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Existence of Solutions for Fractional Impulsive Differential Equations with p-Laplacian Operator

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Abstract

In this paper, we discuss the existence of solutions of boundary value problems for a class of fractional impulsive differential equations. Some fixed point theorems are used to obtain sufficient conditions for the existence of solutions of Impulsive Differential Equations.

Keywords

Fractional Difference Equation, Impulsive, Fixed-Point, Boundary Problem

具p-Laplacian算子的分数阶脉冲微分方程边值问题解的存在性

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摘 要

本文讨论了一类分数阶脉冲微分方程的边值问题解的存在性，应用一些不动点定理给出了脉冲微分方程解的存在性的充分条件。

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关键词

分数阶，脉冲：不动点定理，边值问题

1. 引言

分数阶微分方程在科技、经济和工程等领域都得到了广泛的应用，引起了广大数学学者的关注和研究，并在分数阶微分系统方面得到了很多的研究成果[1]-[9]，含有 p-Laplacian 算子的微分方程也逐渐热门[1] [2] [3]，而对非线性分数阶脉冲微分方程脉冲边值问题的研究相对而言比较少，在文献[4]-[9]中也研究了含有脉冲项的分数阶微分方程的边值问题，因此本文主要研究含有 p-Laplacian 算子的分数阶脉冲微分方程边值问题解的存在性。

在文献[6]中，作者运用 Banach 压缩映射原理和 Leray-Schauder 不动点定理研究了分数阶差分方程边值问题

$$\begin{cases} {}^{C}D_{0+}^{q}u(t)=f\left(t,u(t)\right),1<q\le 2,t\in J', \\ \Delta\left(u(t_k)\right)=I_k\left(u(t_k)\right),\Delta\left(u'(t_k)\right)=Q_k\left(u(t_k)\right), \\ au(0)-bu'(0)=x_0,cu(1)+du'(1)=x_1, \end{cases} \tag{1}$$

解的存在性，这里是标准的 Caputo 分数阶导数，其中 $f\in(C\times R,R)$， $I_k,Q_k\in C(R,R)$，是连续函数。

文献[2]中，作者运用不动点定理研究了如下边值问题

$$\begin{cases} {}^{C}D_{0+}^{\beta}\varphi_p\left({}^{C}D_{0+}^{\beta}x(t)\right)=f\left(t,x(t)\right),1<\alpha\le 2,t\in J', \\ x(0)+x(1)=0,{}^{C}D_{0+}^{\alpha}x(0)+{}^{C}D_{0+}^{\alpha}x(1)=0, \end{cases} \tag{2}$$

解的存在性，这里 ${}^{C}D_{0+}^{\beta}$ 是标准的 Caputo 分数阶导数。

受上述文献启发，本文应用不动点定理研究含 p-Laplacian 算子的分数阶脉冲微分方程边值问题

$$\begin{cases} \varphi_p\left({}^{C}D_{0+}^{\alpha}u(t)\right)=f\left(t,u(t)\right),t\in J' \\ \Delta\left(u(t_k)\right)=I_k\left(u(t_k)\right),\Delta\left(u'(t_k)\right)=Q_k\left(u(t_k)\right), \\ au(0)-bu'(0)=x_0,cu(0)+d\,{}^{C}D_{0+}^{\beta}u(1)=x_1 \end{cases} \tag{3}$$

解的存在性，其中 ${}^{C}D_{0+}^{\alpha}$ 是标准的 Caputo 分数阶导数，

$$1<\alpha\le 2,\quad \alpha-1<\beta\le 1,\quad a\ge 0,\quad b>0,\quad c\ge 0,\quad d>0,$$

$$\left(\delta=bc+a\left(c+\frac{d}{\Gamma(2-\beta)}\right)\right),\quad x_0,x_1\in R,\quad f\in C(J\times R,R),\quad I_k,J_k\in R,$$

$$J=[0,1],\quad 0=t_0<t_1<\cdots<t_k<\cdots<t_m<t_{m+1}=1,\quad J'=J\setminus\{t_1,t_2,\cdots,t_m\}$$

其脉冲项为

$$\Delta u(t_k)=u(t_k^+)-u(t_k^-),\quad \Delta u'(t_k)=u'(t_k^+)-u'(t_k^-)$$

这里$u\left(t_k^+\right)$与$u\left(t_k^-\right)$分别为$u\left(t_k\right)$在$t=t_k$处的右左极限，$k=1,2,\cdots,m$，其中$\varphi_p\left(s\right)=\left|s\right|^{p-2}s$为 p-Laplacian 算子，$p>1$，并且$\varphi_p\left(s\right)$的逆算子$\varphi_q\left(s\right)$，$\frac{1}{p}+\frac{1}{q}=1$。

