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A Lattice Boltzmann Model for the Steady State Compressible Flows

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Abstract

In this paper, a multi-energy-level lattice Boltzmann model for the steady state compressible flows is proposed. Firstly, the Chapman-Enskog expansion and the multi-spatial scale expansion are used to describe the higher-order moment of equilibrium distribution functions and a series of partial differential equations in different spatial scales. Secondly, the modified partial differential equation of the Euler equation with the higher-order truncation error is obtained. Thirdly, comparison between numerical results of the lattice Boltzmann models and exact solution is given. The numerical results agree well with the classical one.

Keywords

Lattice Boltzmann Model, Compressible Flows, Steady State Lattice Boltzmann Equation

一种定常可压缩流动的格子Boltzmann模型

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摘　要

本文给出了一种用于定常可压缩流动的多能级格子Boltzmann模型。我们使用Chapman-Enskog展开和空间多尺度展开技术描述平衡态分布函数的高阶矩和不同空间尺度的系列方程，进而得到了具有高阶误

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差的Euler方程的修正方程。我们还给出了格子Boltzmann模型的数值模拟结果与解析解的比较。结果表明，数值模拟结果与解析解吻合的很好。

关键词

格子Boltzmann模型，可压缩流动，定常格子Boltzmann方程

1. 引言

近 30 年来，格子 Boltmzann 方法(LBM)已经发展成为一种有效的计算流体动力学(CFD)方法[1] [2] [3] [4]。目前，LBM 已经广泛应用于多相流[5]、多组分流[6]、多孔介质流[7]、悬浮流[8]等复杂问题。此外，LBM 可以求解 Burgers 方程[9] [10]、KdV 方程[11]，Lorenz 方程[12]，Schrödinger 方程[13] [14] [15]和 Poisson 方程[16] [17]等线性和非线性偏微分物理方程。

用 LBM 模拟可压缩流动比较困难，因为 LBM 模型本身受到低 Mach 数的限制，其本质是冲量流，与能量相关。近年来，为了消除可压缩流动对低 Mach 数的限制，提出了大量的 LBM 模型[18]-[49]，这些模型都是用来研究非定常可压缩流动的。

在可压缩流动的研究领域，提出了许多明显细化的有限差分模型[50]-[54]。非结构网格有限体积法可以求解复杂边界问题[50]；总变差减小(TVD)模型[51]和本质不振荡(ENO)模型[52]可以减小数值扩散和非物理振荡；无网格法可以跳出对离散网格的限制[53]；水平集法(the level set method)可以追踪移动边界[54]。然而，这些格式在求解可压缩流动时，会产生高清晰度的激波，其中 TVD 格式最为明显。而本文提出的 LBM 模型，是与时间无关的格子 Boltzmann 模型。与以往的 LBM 不同，本文所使用的多尺度是空间多尺度。数值结果表明，LBM 模拟结果与以上经典解法求得的结果吻合的很好。

2. 格子 Boltzmann 模型

2.1. 格子 Boltzmann 方程

首先考虑一个两层正六边形格子，其 12 个格点分别与中心点相连结。假设粒子沿格线运动，粒子速度 $\mathbf{e}_\alpha$ 可分解为 A 和 B 两部分，每一部分具有不同能级 $\varepsilon_A(\alpha=1,\cdots,12)$ 和 $\varepsilon_B(\alpha=13,\cdots,24)$，静止粒子 $(\alpha=0)$ 的能级为 ε_D。即，该速度模型是 25 结点 3 速度 $(0,c,2c)$ 模型，$c=\Delta x/\Delta t$，Δx、Δt 分别为空间步长和时间步长。$|e_\alpha|=c$ ($\alpha=1,\cdots,6$ 和 $\alpha=13,\cdots,18$)；$|e_\alpha|=2c$ ($\alpha=7,\cdots,12$ 和 $\alpha=19,\cdots,24$)，$|e_0|=0$，如图 1 所示。

