)i\right]}{\left(e^{\pi z}+1\right)}\frac{\pi\sin(\pi z)}{z^{2k+1}\cos(\pi z)}\\&=\lim_{z\to(2n+1)i/2}\frac{1}{\pi e^{(2n+1)\pi i}}\frac{\pi}{\left[(2n+1)i\right]^{2k+1}}\tan\left(\pi(2n+1)i\right)\\&=-\frac{i\tanh\left(\pi(2n+1)\right)}{(2n+1)^{2k+1}i^{2k+1}}=-\frac{\tanh\left(\pi(2n+1)\right)}{(2n+1)^{2k+1}i^{2k}}\\&=\mathrm{Res}\left[f;-(2n+1)i\right];\end{aligned}$$

3) 在单极点 $z=\pm(2n+1)/2$，计算 $f(z)$ 的留数

$$\begin{aligned}\mathrm{Res}\left[f;(2n+1)/2\right]&=\lim_{z\to(2n+1)/2}\frac{\pi}{z^{2k+1}\left(e^{\pi z}+1\right)}\frac{\left(z-(2n+1)/2\right)\sin(\pi z)}{\cos(\pi z)}\\&=\frac{\pi}{\left[(2n+1)/2\right]^{2k+1}\left(e^{(2n+1)\pi/2}+1\right)}\frac{\sin\left(\pi(2n+1)/2\right)}{\left[-\pi\sin\left(\pi(2n+1)/2\right)\right]}\\&=-\frac{2^{2k+1}}{(2n+1)^{2k+1}\left(e^{(2n+1)\pi/2}+1\right)};\end{aligned}$$

$$\begin{aligned}\mathrm{Res}\left[f;-(2n+1)/2\right]&=\lim_{z\to-(2n+1)/2}\frac{\pi}{z^{2k+1}\left(e^{\pi z}+1\right)}\frac{\left(z+(2n+1)/2\right)\sin(\pi z)}{\cos(\pi z)}\\&=\frac{\pi}{\left[-(2n+1)/2\right]^{2k+1}\left(e^{-(2n+1)\pi/2}+1\right)}\frac{\sin(\pi z)}{\left[-\pi\sin(\pi z)\right]}\\&=\frac{2^{2k+1}}{(2n+1)^{2k+1}\left(e^{-(2n+1)\pi/2}+1\right)}\\&=\frac{2^{2k+1}}{(2n+1)^{2k+1}\left(e^{(2n+1)\pi/2}+1\right)}+\frac{2^{2k+1}}{(2n+1)^{2k+1}};\end{aligned}$$

根据高斯留数基本定理，从而有

$$0=p(2k)+\sum_{n=-\infty}^{\infty}-\frac{\tanh(\pi(2n+1))}{(2n+1)^{2k+1}i^{2k}}+\sum_{n=-\infty}^{\infty}-\frac{2^{2k+1}}{(2n+1)^{2k+1}\left(e^{(2n+1)\pi/2}+1\right)}+\sum_{n=0}^{\infty}\frac{2^{2k+1}}{(2n+1)^{2k+1}};$$

$$\sum_{n=0}^{\infty}\frac{2\tanh(\pi(2n+1))}{(2n+1)^{2k+1}i^{2k}}+\sum_{n=0}^{\infty}\frac{2^{2k+2}}{(2n+1)^{2k+1}\left(e^{(2n+1)\pi/2}+1\right)}-\sum_{n=0}^{\infty}\frac{2^{2k+1}}{(2n+1)^{2k+1}}=p(2k);$$

令 $k=1,2,3,4$ 整理得到如下级数恒等式

$$\sum_{n=0}^{\infty}-\frac{\tanh(\pi(2n+1))}{(2n+1)^{3}}+\sum_{n=0}^{\infty}\frac{8}{(2n+1)^{3}\left(e^{(2n+1)\pi/2}+1\right)}-\sum_{n=0}^{\infty}\frac{4}{(2n+1)^{3}}=-\frac{1}{8}\pi^{3};$$

$$\sum_{n=0}^{\infty}\frac{\tanh(\pi(2n+1))}{(2n+1)^{5}}+\sum_{n=0}^{\infty}\frac{32}{(2n+1)^{5}\left(e^{(2n+1)\pi/2}+1\right)}-\sum_{n=0}^{\infty}\frac{16}{(2n+1)^{5}}=-\frac{1}{32}\pi^{5};$$