2. 相关定义及引理

在这一节，为了后面的讨论，我们给出一些定义及相关引理。

令$J=\left[0,1\right]$，$0=t_0<t_1<\cdots<t_k<\cdots<t_m<t_{m+1}=1$，$J_0=\left[0,t_1\right]$，$J_1=\left(t_1,t_2\right]$，$\cdots$，$J_m=\left(t_m,1\right]$，$PC\left(J,R\right)=\left\{u:J\to R\middle|u\in C\left(J_k\right),k=0,1,\cdots,m,\text{并且}u\left(t_k^+\right)\text{存在}\right\}$，其空间上范数为$\|u\|=\sup_{t\in J}\left|u\left(t\right)\right|$，有$PC\left(J,R\right)=\left\{u:J\to R\middle|u\in C\left(J_k\right),k=0,1,\cdots,m,\text{并且}u\left(t_k^+\right),u'\left(t_k^+\right)\text{存在}\right\}$，与其范数$\|u\|_{PC'}=\max_{t\in J}\left\{\|u\|,\|u'\|\right\}$，显然$PC$与$PC'$为 Banach 空间。

定理 1： 设E为 Banach 空间，Ω为E上的非空有界闭凸子集，算子$T:\bar{\Omega}\to E$为全连续算子，使得$\|Tu\|\le\|u\|$，$\forall u\in\bar{\Omega}$，则T在$\bar{\Omega}$存在不动点。

定理 2： 设E为 Banach 空间，$T:E\to E$为全连续算子，$V=\left\{u\in E\middle|u=\mu Tu,0<\mu<1\right\}$，为有界集，则$T$在$\bar{\Omega}$存在不动点。

引理 1： [1] 令$\alpha>0$，则分数阶微分方程${}^CD_{0+}^{\alpha}u\left(t\right)=0$有唯一解

$$u\left(t\right)=c_0+c_1t+\cdots+c_{n-1}t^{n-1},c_i\in R,n=\left[\alpha\right]+1$$

引理 2： [1] 令$\alpha>0$，则有

$$I_{0+}^{\alpha}\,{}^CD_{0+}^{\alpha}u\left(t\right)=u\left(t\right)+c_0+c_1t+\cdots+c_{n-1}t^{n-1},c_i\in R,i=0,1,\cdots,n-1,n=\left[\alpha\right]+1$$

对于边值问题(3)，可通过$\varphi_p\left(s\right)$的逆算子$\varphi_q\left(s\right)$将其转化为等价的边值问题

$$\begin{cases}{}^CD_{0+}^{\alpha}u\left(t\right)=\varphi_q\left(f\left(t,u\left(t\right)\right)\right),t\in J'\\ \Delta\left(u\left(t_k\right)\right)=I_k\left(u\left(t_k\right)\right),\Delta\left(u'\left(t_k\right)\right)=Q_k\left(u\left(t_k\right)\right),\\ au\left(0\right)-bu'\left(0\right)=x_0,cu\left(0\right)+d\,{}^CD_{0+}^{\beta}u\left(1\right)=x_1\end{cases}\tag{4}$$

引理 3： 对于给定函数$y\in C\left[0,1\right]$，$u\left(t\right)$为如下边值问题的解

$$\begin{cases}{}^CD_{0+}^{\alpha}u\left(t\right)=\varphi_q\left(y\left(t\right)\right),t\in J'\\ \Delta\left(u\left(t_k\right)\right)=I_k\left(u\left(t_k\right)\right),\Delta\left(u'\left(t_k\right)\right)=Q_k\left(u\left(t_k\right)\right),\\ au\left(0\right)-bu'\left(0\right)=x_0,cu\left(0\right)+d\,{}^CD_{0+}^{\beta}u\left(1\right)=x_1\end{cases}\tag{5}$$

当且仅当

$$u\left(t\right)=\begin{cases}\frac{1}{\Gamma\left(\alpha\right)}\int_0^t\left(t-s\right)^{\alpha-1}\varphi_q\left(y\left(s\right)\right)\mathrm{d}t-c_1-c_2t, & t\in J_0\\ \frac{1}{\Gamma\left(\alpha\right)}\int_{t_k}^t\left(t-s\right)^{\alpha-1}\varphi_q\left(y\left(s\right)\right)\mathrm{d}t+\sum_{i=1}^{k}\frac{t-t_k}{\Gamma\left(\alpha-1\right)}\int_{t_{-1k}}^{t_k}\left(t-s\right)^{\alpha-2}\varphi_q\left(y\left(s\right)\right)\mathrm{d}t & \\ +\sum_{i=1}^{k-1}\frac{t_k-t_i}{\Gamma\left(\alpha-1\right)}\int_{t_{-1k}}^{t_k}\left(t-s\right)^{\alpha-2}\varphi_q\left(y\left(s\right)\right)\mathrm{d}t+\sum_{i=1}^{k}I_i\left(u\left(t_i\right)\right)+\sum_{i=1}^{k}\left(t-t_k\right)Q_i\left(u\left(t_i\right)\right) & \\ +\sum_{i=1}^{k-1}\left(t_k-t_i\right)Q_i\left(u\left(t_i\right)\right)-c_1-c_2t, & t\in J_k\end{cases}$$