定义 $f_\alpha(\mathbf{x},t)$ 为 $\mathbf{x}$ 位置，t 时刻的分布函数，其速度为 $\mathbf{e}_\alpha$，则每个位置的宏观质量、动量和能量分别为

$$\rho=\sum_\alpha f_\alpha, \tag{1}$$

$$\rho u_j=\sum_\alpha e_{\alpha j} f_\alpha, \tag{2}$$

$$\frac{1}{2}\rho u^2+\rho E=\sum_\alpha f_\alpha \varepsilon_\alpha, \tag{3}$$

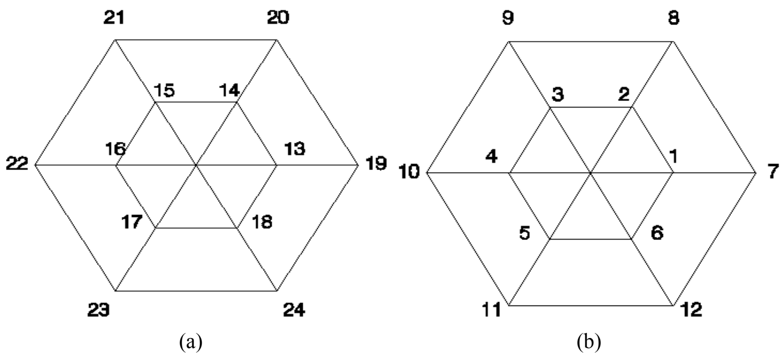


Figure 1. Schematic of lattice. (a) type A, (b) type B
图 1. 格子示意图。(a) A 型，(b) B 型

在公式(3)中，E 为单位质量的内能。ε_α 为能级[41]，且具有三个值 ε_A，ε_B 和 ε_D。则定常格子 Boltzmann 方程[55] [56]可表示为

$$f_\alpha\left(\mathbf{x}+\varepsilon\mathbf{e}_\alpha\right)-f_\alpha\left(\mathbf{x}\right)=-\frac{1}{\tau}\left[f_\alpha\left(\mathbf{x}\right)-f_\alpha^{eq}\left(\mathbf{x}\right)\right] \tag{4}$$

其中 τ 是松弛因子，$f_\alpha^{eq}\left(\mathbf{x}\right)$ 是 $\mathbf{x}$ 位置的局部平衡态分布函数，其速度为 $\mathbf{e}_\alpha$。设平衡态分布函数 f_α^{eq} 具有如下形式

$$f_\alpha^{eq}=A_0'\rho+A_2'\rho u_je_{\alpha j}+A_5'\rho u_iu_je_{\alpha i}e_{\alpha j}+A_6'\rho u^2+A_7'\rho u_iu_ju_ke_{\alpha i}e_{\alpha j}e_{\alpha k},$$
$$\left(\alpha=1,\cdots,6\right),\varepsilon_\alpha=\varepsilon_A;$$

$$f_\alpha^{eq}=A_0''\rho+A_2''\rho u_je_{\alpha j}+A_5''\rho u_iu_je_{\alpha i}e_{\alpha j}+A_6''\rho u^2+A_7''\rho u_iu_ju_ke_{\alpha i}e_{\alpha j}e_{\alpha k},$$
$$\left(\alpha=7,\cdots,12\right),\varepsilon_\alpha=\varepsilon_A;$$

$$f_\alpha^{eq}=B_0'\rho+B_2'\rho u_je_{\alpha j}+B_5'\rho u_iu_je_{\alpha i}e_{\alpha j}+B_6'\rho u^2+B_7'\rho u_iu_ju_ke_{\alpha i}e_{\alpha j}e_{\alpha k},$$
$$\left(\alpha=13,\cdots,18\right),\varepsilon_\alpha=\varepsilon_B;$$

$$f_\alpha^{eq}=B_0''\rho+B_2''\rho u_je_{\alpha j}+B_5''\rho u_iu_je_{\alpha i}e_{\alpha j}+B_6''\rho u^2+B_7''\rho u_iu_ju_ke_{\alpha i}e_{\alpha j}e_{\alpha k},$$
$$\left(\alpha=19,\cdots,24\right),\varepsilon_\alpha=\varepsilon_B;$$

$$f_0^{eq}=D_0\rho+D_6\rho u^2,$$
$$\left(\alpha=0\right),\varepsilon_\alpha=\varepsilon_D.$$