$$\sum_{n=0}^{\infty}-\frac{\tanh(\pi(2n+1))}{(2n+1)^{7}}+\sum_{n=0}^{\infty}\frac{128}{(2n+1)^{7}\left(e^{(2n+1)\pi/2}+1\right)}-\sum_{n=0}^{\infty}\frac{64}{(2n+1)^{7}}=-\frac{41\pi^{7}}{2880};$$

$$\sum_{n=0}^{\infty}\frac{\tanh(\pi(2n+1))}{(2n+1)^{9}}+\sum_{n=0}^{\infty}\frac{512}{(2n+1)^{9}\left(e^{(2n+1)\pi/2}+1\right)}-\sum_{n=0}^{\infty}\frac{256}{(2n+1)^{9}}=-\frac{43}{7680}\pi^{9}。$$

命题 8 无穷级数恒等式

$$\sum_{n=0}^{\infty}\frac{4\coth((2n+1)\pi/2)}{(2n+1)}+\sum_{n=1}^{\infty}\frac{2}{n\left(e^{2n\pi}+1\right)}-\sum_{n=1}^{\infty}\frac{1}{n}=\frac{\pi}{2}\tag{1}$$

$$\sum_{n=0}^{\infty}-\frac{16\coth((2n+1)\pi/2)}{(2n+1)^{3}}+\sum_{n=1}^{\infty}\frac{2}{n^{3}\left(e^{2n\pi}+1\right)}-\sum_{n=1}^{\infty}\frac{1}{n^{3}}=-\frac{1}{3}\pi^{3}\tag{2}$$

$$\sum_{n=0}^{\infty}\frac{64\coth((2n+1)\pi/2)}{(2n+1)^{5}}+\sum_{n=1}^{\infty}\frac{2}{n^{5}\left(e^{2n\pi}+1\right)}-\sum_{n=1}^{\infty}\frac{1}{n^{5}}=\frac{\pi^{5}}{9}\tag{3}$$

$$\sum_{n=0}^{\infty}-\frac{256\coth((2n+1)\pi/2)}{(2n+1)^{7}}+\sum_{n=1}^{\infty}\frac{2}{n^{7}\left(e^{2n\pi}+1\right)}-\sum_{n=1}^{\infty}\frac{1}{n^{7}}=-\frac{44}{945}\pi^{7}\tag{4}$$

证明 变换核函数，$f(z)=\dfrac{\pi\cos(\pi z)}{z^{2k+1}\left(e^{2\pi z}+1\right)\sin(\pi z)}$ 有单极点 $z=\pm(2n+1)i/2$， $z=\pm n$ 。

1) 函数展开成幂级数

$$f(z)=\frac{\pi\cot(\pi z)}{z^{2k+1}\left(e^{2\pi z}+1\right)}=\frac{\pi\left[1/(\pi z)-(\pi z)/3-(\pi z)^3/45+\cdots\right]}{z^{2k+1}\left[2+(2\pi z)+(2\pi z)^2/2+(2\pi z)^3/6+\cdots\right]}$$
$$=\frac{\left[1-(\pi z)^2/3-(\pi z)^4/45+\cdots\right]}{2z^{2k+2}\left[1+\pi z+(2\pi z)^2/4+(2\pi z)^3/12+\cdots\right]}$$
$$=\frac{1}{2z^{2k+2}}\left(1-\frac{1}{3}(\pi z)^2-\frac{1}{45}(\pi z)^4-\frac{2}{945}(\pi z)^6+\cdots\right)$$
$$\cdot\left(1-(\pi z)+\frac{1}{3}(\pi z)^3-\frac{2}{15}(\pi z)^5+\frac{17}{315}(\pi z)^7+\cdots\right);$$

$f(z)$的留数$\mathrm{Res}[f;0]=q(2k+2)$；其中$q(2)=-\dfrac{\pi}{2}$； $q(4)=\dfrac{1}{3}\pi^3$； $q(6)=-\dfrac{\pi^5}{9}$； $q(8)=\dfrac{44}{945}\pi^7$；