其中

$$c_1=\frac{1}{\delta}\left(\frac{bc}{\Gamma(\alpha)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\varphi_q(y(s))\mathrm{d}s+\frac{bc(1-t_m)}{\Gamma(\alpha-1)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q(y(s))\mathrm{d}s\right.$$
$$+\sum_{i=1}^{m-1}\frac{bc(t_m-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q(y(s))\mathrm{d}s+\sum_{i=1}^{m}bcI_i(u(t_i))+\sum_{i=1}^{m}bc(1-t_m)Q_i(u(t_i))$$
$$+\sum_{i=1}^{m-1}bc(t_m-t_i)Q_i(u(t_i))+\sum_{i=1}^{m+1}\frac{bd}{\Gamma(\alpha-1)\Gamma(2-\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q(y(s))\mathrm{d}s$$
$$\left.+\frac{bd}{\Gamma(2-\beta)}\sum_{i=1}^{m}Q_i(u(t_i))-bx_1-cx_0-\frac{dx_0}{\Gamma(2-\beta)}\right)$$

$$c_2=\frac{1}{\delta}\left(\frac{ac}{\Gamma(\alpha)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\varphi_q(y(s))\mathrm{d}s+\frac{ac(1-t_m)}{\Gamma(\alpha-1)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q(y(s))\mathrm{d}s\right.$$
$$+\sum_{i=1}^{m-1}\frac{ac(t_m-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q(y(s))\mathrm{d}s+\sum_{i=1}^{m}acI_i(u(t_i))+\sum_{i=1}^{m}ac(1-t_m)Q_i(u(t_i))$$
$$+\sum_{i=1}^{m-1}ac(t_m-t_i)Q_i(u(t_i))+\sum_{i=1}^{m+1}\frac{ad}{\Gamma(\alpha-1)\Gamma(2-\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q(y(s))\mathrm{d}s$$
$$\left.+\frac{ad}{\Gamma(2-\beta)}\sum_{i=1}^{m}Q_i(u(t_i))-ax_1-cx_0\right)$$

证明：设$u(t)$为(5)的解，由(2)可得

$$u(t)=I_{0+}^{\alpha}y(t)-c_1-c_2t=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\varphi_q(y(s))\mathrm{d}t-c_1-c_2t,t\in J_0,c_1,c_2\in R$$

则有

$$u'(t)=\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-1}\varphi_q(y(s))\mathrm{d}t-c_2 \tag{6}$$

若$t\in J_1$，有

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_{t_1}^t(t-s)^{\alpha-1}\varphi_q(y(s))\mathrm{d}s-d_1-d_2(t-t_1)$$
$$u'(t)=\frac{1}{\Gamma(\alpha-1)}\int_{t_1}^t(t-s)^{\alpha-2}\varphi_q(y(s))\mathrm{d}s-d_2$$

其中$d_1,d_2\in R$，可得

$$u(t^-)=\frac{1}{\Gamma(\alpha)}\int_{t_1}^t(t-s)^{\alpha-1}\varphi_q(y(s))\mathrm{d}s-c_1-c_2t_1,u(t^+)=-d_1$$

$$u'(t^-)=\frac{1}{\Gamma(\alpha-1)}\int_{t_1}^t(t-s)^{\alpha-2}\varphi_q(y(s))\mathrm{d}s-c_2,u'(t^+)=-d_2$$

由上述定义$\Delta u(t_k)=u(t_k^+)-u(t_k^-)$，$\Delta u'(t_k)=u'(t_k^+)-u'(t_k^-)$，则

$$-d_1=\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\varphi_q(y(s))\mathrm{d}s-c_1-c_2t_1+I_1(u(t_1))$$

$$-d_2=\frac{1}{\Gamma(\alpha-1)}\int_0^{t_1}(t_1-s)^{\alpha-2}\varphi_q(y(s))\mathrm{d}s-c_2+Q_1(u(t_1))$$