其中 A_β'、A_β''、B_β'、$B_\beta''\left(\beta=0,2,5,6,7\right)$，$D_\beta\left(\beta=0,6\right)$ 是由质量守恒、动量守恒、能量守恒和高阶矩的关系所确定的待定系数[46]，即

$$\sum_\alpha f_\alpha^{eq}=\rho, \tag{5}$$

$$\sum_\alpha f_\alpha^{eq}e_{\alpha j}=\rho u_j, \tag{6}$$

$$\sum_\alpha f_\alpha^{eq}\varepsilon_\alpha=\frac{1}{2}\rho u^2+\rho E, \tag{7}$$

$$\pi_{ij}^0\equiv\sum_\alpha f_\alpha^{eq}e_{\alpha i}e_{\alpha j}=\rho u_iu_j+p\delta_{ij}, \tag{8}$$

$$Q_j^0\equiv\sum_\alpha f_\alpha^{eq}e_{\alpha j}\varepsilon_\alpha=\left(\frac{1}{2}\rho u^2+\rho E+p\right)u_j, \tag{9}$$

$$P_{ijk}^{0} \equiv \sum_{\alpha} f_{\alpha}^{eq} e_{\alpha i} e_{\alpha j} e_{\alpha k} = 0 , \tag{10}$$

$$R_{jk}^{0} \equiv \sum_{\alpha} f_{\alpha}^{eq} e_{\alpha j} e_{\alpha k} \varepsilon_{\alpha} = 0 . \tag{11}$$

其中 p 为理想气体的压力

$$p = (\gamma - 1) \rho E , \tag{12}$$

γ 为比热比。

2.2. 平衡态分布中的系数

Navier-Stokes 方程的恢复与矩的对称性有关

$$T_{i_1,\cdots,i_n}^{n} = \sum_{\alpha} e_{\alpha i_1} \cdots e_{\alpha i_n} . \tag{13}$$

本文中的六个矩 $T^1,\cdots,T^6$ 可由文献[57]得出

$$T_{ij}^{2} = \frac{bc^2}{D} \delta_{ij} , \tag{14}$$

$$T_{ijkm}^{4} = \frac{bc^4}{D(D+2)} \Delta_{ijkm}^{4} , \tag{15}$$

其中，$\Delta_{ijkm}^{4} = \delta_{ij}\delta_{km} + \delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk}$。

$$T_{ijklmn}^{6} = \frac{bc^6}{D(D+2)(D+4)} \Delta_{ijklmn}^{6} , \tag{16}$$

$$\begin{aligned} \Delta_{ijklmn}^{6} = {} & \delta_{ij}\Delta_{klmn} + \delta_{ik}\Delta_{jlmn} + \delta_{il}\Delta_{jkmn} + \delta_{im}\Delta_{jkln} + \delta_{in}\Delta_{jklm} \\ & + \delta_{jk}\Delta_{ilmn} + \delta_{jl}\Delta_{ikmn} + \delta_{jm}\Delta_{ikln} + \delta_{jn}\Delta_{iklm} + \delta_{kl}\Delta_{ijmn} \\ & + \delta_{km}\Delta_{ijln} + \delta_{kn}\Delta_{ijlm} + \delta_{lm}\Delta_{ijkn} + \delta_{ln}\Delta_{ijkm} + \delta_{mn}\Delta_{ijkl} . \end{aligned} \tag{17}$$

所以

$$\Delta_{ijklmn} \rho u_l u_m u_n = 18 \rho u_i u_j u_k + 9 \rho u^2 u_k \delta_{ij} + 9 \rho u^2 u_j \delta_{ik} + 9 \rho u^2 u_i \delta_{jk} . \tag{18}$$

将平衡态分布函数带入(5)~(11)式，再使用(14)~(18)式，即可以得到关于系数 A'_β，A''_β，B'_β，B''_β $(\beta = 0,2,5,6,7)$，$D_\beta\ (\beta = 0,6)$ 的线性方程组

$$b\left(A'_0 + A''_0 + B'_0 + B''_0\right) + D_0 = 1 , \tag{19}$$

$$\frac{b}{D}\left(A'_5 c_1^2 + A''_5 c_2^2 + B'_5 c_1^2 + B''_5 c_2^2\right) + b\left(A'_6 + A''_6 + B'_6 + B''_6\right) + D_6 = 0 , \tag{20}$$

$$\frac{b}{D}\left(A'_2 c_1^2 + A''_2 c_2^2 + B'_2 c_1^2 + B''_2 c_2^2\right) = 1 , \tag{21}$$