2) 在单极点$z=\pm(2n+1)i/2$，计算$f(z)$的留数，用洛必达法则可得

$$\mathrm{Res}\left[f;(2n+1)i/2\right]=\lim_{z\to(2n+1)i/2}\frac{\left[z-(2n+1)i/2\right]}{\left(e^{2\pi z}+1\right)}\frac{\pi\cot(\pi z)}{z^{2k+1}}$$
$$=\lim_{z\to(2n+1)i/2}\frac{1}{2\pi e^{(2n+1)\pi i}}\frac{\pi\cot\left((2n+1)\pi i/2\right)}{\left[(2n+1)i/2\right]^{2k+1}}=\frac{-2^{2k+1}i\coth\left((2n+1)\pi/2\right)}{-(2n+1)^{2k+1}i^{2k+1}}$$
$$=\frac{2^{2k+1}\coth\left((2n+1)\pi/2\right)}{(2n+1)^{2k+1}i^{2k}}=\mathrm{Res}\left[f;-(2n+1)i/2\right];$$

3) 在单极点$z=\pm n$，计算$f(z)$的留数

$$\mathrm{Res}[f;n]=\lim_{z\to n}\frac{\pi}{z^{2k+1}\left(e^{2\pi z}+1\right)}\frac{(z-n)\cos(\pi z)}{\sin(\pi z)}=\frac{\pi}{n^{2k+1}\left(e^{2n\pi}+1\right)}\frac{\cos(n\pi)}{\left[\pi\cos(\pi n)\right]}=\frac{1}{n^{2k+1}\left(e^{2n\pi}+1\right)};$$

$$\mathrm{Res}[f;-n]=\lim_{z\to -n}\frac{\pi}{z^{2k+1}\left(e^{2\pi z}+1\right)}\frac{(z+n)\cos(\pi z)}{\sin(\pi z)}$$
$$=\frac{\pi}{-n^{2k+1}\left(e^{-2n\pi}+1\right)}\frac{\cos(-\pi n)}{\left[\pi\cos(-\pi n)\right]}=\frac{e^{2n\pi}}{-n^{2k+1}\left(e^{2n\pi}+1\right)}=\frac{1}{n^{2k+1}\left(e^{2n\pi}+1\right)}-\frac{1}{n^{2k+1}};$$

根据高斯留数基本定理，从而有

$$0=q(2k+2)+\sum_{n=-\infty}^{\infty}\frac{2^{2k+1}\coth((2n+1)\pi/2)}{(2n+1)^{2k+1}i^{2k}}+\sum_{k=-\infty}^{\infty}\frac{1}{n^{2k+1}(e^{2n\pi}+1)}+\sum_{n=1}^{\infty}-\frac{1}{n^{2k+1}};$$

$$\sum_{n=0}^{\infty}\frac{2^{2k+2}\coth((2n+1)\pi/2)}{(2n+1)^{2k+1}i^{2k}}+\sum_{n=1}^{\infty}\frac{2}{n^{2k+1}(e^{2n\pi}+1)}-\sum_{n=1}^{\infty}\frac{1}{n^{2k+1}}=-q(2k+2);$$

令 $k=0,1,2,3$ 整理得到如下级数恒等式

$$\sum_{n=0}^{\infty}\frac{4\coth((2n+1)\pi/2)}{(2n+1)}+\sum_{n=1}^{\infty}\frac{2}{n(e^{2n\pi}+1)}-\sum_{n=1}^{\infty}\frac{1}{n}=\frac{\pi}{2};$$

$$\sum_{n=0}^{\infty}-\frac{16\coth((2n+1)\pi/2)}{(2n+1)^3}+\sum_{n=1}^{\infty}\frac{2}{n^3(e^{2n\pi}+1)}-\sum_{n=1}^{\infty}\frac{1}{n^3}=-\frac{1}{3}\pi^3;$$

$$\sum_{n=0}^{\infty}\frac{64\coth((2n+1)\pi/2)}{(2n+1)^5}+\sum_{n=1}^{\infty}\frac{2}{n^5(e^{2n\pi}+1)}-\sum_{n=1}^{\infty}\frac{1}{n^5}=\frac{\pi^5}{9};$$

$$\sum_{n=0}^{\infty}-\frac{256\coth((2n+1)\pi/2)}{(2n+1)^7}+\sum_{n=1}^{\infty}\frac{2}{n^7(e^{2n\pi}+1)}-\sum_{n=1}^{\infty}\frac{1}{n^7}=-\frac{44}{945}\pi^7。$$