因此

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\varphi_q(y(s))\mathrm{d}t+\sum_{i=1}^{k}\frac{t-t_k}{\Gamma(\alpha-1)}\int_{t_{-1k}}^{t_k}(t-s)^{\alpha-2}\varphi_q(y(s))\mathrm{d}t$$
$$+\sum_{i=1}^{k-1}\frac{t_k-t_i}{\Gamma(\alpha-1)}\int_{t_{-1k}}^{t_k}(t-s)^{\alpha-2}\varphi_q(y(s))\mathrm{d}t+\sum_{i=1}^{k}I_i(u(t_i))+\sum_{i=1}^{k}(t-t_k)Q_i(u(t_i))$$
$$+\sum_{i=1}^{k-1}(t_k-t_i)Q_i(u(t_i))-c_1-c_2t$$

由边值条件 $au(0)-bu'(0)=x_0$， $cu(0)+d\,{}^{C}D_{0+}^{\beta}u(1)=x_1$，

$$-ac_1+bc_2=x_0$$

$$cc_1+\left(c+\frac{1}{\Gamma(2-\beta)}\right)$$
$$=\frac{c}{\Gamma(\alpha)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\varphi_q(y(s))\mathrm{d}s+\sum_{i=1}^{k}\frac{c(1-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q(y(s))\mathrm{d}s$$
$$+\sum_{i=1}^{m}cI_i(u(t_i))+\sum_{i=1}^{m}c(1-t_i)Q_i(u(t_i))+\sum_{i=1}^{m-1}bc(t_m-t_i)Q_i(u(t_i))$$
$$+\sum_{i=1}^{m}\frac{d}{\Gamma(2-\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q(y(s))\mathrm{d}s+\frac{d}{\Gamma(2-\beta)}\sum_{i=1}^{m}Q_i(u(t_i))-x_1$$

可得

$$c_1=\frac{1}{\delta}\left(\frac{bc}{\Gamma(\alpha)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\varphi_q(y(s))\mathrm{d}s+\frac{bc(1-t_m)}{\Gamma(\alpha-1)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q(y(s))\mathrm{d}s\right.$$
$$+\sum_{i=1}^{m-1}\frac{bc(t_m-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q(y(s))\mathrm{d}s+\sum_{i=1}^{m}bcI_i(u(t_i))+\sum_{i=1}^{m}bc(1-t_m)Q_i(u(t_i))$$
$$+\sum_{i=1}^{m-1}bc(t_m-t_i)Q_i(u(t_i))+\sum_{i=1}^{m+1}\frac{bd}{\Gamma(\alpha-1)\Gamma(2-\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q(y(s))\mathrm{d}s$$
$$\left.+\frac{bd}{\Gamma(2-\beta)}\sum_{i=1}^{m}Q_i(u(t_i))-bx_1-cx_0-\frac{dx_0}{\Gamma(2-\beta)}\right)$$

$$c_2=\frac{1}{\delta}\left(\frac{ac}{\Gamma(\alpha)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\varphi_q(y(s))\mathrm{d}s+\frac{ac(1-t_m)}{\Gamma(\alpha-1)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q(y(s))\mathrm{d}s\right.$$
$$+\sum_{i=1}^{m-1}\frac{ac(t_m-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q(y(s))\mathrm{d}s+\sum_{i=1}^{m}acI_i(u(t_i))+\sum_{i=1}^{m}ac(1-t_m)Q_i(u(t_i))$$
$$+\sum_{i=1}^{m-1}ac(t_m-t_i)Q_i(u(t_i))+\sum_{i=1}^{m+1}\frac{ad}{\Gamma(\alpha-1)\Gamma(2-\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q(y(s))\mathrm{d}s$$
$$\left.+\frac{ad}{\Gamma(2-\beta)}\sum_{i=1}^{m}Q_i(u(t_i))-ax_1-cx_0\right)$$

命题得证。

3. 主要结果

定义算子 $T:PC(J,R)\to PC(J,R)$ 如下

$$Tu(t)=\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\varphi_q\left(f(s,u)\right)\mathrm{d}t+\sum_{i=1}^{k}\frac{1}{\Gamma(\alpha-1)}\int_{t_{-1k}}^{t_k}(t-s)^{\alpha-2}\varphi_q\left(f(s,u)\right)\mathrm{d}t$$
$$+\sum_{i=1}^{k-1}\frac{t-t_k}{\Gamma(\alpha-1)}\int_{t_{-1k}}^{t_k}(t-s)^{\alpha-2}\varphi_q\left(f(s,u)\right)\mathrm{d}t+\sum_{i=1}^{k-1}\frac{t_k-t_i}{\Gamma(\alpha-1)}\int_{t_{-1k}}^{t_k}(t-s)^{\alpha-2}\varphi_q\left(f(s,u)\right)\mathrm{d}t$$
$$+\sum_{i=1}^{k}I_i\left(u(t_i)\right)+\sum_{i=1}^{k}(t-t_k)Q_i\left(u(t_i)\right)+\sum_{i=1}^{k-1}(t_k-t_i)Q_i\left(u(t_i)\right)-c_1-c_2t$$