$$\frac{b}{D(D+2)}\left(A'_7 c_1^4 + A''_7 c_2^4 + B'_7 c_1^4 + B''_7 c_2^4\right) = 0 , \tag{22}$$

$$b\left(A'_0 \varepsilon_A + A''_0 \varepsilon_A + B'_0 \varepsilon_B + B''_0 \varepsilon_B\right) + D_0 \varepsilon_D = E , \tag{23}$$

$$\begin{aligned} & \frac{b}{D}\left(A'_5 \varepsilon_A c_1^2 + A''_5 \varepsilon_A c_2^2 + B'_5 \varepsilon_B c_1^2 + B''_5 \varepsilon_B c_2^2\right) \\ & + b\left(A'_6 \varepsilon_A + A''_6 \varepsilon_A + B'_6 \varepsilon_B + B''_6 \varepsilon_B\right) + D_6 \varepsilon_D = \frac{1}{2} , \end{aligned} \tag{24}$$

$$\frac{\rho b}{D}\left(A_0'c_1^2 + A_0''c_2^2 + B_0'c_1^2 + B_0''c_2^2\right) = p\,, \tag{25}$$

$$\frac{2b}{D(D+2)}\left(A_5'c_1^4 + A_5''c_2^4 + B_5'c_1^4 + B_5''c_2^4\right) = 1\,, \tag{26}$$

$$\frac{b}{D}\left(A_6'c_1^2 + A_6''c_2^2 + B_6'c_1^2 + B_6''c_2^2\right) + \frac{b}{D(D+2)}\left(A_5'c_1^4 + A_5''c_2^4 + B_5'c_1^4 + B_5''c_2^4\right) = 0\,, \tag{27}$$

$$\frac{b}{D}\left(A_2'\varepsilon_A c_1^2 + A_2''\varepsilon_A c_2^2 + B_2'\varepsilon_B c_1^2 + B_2''\varepsilon_B c_2^2\right) = \gamma E\,, \tag{28}$$

$$\frac{3b}{D(D+2)}\left(A_7'\varepsilon_A c_1^4 + A_7''\varepsilon_A c_2^4 + B_7'\varepsilon_B c_1^4 + B_7''\varepsilon_B c_2^4\right) = \frac{1}{2}\,, \tag{29}$$

$$\begin{aligned} &\frac{b}{D(D+2)}\left(A_2'c_1^4 + A_2''c_2^4 + B_2'c_1^4 + B_2''c_2^4\right) \\ &+ \frac{9bu^2}{D(D+2)(D+4)}\left(A_7'c_1^6 + A_7''c_2^6 + B_7'c_1^6 + B_7''c_2^6\right) = 0, \end{aligned} \tag{30}$$

$$\frac{18b}{D(D+2)(D+4)}\left(A_7'c_1^6 + A_7''c_2^6 + B_7'c_1^6 + B_7''c_2^6\right) = 0\,, \tag{31}$$

$$\frac{b}{D}\left(A_0'\varepsilon_A c_1^2 + A_0''\varepsilon_A c_2^2 + B_0'\varepsilon_B c_1^2 + B_0''\varepsilon_B c_2^2\right) = 0\,, \tag{32}$$

$$\frac{2b}{D(D+2)}\left(A_5'\varepsilon_A c_1^4 + A_5''\varepsilon_A c_2^4 + B_5'\varepsilon_B c_1^4 + B_5''\varepsilon_B c_2^4\right) = 0\,, \tag{33}$$

$$\begin{aligned} &\frac{b}{D}\left(A_6'\varepsilon_A c_1^2 + A_6''\varepsilon_A c_2^2 + B_6'\varepsilon_B c_1^2 + B_6''\varepsilon_B c_2^2\right) \\ &+ \frac{b}{D(D+2)}\left(A_5'c_1^4\varepsilon_A + A_5''c_2^4\varepsilon_A + B_5'c_1^4\varepsilon_B + B_5''c_2^4\varepsilon_B\right) = 0. \end{aligned} \tag{34}$$