命题 9 无穷级数恒等式

$$\sum_{n=0}^{\infty}\frac{2\coth((2n+1)\pi)}{(2n+1)}+\sum_{n=1}^{\infty}\frac{2}{n(e^{n\pi}+1)}-\sum_{n=1}^{\infty}\frac{1}{n}=\frac{\pi}{4} \tag{1}$$

$$\sum_{n=0}^{\infty}-\frac{2\coth((2n+1)\pi)}{(2n+1)^3}+\sum_{n=1}^{\infty}\frac{2}{n^3(e^{n\pi}+1)}-\sum_{n=1}^{\infty}\frac{1}{n^3}=-\frac{5}{48}\pi^3 \tag{2}$$

$$\sum_{n=0}^{\infty}\frac{2\coth((2n+1)\pi)}{(2n+1)^5}+\sum_{n=1}^{\infty}\frac{2}{n^5(e^{n\pi}+1)}-\sum_{n=1}^{\infty}\frac{1}{n^5}=\frac{\pi^5}{288} \tag{3}$$

$$\sum_{n=0}^{\infty}-\frac{2\coth((2n+1)\pi)}{(2n+1)^7}+\sum_{n=1}^{\infty}\frac{2}{n^7(e^{n\pi}+1)}-\sum_{n=1}^{\infty}\frac{1}{n^7}=-\frac{47}{48384}\pi^7 \tag{4}$$

证明 变换核函数 $f(z)=\dfrac{\pi\cos(\pi z)}{z^{2k+1}(e^{\pi z}+1)\sin(\pi z)}$ 有单极点 $z=\pm(2n+1)i$， $z=\pm n$ 。

1) 展开成幂级数

$$f(z)=\frac{\pi\cot(\pi z)}{z^{2k+1}(e^{\pi z}+1)}=\frac{\pi\left(1/(\pi z)-(\pi z)/3-(\pi z)^3/45-\cdots\right)}{z^{2k+1}\left(2+(\pi z)+(\pi z)^2/2+\cdots\right)}$$

$$=\frac{\pi\left(1-(\pi z)^2/3-(\pi z)^4/45-\cdots\right)}{z^{2k+2}\left(2+(\pi z)+(\pi z)^2/2+(\pi z)^3/6+(\pi z)^3/24+\cdots\right)}$$

$$=\frac{1}{2z^{2k+2}}\left(1-\frac{1}{3}(\pi z)^2-\frac{1}{45}(\pi z)^4-\frac{2}{945}(\pi z)^6+\cdots\right)$$

$$\cdot\left(1-\frac{1}{2}(\pi z)+\frac{1}{24}(\pi z)^3-\frac{1}{240}(\pi z)^5+\frac{17}{40320}(\pi z)^7+\cdots\right);$$

$f(z)$ 留数 $\mathrm{Res}[f;0]=q(2k+2)$；其中 $q(2)=-\dfrac{\pi}{4}$；$q(4)=\dfrac{5}{48}\pi^3$；$q(6)=-\dfrac{\pi^5}{288}$； $q(8)=\dfrac{47}{48384}\pi^7$；

2) 在单极点 $z=\pm(2n+1)i$，计算 $f(z)$ 的留数

$$\mathrm{Res}[f;(2n+1)i]=\lim_{z\to(2n+1)i}\frac{z-(2n+1)i}{(e^{\pi z}+1)}\frac{\pi\cot(\pi z)}{z^{2k+1}}=\frac{1}{\pi e^{\pi(2n+1)i}}\frac{\pi\cot((2n+1)\pi i)}{[(2n+1)i]^{2k+1}}$$

$$=\frac{-i\coth((2n+1)\pi)}{-i^{2k+1}(2n+1)^{2k+1}}=\frac{\coth((2n+1)\pi)}{(2n+1)^{2k+1}i^{2k}}=\mathrm{Res}[f;-(2n+1)i];$$

3) 在单极点 $z=\pm n$， 计算 $f(z)$ 的留数

$$\mathrm{Res}[f;n]=\lim_{z\to n}\frac{\pi}{z^{2k+1}(e^{\pi z}+1)}\frac{(z-n)\cos(\pi z)}{\sin(\pi z)}=\lim_{z\to n}\frac{\pi}{z^{2k+1}(e^{\pi z}+1)}\frac{\cos(\pi z)}{[\pi\cos(\pi z)]}=\frac{1}{n^{2k+1}(e^{n\pi}+1)};$$

$$\mathrm{Res}[f;-n]=\lim_{z\to -n}\frac{\pi}{z^{2k+1}(e^{\pi z}+1)}\frac{(z+n)\cos(\pi z)}{\sin(\pi z)}$$

$$=\frac{\pi}{-n^{2k+1}(e^{-n\pi}+1)}\frac{\cos(\pi z)}{[\pi\cos(\pi z)]}=\frac{e^{n\pi}}{-n^{2k+1}(e^{n\pi}+1)}=-\frac{1}{n^{2k+1}}+\frac{1}{n^{2k+1}(e^{n\pi}+1)};$$