其中

$$c_1=\frac{1}{\delta}\left(\frac{bc}{\Gamma(\alpha)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\varphi_q\left(f(s,u)\right)\mathrm{d}s+\frac{bc(1-t_m)}{\Gamma(\alpha-1)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q\left(f(s,u)\right)\mathrm{d}s\right.$$
$$+\sum_{i=1}^{m-1}\frac{bc(t_m-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q\left(f(s,u)\right)\mathrm{d}s+\sum_{i=1}^{m}bcI_i\left(u(t_i)\right)+\sum_{i=1}^{m}bc(1-t_m)Q_i\left(u(t_i)\right)$$
$$+\sum_{i=1}^{m-1}bc(t_m-t_i)Q_i\left(u(t_i)\right)+\sum_{i=1}^{m+1}\frac{bd}{\Gamma(\alpha-1)\Gamma(2-\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q\left(y(s)\right)\mathrm{d}s$$
$$\left.+\frac{bd}{\Gamma(2-\beta)}\sum_{i=1}^{m}Q_i\left(u(t_i)\right)-bx_1-cx_0-\frac{dx_0}{\Gamma(2-\beta)}\right)$$

$$c_2=\frac{1}{\delta}\left(\frac{ac}{\Gamma(\alpha)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\varphi_q\left(f(s,u)\right)\mathrm{d}s+\frac{ac(1-t_m)}{\Gamma(\alpha-1)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q\left(f(s,u)\right)\mathrm{d}s\right.$$
$$+\sum_{i=1}^{m-1}\frac{ac(t_m-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q\left(f(s,u)\right)\mathrm{d}s+\sum_{i=1}^{m}acI_i\left(u(t_i)\right)+\sum_{i=1}^{m}ac(1-t_m)Q_i\left(u(t_i)\right)$$
$$+\sum_{i=1}^{m-1}ac(t_m-t_i)Q_i\left(u(t_i)\right)+\sum_{i=1}^{m+1}\frac{ad}{\Gamma(\alpha-1)\Gamma(2-\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\varphi_q\left(f(s,u)\right)\mathrm{d}s$$
$$\left.+\frac{ad}{\Gamma(2-\beta)}\sum_{i=1}^{m}Q_i\left(u(t_i)\right)-ax_1-cx_0\right)$$

由**引理 3**，令 $y(t)=f\left(t,u(t)\right)$ 则边值问题(3)有解等价存在不动点，即 $u=Tu$，于是(3)有解当且仅当 $u=Tu$。

定理 3： 令 $\lim\limits_{u\to0}\frac{\varphi_q\left(f(t,u)\right)}{u}=0$，$\lim\limits_{u\to0}\frac{I_k(u)}{u}=0$，$\lim\limits_{u\to0}\frac{Q_k(u)}{u}=0$，并且有

$$\Lambda=\frac{1}{\delta}\frac{\delta_1}{\Gamma(\alpha+1)}(m+1)\left(c(a+b)+\delta\right)+\frac{1}{\delta}\frac{\delta_1}{\Gamma(\alpha+1)}\left((2m-1)\left(c(a+b)+\delta\right)+\frac{(m+1)(a+b)d}{\Gamma(2-\beta)}\right)$$
$$+\frac{m\left(c(a+b)+\delta\right)}{\delta}\delta_2+\frac{1}{\delta}\left((2m-1)\left(c(a+b)+\delta\right)+\frac{md(a+b)}{\Gamma(2-\beta)}\right)\delta_3$$
$$+\frac{1}{\delta}\left((a+b)|x_1|+\frac{2c+d}{\Gamma(2-\beta)}|x_0|\right)\le1$$

证明： 首先，证明 $T:PC(J,R)\to PC(J,R)$ 是全连续算子，T 作用于 f,I_k,Q_k 均连续，

设 Ω 为 $T:PC(J,R)\to PC(J,R)$ 的有界子集，则存在常数 $L_i>0\,(i=1,2,3,4)$ 使得 $\left|f(t,u)\right|\le L_1$，$\left|I_k(u)\right|\le L_2$，$\left|Q_k(u)\right|\le L_3$，$\left|\varphi_q\left(f(t,u)\right)\right|\le L_4$，又