其中，$D(=2)$ 是空间维度，b （$=6$，正六边形格子）是每个位置的节点数。

为了得到平衡态分布函数中的系数，根据气体静止时的各向同性，我们提出了一些补充条件。如引入下列假设

$$A_0'\varepsilon_A + B_0'\varepsilon_B = 4\left(A_0''\varepsilon_A + B_0''\varepsilon_B\right) = 0\,, \tag{35}$$

$$\left(A_2'\varepsilon_A + B_2'\varepsilon_B\right) = 4\left(A_2''\varepsilon_A + B_2''\varepsilon_B\right) = \frac{\gamma ED}{2bc^2}\,, \tag{36}$$

$$A_5'\varepsilon_A + B_5'\varepsilon_B = 16\left(A_5''\varepsilon_A + B_5''\varepsilon_B\right) = 0\,, \tag{37}$$

$$A_6' + B_6' = 4\left(A_6'' + B_6''\right) = -\frac{D}{4bc^2}\,, \tag{38}$$

$$A_6'\varepsilon_A + B_6'\varepsilon_B = 4\left(A_6''\varepsilon_A + B_6''\varepsilon_B\right) = 0\,, \tag{39}$$

$$\left(A_7'\varepsilon_A + B_7'\varepsilon_B\right) = 16\left(A_7''\varepsilon_A + B_7''\varepsilon_B\right) = \frac{D(D+2)}{12bc^4}, \tag{40}$$

得到系数

$$A_0' = -\frac{1}{\varepsilon_A - \varepsilon_B}\frac{1}{3}\left[\frac{4}{b}\left(1 - \frac{E}{\varepsilon_D}\right) - \frac{pD}{\rho bc^2}\right]\varepsilon_B\,, \tag{41}$$

$$B_0' = \frac{1}{\varepsilon_A - \varepsilon_B}\frac{1}{3}\left[\frac{4}{b}\left(1 - \frac{E}{\varepsilon_D}\right) - \frac{pD}{\rho bc^2}\right]\varepsilon_A, \tag{42}$$

$$A_0'' = -\frac{1}{\varepsilon_A - \varepsilon_B}\frac{1}{3}\left[-\frac{1}{b}\left(1 - \frac{E}{\varepsilon_D}\right) + \frac{pD}{\rho bc^2}\right]\varepsilon_B, \tag{43}$$

$$B_0'' = \frac{1}{\varepsilon_A - \varepsilon_B}\frac{1}{3}\left[-\frac{1}{b}\left(1 - \frac{E}{\varepsilon_D}\right) + \frac{pD}{\rho bc^2}\right]\varepsilon_A, \tag{44}$$

$$D_0 = \frac{E}{\varepsilon_D}, \tag{45}$$

$$A_2' = -\frac{1}{\varepsilon_A - \varepsilon_B}\left(\frac{1}{3}\frac{4D}{bc^2}\varepsilon_B - \frac{\gamma ED}{2bc^2}\right), \tag{46}$$

$$B_2' = \frac{1}{\varepsilon_A - \varepsilon_B}\left(\frac{1}{3}\frac{4D}{bc^2}\varepsilon_A - \frac{\gamma ED}{2bc^2}\right), \tag{47}$$

$$A_2'' = -\frac{1}{\varepsilon_A - \varepsilon_B}\left(-\frac{1}{12}\frac{D}{bc^2}\varepsilon_B - \frac{\gamma ED}{8bc^2}\right), \tag{48}$$

$$B_2'' = \frac{1}{\varepsilon_A - \varepsilon_B}\left(-\frac{1}{12}\frac{D}{bc^2}\varepsilon_A - \frac{\gamma ED}{8bc^2}\right), \tag{49}$$

$$A_5' = -\frac{1}{\varepsilon_A - \varepsilon_B}\frac{1}{3}\left[\frac{5D^2}{4bc^4} - \frac{D(D+2)}{2bc^4}\right]\varepsilon_B, \tag{50}$$

$$B_5' = \frac{1}{\varepsilon_A - \varepsilon_B}\frac{1}{3}\left[\frac{5D^2}{4bc^4} - \frac{D(D+2)}{2bc^4}\right]\varepsilon_A, \tag{51}$$

$$A_5'' = -\frac{1}{\varepsilon_A - \varepsilon_B}\frac{1}{12}\left[-\frac{5D^2}{16bc^4} + \frac{D(D+2)}{2bc^4}\right]\varepsilon_B, \tag{52}$$

$$B_5'' = \frac{1}{\varepsilon_A - \varepsilon_B}\frac{1}{12}\left[-\frac{5D^2}{16bc^4} + \frac{D(D+2)}{2bc^4}\right]\varepsilon_A, \tag{53}$$