根据高斯留数基本定理，从而有

$$0=q(2k+2)+\sum_{n=-\infty}^{\infty}\frac{\coth((2n+1)\pi)}{(2n+1)^{2k+1}i^{2k}}+\sum_{n=-\infty}^{\infty}\frac{1}{n^{2k+1}(e^{n\pi}+1)}+\sum_{n=1}^{\infty}-\frac{1}{n^{2k+1}};$$

$$\sum_{n=0}^{\infty}\frac{2\coth((2n+1)\pi)}{(2n+1)^{2k+1}i^{2k}}+\sum_{n=1}^{\infty}\frac{2}{n^{2k+1}(e^{n\pi}+1)}-\sum_{n=1}^{\infty}\frac{1}{n^{2k+1}}=-q(2k+2);$$

令 $k=0,1,2,3$ 整理得到如下级数恒等式

$$\sum_{n=0}^{\infty}\frac{2\coth((2n+1)\pi)}{(2n+1)}+\sum_{n=1}^{\infty}\frac{2}{n(e^{n\pi}+1)}-\sum_{n=1}^{\infty}\frac{1}{n}=\frac{\pi}{4};$$

$$\sum_{n=0}^{\infty}-\frac{2\coth((2n+1)\pi)}{(2n+1)^3}+\sum_{n=1}^{\infty}\frac{2}{n^3(e^{n\pi}+1)}-\sum_{n=1}^{\infty}\frac{1}{n^3}=-\frac{5}{48}\pi^3;$$

$$\sum_{n=0}^{\infty}\frac{2\coth((2n+1)\pi)}{(2n+1)^5}+\sum_{n=1}^{\infty}\frac{2}{n^5(e^{n\pi}+1)}-\sum_{n=1}^{\infty}\frac{1}{n^5}=\frac{\pi^5}{288};$$

$$\sum_{n=0}^{\infty}-\frac{2\coth((2n+1)\pi)}{(2n+1)^7}+\sum_{n=1}^{\infty}\frac{2}{n^7(e^{n\pi}+1)}-\sum_{n=1}^{\infty}\frac{1}{n^7}=-\frac{47}{48384}\pi^7。$$

命题 10 无穷级数恒等式

$$\sum_{n=1}^{\infty}-\frac{\coth(n\pi)}{n}+\sum_{n=1}^{\infty}\frac{2}{n(e^{2n\pi}-1)}+\sum_{n=1}^{\infty}\frac{1}{n}=0 \tag{1}$$

$$\sum_{n=1}^{\infty}\frac{\coth(n\pi)}{n^3}+\sum_{n=1}^{\infty}\frac{2}{n^3(e^{2n\pi}-1)}+\sum_{n=1}^{\infty}\frac{1}{n^3}=\frac{7}{90}\pi^3 \tag{2}$$

$$\sum_{n=1}^{\infty}-\frac{\coth(n\pi)}{n^5}+\sum_{n=1}^{\infty}\frac{2}{n^5(e^{2n\pi}-1)}+\sum_{n=1}^{\infty}\frac{1}{n^5}=0 \tag{3}$$

证明 变换核函数，$f(z)=\dfrac{\pi\cos(\pi z)}{z^{2k+1}(e^{2\pi z}-1)\sin(\pi z)}$ 有单极点 $z=\pm n$， $z=\pm ni$ 。