$$
\begin{aligned}
|c_1| \le & \frac{1}{\delta}\left(\frac{bc}{\Gamma(\alpha)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\left|\varphi_q\left(f(s,u)\right)\right|\mathrm{d}s+\frac{bc(1-t_m)}{\Gamma(\alpha-1)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|\varphi_q\left(f(s,u)\right)\right|\mathrm{d}s\right.\\
&+\sum_{i=1}^{m-1}\frac{bc(t_m-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|\varphi_q\left(f(s,u)\right)\right|\mathrm{d}s+\sum_{i=1}^{m}bc\left|I_i\left(u(t_i)\right)\right|+\sum_{i=1}^{m}bc(1-t_m)\left|Q_i\left(u(t_i)\right)\right|\\
&+\sum_{i=1}^{m-1}bc(t_m-t_i)\left|Q_i\left(u(t_i)\right)\right|+\sum_{i=1}^{m+1}\frac{bd}{\Gamma(\alpha-1)\Gamma(2-\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|\varphi_q\left(f(s,u)\right)\right|\mathrm{d}s\\
&\left.+\frac{bd}{\Gamma(2-\beta)}\sum_{i=1}^{m}\left|Q_i\left(u(t_i)\right)\right|+b|x_1|+c|x_0|+\frac{d|x_0|}{\Gamma(2-\beta)}\right)\\
\le & \frac{1}{\delta}\left(\frac{bcL_4}{\Gamma(\alpha)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\mathrm{d}s+\frac{bcL_4}{\Gamma(\alpha-1)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\mathrm{d}s+\sum_{i=1}^{m-1}\frac{bcL_4}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\mathrm{d}s\right.\\
&+\sum_{i=1}^{m}bcL_2+\sum_{i=1}^{m}bc(1-t_m)L_3+\sum_{i=1}^{m-1}bc(t_m-t_i)L_3+\sum_{i=1}^{m+1}\frac{bdL_4}{\Gamma(\alpha-1)\Gamma(2-\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\mathrm{d}s\\
&\left.+\frac{bd}{\Gamma(2-\beta)}\sum_{i=1}^{m}L_3+b|x_1|+c|x_0|+\frac{d|x_0|}{\Gamma(2-\beta)}\right)\\
\le & \frac{1}{\delta}\frac{bc(m+1)L_4}{\Gamma(\alpha+1)}+\frac{1}{\delta}\frac{bcmL_4}{\Gamma(\alpha)}+\frac{1}{\delta}\sum_{i=1}^{m-1}\frac{bcmL_4}{\Gamma(\alpha)}+\frac{bcmL_2}{\delta}+\frac{bcmL_3}{\delta}+\frac{bc(m-1)L_3}{\delta}\\
&+\frac{1}{\delta}\frac{bd(m+1)L_4}{\Gamma(\alpha)\Gamma(2-\beta)}+\frac{1}{\delta}\frac{bdmL_3}{\Gamma(2-\beta)}+\frac{1}{\delta}b|x_1|+\frac{1}{\delta}c|x_0|+\frac{1}{\delta}\frac{d|x_0|}{\Gamma(2-\beta)}
\end{aligned}
$$

$$
\begin{aligned}
|c_2| \le & \frac{1}{\delta}\left(\frac{ac}{\Gamma(\alpha)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\left|\varphi_q\left(f(s,u)\right)\right|\mathrm{d}s+\frac{ac(1-t_m)}{\Gamma(\alpha-1)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|\varphi_q\left(f(s,u)\right)\right|\mathrm{d}s\right.\\
&+\sum_{i=1}^{m-1}\frac{ac(t_m-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|\varphi_q\left(f(s,u)\right)\right|\mathrm{d}s+\sum_{i=1}^{m}ac\left|I_i\left(u(t_i)\right)\right|+\sum_{i=1}^{m}ac(1-t_m)\left|Q_i\left(u(t_i)\right)\right|\\
&+\sum_{i=1}^{m-1}ac(t_m-t_i)\left|Q_i\left(u(t_i)\right)\right|+\sum_{i=1}^{m+1}\frac{ad}{\Gamma(\alpha-1)\Gamma(2-\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\left|\varphi_q\left(f(s,u)\right)\right|\mathrm{d}s\\
&\left.+\frac{ad}{\Gamma(2-\beta)}\sum_{i=1}^{m}\left|Q_i\left(u(t_i)\right)\right|-a|x_1|-c|x_0|\right)\\
\le & \frac{1}{\delta}\left(\frac{acL_4}{\Gamma(\alpha)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\mathrm{d}s+\frac{acL_4}{\Gamma(\alpha-1)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\mathrm{d}s+\sum_{i=1}^{m-1}\frac{acL_4}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\mathrm{d}s\right.\\
&+\sum_{i=1}^{m}acL_2+\sum_{i=1}^{m}ac(1-t_m)L_3+\sum_{i=1}^{m-1}ac(t_m-t_i)L_3+\sum_{i=1}^{m+1}\frac{adL_4}{\Gamma(\alpha-1)\Gamma(2-\beta)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\mathrm{d}s\\
&\left.+\frac{ad}{\Gamma(2-\beta)}\sum_{i=1}^{m}L_3-a|x_1|-c|x_0|\right)\\
\le & \frac{1}{\delta}\frac{ac(m+1)L_4}{\Gamma(\alpha+1)}+\frac{1}{\delta}\frac{acmL_4}{\Gamma(\alpha)}+\frac{1}{\delta}\frac{ac(m-1)L_4}{\Gamma(\alpha)}+\frac{acmL_2}{\delta}+\frac{acmL_3}{\delta}\\
&+\frac{ac(m-1)L_3}{\delta}+\frac{1}{\delta}\frac{ad(m+1)L_4}{\Gamma(\alpha)\Gamma(2-\beta)}+\frac{1}{\delta}\frac{admL_3}{\Gamma(2-\beta)}+\frac{1}{\delta}a|x_1|+\frac{1}{\delta}c|x_0|
\end{aligned}
$$