$$A_6' = -\frac{1}{\varepsilon_A - \varepsilon_B}\left(-\frac{D}{4bc^2}\right)\varepsilon_B, \tag{54}$$

$$B_6' = \frac{1}{\varepsilon_A - \varepsilon_B}\left(-\frac{D}{4bc^2}\right)\varepsilon_A, \tag{55}$$

$$A_6'' = -\frac{1}{\varepsilon_A - \varepsilon_B}\left(-\frac{D}{16bc^2}\right)\varepsilon_B, \tag{56}$$

$$B_6'' = \frac{1}{\varepsilon_A - \varepsilon_B}\left(-\frac{D}{16bc^2}\right)\varepsilon_A, \tag{57}$$

$$D_6 = \frac{1}{2\varepsilon_D}, \tag{58}$$

$$A_7' = \frac{1}{\varepsilon_A - \varepsilon_B}\frac{D(D+2)}{12bc^4}, \tag{59}$$

$$B_7' = -\frac{1}{\varepsilon_A - \varepsilon_B}\frac{D(D+2)}{12bc^4}, \tag{60}$$

$$A_7'' = \frac{1}{\varepsilon_A - \varepsilon_B}\frac{D(D+2)}{192bc^4}, \tag{61}$$

$$B_7'' = -\frac{1}{\varepsilon_A - \varepsilon_B}\frac{D(D+2)}{192bc^4}. \tag{62}$$

2.3. 宏观方程

假设 Knudsen 数 ε 为小量，对分布函数 $f_\alpha(\mathbf{x})$ 进行 Chapman- Enskog 展开[58]

$$f_\alpha = \sum_{n=0}^{\infty}\varepsilon^n f_\alpha^{(n)} = f_\alpha^{(0)} + \varepsilon f_\alpha^{(1)} + \varepsilon^2 f_\alpha^{(2)} + \cdots, \tag{63}$$

在(63)式中，$f_\alpha^{(0)}$ 表示 f_α^{eq}。

引入 $\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_3$ 等空间尺度，$x_{si} = \varepsilon^3 x_s$，即

$$\frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_{0i}} + \varepsilon\frac{\partial}{\partial x_{1i}} + \varepsilon^2\frac{\partial}{\partial x_{2i}} + \varepsilon^3\frac{\partial}{\partial x_{3i}} + O(\varepsilon^4). \tag{64}$$

对(4)式进行 Taylor 展开，保留到 $O(\varepsilon^5)$，可得到不同空间尺度的一系列偏微分方程组，即

$$\Delta_0 f_\alpha^{(0)} = -\frac{1}{\tau} f_\alpha^{(1)}, \tag{65}$$

$$\Delta_1 f_\alpha^{(0)} + C_2\Delta_0^2 f_\alpha^{(0)} = -\frac{1}{\tau} f_\alpha^{(2)}, \tag{66}$$

$$C_3\Delta_0^3 f_\alpha^{(0)} + 2C_2\Delta_0\Delta_1 f_\alpha^{(0)} + \Delta_2 f_\alpha^{(0)} = -\frac{1}{\tau} f_\alpha^{(3)}, \tag{67}$$

$$C_4\Delta_0^4 f_\alpha^{(0)} + 3C_3\Delta_0^2\Delta_1 f_\alpha^{(0)} + 2C_2\Delta_0\Delta_2 f_\alpha^{(0)} + \Delta_3 f_\alpha^{(0)} + C_2\Delta_1^2 f_\alpha^{(0)} = -\frac{1}{\tau} f_\alpha^{(4)}. \tag{68}$$

其中，$\Delta_s \equiv e_{\alpha_j}\frac{\partial}{\partial x_{sj}}$，$s = 0,1,2,3$。(65)~(68)式被称为不同空间尺度的系列方程，可用于一维、二维、三维等情况。(11)~(16)式中关于松弛因子 τ 的多项式为

$$C_2 = \frac{1}{2} - \tau, \tag{69}$$

$$C_3 = \tau^2 - \tau + \frac{1}{6}, \tag{70}$$

$$C_4 = -\tau^3 + \frac{3}{2}\tau^2 - \frac{7}{12}\tau + \frac{1}{24}. \tag{71}$$