1) 展开成幂级数

$$f(z)=\frac{\pi\cot(\pi z)}{z^{2k+1}\left(e^{2\pi z}-1\right)}=\frac{\pi\left[1/(\pi z)-(\pi z)/3-(\pi z)^3/45+\cdots\right]}{z^{2k+1}\left[(2\pi z)+(2\pi z)^2/2+(2\pi z)^3/6+\cdots\right]}$$

$$=\frac{\left[1-(\pi z)^2/3-(\pi z)^4/45+\cdots\right]}{2\pi z^{2k+3}\left[1+\pi z+2(\pi z)^3/3+(\pi z)^3/3+2(\pi z)^4/15+\cdots\right]}$$

$$=\frac{1}{2\pi z^{2k+3}}\left(1-\frac{1}{3}(\pi z)^2-\frac{1}{45}(\pi z)^4-\frac{2}{945}(\pi z)^6+\cdots\right)$$

$$\cdot\left(1-\pi z+\frac{1}{3}(\pi z)^2-\frac{1}{45}(\pi z)^4+\frac{2}{945}(\pi z)^6+\cdots\right);$$

$f(z)$的留数$\mathrm{Res}[f;0]=q(2k+3)$；其中$q(3)=0$； $q(5)=-\frac{7}{90}\pi^3$； $q(7)=0$；

2）在单极点$z=\pm n$，计算$f(z)$的留数，用洛必达法则可得

$$\mathrm{Res}[f;n]=\lim_{z\to n}\frac{\pi}{z^{2k+1}\left(e^{2\pi z}-1\right)}\frac{[z-n]\cos(\pi z)}{\sin(\pi z)}$$

$$=\lim_{z\to n}\frac{\pi}{n^{2k+1}\left(e^{2n\pi}-1\right)}\frac{\cos(n\pi)}{[\pi\cos(\pi n)]}=\frac{1}{n^{2k+1}\left(e^{2n\pi}-1\right)};$$

$$\mathrm{Res}[f;-n]=\lim_{z\to -n}\frac{\pi}{z^{2k+1}\left(e^{2\pi z}-1\right)}\frac{[z+n]\cos(-\pi n)}{\pi\sin(-\pi n)}$$

$$=\frac{\pi}{-n^{2k+1}\left(e^{-2n\pi}-1\right)}\frac{\cos(-\pi n)}{[\pi\cos(-\pi n)]}=\frac{1}{n^{2k+1}\left(e^{2n\pi}-1\right)}+\frac{1}{n^{2k+1}};$$

3）在单极点$z=\pm ni$，计算$f(z)$的留数

$$\mathrm{Res}[f;ni]=\lim_{z\to n}\frac{(z-ni)\pi\cot(\pi z)}{\left(e^{2\pi z}-1\right)z^{2k+1}}=\frac{\pi}{2\pi e^{2ni\pi}}\frac{\cot(\pi ni)}{[ni]^{2k+1}}=\frac{-i\coth(n\pi)}{2n^{2k+1}i^{2k+1}}=-\frac{\coth(n\pi)}{2n^{2k+1}i^{2k}}=\mathrm{Res}[f;-ni];$$

根据高斯留数基本定理，从而有

$$0=q(2k+3)+\sum_{n=-\infty}^{\infty}-\frac{\coth(n\pi)}{2n^{2k+1}i^{2k}}+\sum_{k=-\infty}^{\infty}\frac{1}{n^{2k+1}\left(e^{2n\pi}-1\right)}+\sum_{n=1}^{\infty}\frac{1}{n^{2k+1}};$$

$$\sum_{n=1}^{\infty}-\frac{\coth(n\pi)}{n^{2k+1}i^{2k}}+\sum_{n=1}^{\infty}\frac{2}{n^{2k+1}\left(e^{2n\pi}-1\right)}+\sum_{n=1}^{\infty}\frac{1}{n^{2k+1}}=-q(2k+3);$$

令 $k=0,1,2$ 整理得到如下级数恒等式

$$\sum_{n=1}^{\infty}-\frac{\coth(n\pi)}{n}+\sum_{n=1}^{\infty}\frac{2}{n\left(e^{2n\pi}-1\right)}+\sum_{n=1}^{\infty}\frac{1}{n}=0\text{；}$$

$$\sum_{n=1}^{\infty}\frac{\coth(n\pi)}{n^3}+\sum_{n=1}^{\infty}\frac{2}{n^3\left(e^{2n\pi}-1\right)}+\sum_{n=1}^{\infty}\frac{1}{n^3}=\frac{7}{90}\pi^3\text{；}$$

$$\sum_{n=1}^{\infty}-\frac{\coth(n\pi)}{n^5}+\sum_{n=1}^{\infty}\frac{2}{n^5\left(e^{2n\pi}-1\right)}+\sum_{n=1}^{\infty}\frac{1}{n^5}=0\text{。}$$