$$|Tu(t)| \le \frac{1}{\delta}\frac{L_4}{\Gamma(\alpha+1)}(m+1)(c(a+b)+\delta)+\frac{1}{\delta}\frac{L_4}{\Gamma(\alpha+1)}\left((2m-1)(c(a+b)+\delta)+\frac{(m+1)(a+b)d}{\Gamma(2-\beta)}\right)$$
$$+\frac{m(c(a+b)+\delta)}{\delta}L_2+\frac{m}{\delta}\left((2p-1)(\delta+c(a+b))+\frac{md(a+b)}{\Gamma(2-\beta)}\right)L_3+\frac{1}{\delta}\left((a+b)|x_1|+\frac{2c+d}{\Gamma(2-\beta)}|x_0|\right):=L$$

因此$\|Tu\| \le L$。

另一方面

$$|Tu'(t)| \le \frac{1}{\delta}\frac{ac(m+1)L_4}{\Gamma(\alpha+1)}+\frac{acmL_2}{\delta}+\frac{L_3}{\delta}\left((m+1)\left(\delta+\frac{ad}{\Gamma(2-\beta)}\right)+ac(2m-1)\right)+\frac{(a|x_1|+c|x_0|)}{\delta}:=\bar{L}$$

因此对于$t_1,t_2 \in J_k, t_1 \le t_2, 0 \le k \le m$有

$$|Tu(t_2)-Tu(t_1)| \le \int_{t_1}^{t_2}(Tu)'(s)\mathrm{d}s \le \bar{L}(t_2-t_1)$$

则t在每一个子空间等度连续，由 Arzela-Ascoli 定理，T为全连续算子，由条件$\lim_{u\to 0}\frac{\varphi_q(f(t,u))}{u}=0$，$\lim_{u\to 0}\frac{I_k(u)}{u}=0$，$\lim_{u\to 0}\frac{Q_k(u)}{u}=0$，存在常数$r>0, \delta_i>0 (i=1,2,3)$有$|\varphi_q(f(t,u))| \le \delta_1|u|$，$|I_i(u)| \le \delta_2|u|$，$|Q_i(u)| \le \delta_3|u|$，由

$$\Lambda=\frac{1}{\delta}\frac{\delta_1}{\Gamma(\alpha+1)}(m+1)(c(a+b)+\delta)+\frac{1}{\delta}\frac{\delta_1}{\Gamma(\alpha+1)}\left((2m-1)(c(a+b)+\delta)+\frac{(m+1)(a+b)d}{\Gamma(2-\beta)}\right)$$
$$+\frac{m(c(a+b)+\delta)}{\delta}\delta_2+\frac{1}{\delta}\left((2m-1)(c(a+b)+\delta)+\frac{md(a+b)}{\Gamma(2-\beta)}\right)\delta_3+\frac{1}{\delta}\left((a+b)|x_1|+\frac{2c+d}{\Gamma(2-\beta)}|x_0|\right)\le 1$$

对于$u \in PC(J,R)$，定义$\Omega=\{u \in PC(J,R) \mid \|u\|<r\}$，$u \in \partial\Omega$时，$\|u\|=r$，有

$$|Tu(t)| \le \left\{\frac{1}{\delta}\frac{\delta_1}{\Gamma(\alpha+1)}(m+1)(c(a+b)+\delta)+\frac{1}{\delta}\frac{\delta_1}{\Gamma(\alpha+1)}\left((2m-1)(c(a+b)+\delta)+\frac{(m+1)(a+b)d}{\Gamma(2-\beta)}\right)\right.$$
$$\left.+\frac{m(c(a+b)+\delta)}{\delta}\delta_2+\frac{1}{\delta}\left((2m-1)(c(a+b)+\delta)+\frac{md(a+b)}{\Gamma(2-\beta)}\right)\delta_3+\frac{1}{\delta}\left((a+b)|x_1|+\frac{2c+d}{\Gamma(2-\beta)}|x_0|\right)\right\}\|u\|_{PC'}$$