公式(69)~(71)称为 Chapman 多项式，与文献[59]所得到的结果完全一致。它们可用来表示修正的偏微分方程的扩散项和色散项系数。

我们很容易得到 $\mathbf{x}_0$ 尺度的守恒率方程

$$\frac{\partial \rho u_j}{\partial x_{0j}} = 0, \tag{72}$$

$$\frac{\partial \pi_{ij}^0}{\partial x_{0j}} = 0, \tag{73}$$

$$\frac{\partial Q_j^0}{\partial x_{0j}} = 0. \tag{74}$$

将(65) + (66) $\times\varepsilon$ 两端分别乘以 1，$e_{\alpha i}$ 和 ε_α，并分别对 α 求和，即可得到具有二阶精度的定常 Euler 方程

$$\frac{\partial \rho u_j}{\partial x_j} = O\left(k^2\right), \tag{75}$$

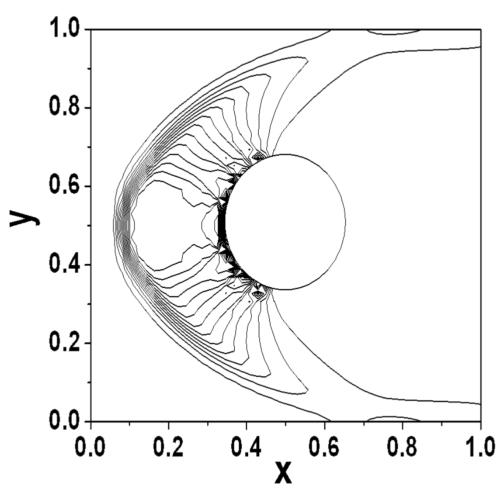
$$\frac{\partial \pi_{ij}^0}{\partial x_j} = O\left(k^2\right), \tag{76}$$

$$\frac{\partial Q_j^0}{\partial x_j} = O\left(k^2\right). \tag{77}$$

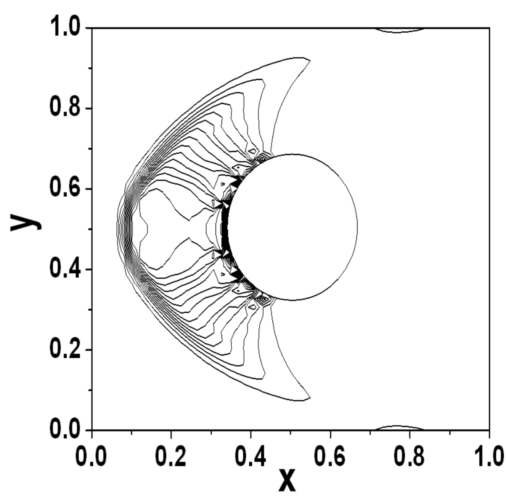
3. 数值例子

3.1. 4Mach 数圆柱绕流[46]

把求解域 $[0,1]\times[0,1]$ 分成 100×100 的均匀网格。圆柱横截面的半径为 $r=0.15$，圆心坐标为 $(0.5,0.5)$。迭代的初始条件为 $\rho=1.4$，$u=4.0$，$v=0.0$，$p=1.0$，即入流 Mach 数为 $M_\infty=4$，$\gamma=1.4$。入口边界、上边界和下边界的边界条件与初始条件相同；出口边界条件为 Neumann 条件，即 $\partial F/\partial n=0$，其中 F 为 ρ、u、v 或 p；n 为法线方向的矢量。圆柱表面为粘性边界条件，即 $u=v=0$。图 2 给出了 Mach 数为 4 时圆柱绕流的模拟结果，其中(a)为密度等值线图，(b)为压力等值线图，x、y 轴为流场计算区域的坐标。我们在等值线中发现了一些大梯度区振荡，这是由边界区域的精度较低导致的，因为我们使用了矩形网格构建圆柱边界。由于流体是超音速流，因此边界的处理对激波没有影响。为了与以前 LBM 模型所得的结果作比较，我们选择了文献[41]中的一阶 LBM 模型和文献[46]中的二阶 LBM 模型，并把文献[60]中有限谱 ENO 模型(FSEM)所得的结果作为解析解。表 1 给出了 $x=0.25$ 时本文所使用的 LBM 模型与其它模型的 L_1 范数误差。从表 1 中可以发现，本模型的精度较文献[41]和文献[46]略有提高。



(a)



(b)

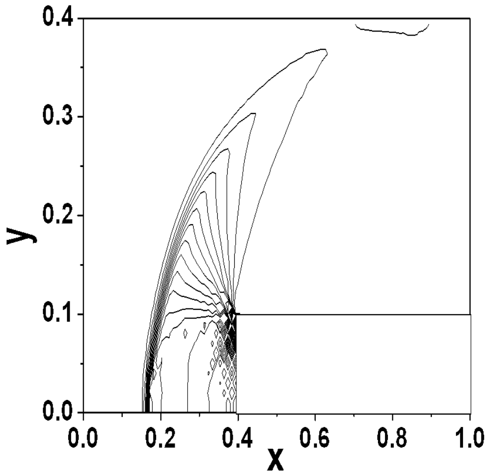
Figure 2. Numerical result of 4 Mach number incoming flow around a circular cylinder. (a) is the density contours, (b) is the pressure contours

图 2. 4 Mach 数圆柱绕流模拟结果。(a) 密度等值线图，(b) 压力等值线图

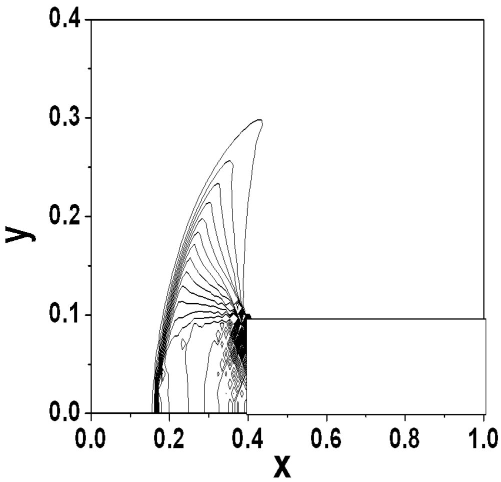
Table 1. The L_1 norm errors of the flows around a circular cylinder with 4 Mach number incoming at line $x = 0.25$

表 1. $x = 0.25$ 时，4 Mach 数圆柱绕流的 L_1 范数误差

	密度	速度	压强
LBM (文献[46])	0.0235	0.0251	0.00325
LBM (文献[41])	0.0738	0.0771	0.01077
本文模型	0.0136	0.0238	0.00193



(a)



(b)

Figure 3. Numerical result of 3 Mach number incoming flow around a rectangle. (a) is the density contours, (b) is the pressure contours

图 3. 3 Ma 数前台阶流模拟结果。(a) 密度等值线图，(b) 压力等值线图

3.2. 3Mach 数前台阶流

把求解域 $[0,1]\times[-0.4,0.4]$ 分成 100×100 的均匀网格。矩形位于 $[0.4,1.0]\times[-0.1,0.1]$。迭代的初始条件为 $\rho=1.4$，$u=3.0$，$v=0.0$，$p=1.0$，即入流 Mach 数为 $M_\infty=3$，$\gamma=1.4$。边界条件与图 2 相同。图 3 给出了入流 Mach 数为 3 时，绕前台阶流动的数值模拟结果。其中图 3(a)为密度等值线图，图 3(b)为压力等值线图。其它参数为 $\gamma=1.4$，$c=3.0$，$\tau=1.51$，$\varepsilon_A=2c^2$，$\varepsilon_B=0.6c^2$，$\varepsilon_D=0.13c^2$，网格数为 100×100，Mach 数为 3，等值线条数为 30。数值结果表明，LBM 模型所得到的结果与经典结果吻合的较好。

4. 结论

本文提出了用于定常可压缩流动的多能级格子 Boltzmann 模型，并得出以下结论：

首先，给出了定常格子 Boltzmann 方程，得到了不同空间尺度的一系列偏微分方程。使用了 Chapman-Enskog 展开和空间多尺度展开描述平衡态分布函数的高阶矩。

其次，得到了具有高阶截断误差的 Euler 方程的修正方程。

本文仍有后续问题需要解决：(1) 模型精度问题。本模型中的数值耗散和色散，以及数值现象与参数之间的关系仍需要研究；(2) 固定边界的边界条件及更高 Mach 数的流动仍需更细致地模拟。这些问题将在以后的文章中讨论。

最后，我们需要指出，建立用于模拟可压缩流动乃至高 Mach 数流动的格子 Boltzmann 模型仍是需要解决的问题。

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