有$\|Tu\|_{PC'} \le \|u\|_{PC'}$，由**定理 1** 算子$T$至少存在一个不动点，则(3)至少存在一个解。

定理 4： 假设存在正常数$L_i \ge 0 (i=1,2,3,4)$使得$|f(t,u)| \le L_1$，$|I_k(u)| \le L_2$，$|Q_k(u)| \le L_3$，$|\varphi_q(f(t,u))| \le L_4$，则(3)至少存在一个解。

证明：由**定理 3**，算子T为全连续算子，此时假设$V=\{u \in E \mid u=\mu Tu, 0<\mu<u\}$为有界集，令$u \in V$，我们有

$$u(t)=\frac{\mu}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\varphi_q(f(s,u))\mathrm{d}t+\frac{\mu}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_k}^{t}(t-s)^{\alpha-1}\varphi_q(f(s,u))\mathrm{d}t$$
$$+\sum_{i=1}^{k}\frac{\mu(t-t_k)}{\Gamma(\alpha-1)}\int_{t_{-1k}}^{t_k}(t-s)^{\alpha-2}\varphi_q(f(s,u))\mathrm{d}t+\sum_{i=1}^{k-1}\frac{\mu(t_k-t_i)}{\Gamma(\alpha-1)}\int_{t_{-1k}}^{t_k}(t-s)^{\alpha-2}\varphi_q(f(s,u))\mathrm{d}t$$
$$+\sum_{i=1}^{k}\mu I_i(u(t_i))+\sum_{i=1}^{k}\mu(t-t_k)Q_i(u(t_i))+\sum_{i=1}^{k-1}\mu(t_k-t_i)Q_i(u(t_i))-\mu c_1-\mu c_2 t$$

由条件，有

$$
\begin{aligned}
|u(t)| &= \mu|Tu(t)| \\
&\le \frac{1}{\delta}\frac{\delta_1}{\Gamma(\alpha+1)}(m+1)(c(a+b)+\delta)+\frac{1}{\delta}\frac{\delta_1}{\Gamma(\alpha+1)}\left((2m-1)(c(a+b)+\delta)+\frac{(m+1)(a+b)d}{\Gamma(2-\beta)}\right) \\
&\quad +\frac{m(c(a+b)+\delta)}{\delta}\delta_2+\frac{1}{\delta}\left((2m-1)(c(a+b)+\delta)+\frac{md(a+b)}{\Gamma(2-\beta)}\right)\delta_3+\frac{1}{\delta}\left((a+b)|x_1|+\frac{2c+d}{\Gamma(2-\beta)}|x_0|\right)
\end{aligned}
$$

则对 $t\in J$，

$$
\begin{aligned}
\|u(t)\|_{PC'} &\le \frac{1}{\delta}\frac{\delta_1}{\Gamma(\alpha+1)}(m+1)(c(a+b)+\delta)+\frac{1}{\delta}\frac{\delta_1}{\Gamma(\alpha+1)}\left((2m-1)(c(a+b)+\delta)+\frac{(m+1)(a+b)d}{\Gamma(2-\beta)}\right) \\
&\quad +\frac{m(c(a+b)+\delta)}{\delta}\delta_2+\frac{1}{\delta}\left((2m-1)(c(a+b)+\delta)+\frac{md(a+b)}{\Gamma(2-\beta)}\right)\delta_3+\frac{1}{\delta}\left((a+b)|x_1|+\frac{2c+d}{\Gamma(2-\beta)}|x_0|\right)
\end{aligned}
$$

又由 V 为有界集，故由**定理 2** 算子 T 至少存在一个不动点，则(3)至少存在一个解。

4. 例子

对于如下具 p-Laplacian 算子的非线性分数阶脉冲微分方程边值问题

$$
\begin{cases}
\varphi_{5/3}\left({}^{C}D_{0+}^{\frac{5}{4}}u(t)\right)=\dfrac{\cos t}{\left(t+5\sqrt{2}\right)^2}\dfrac{u(t)}{1+u(t)}, \\
\Delta\left(u\left(\dfrac{1}{2}\right)\right)=\dfrac{\left|u\left(\frac{1}{2}\right)\right|}{20+\left|u\left(\frac{1}{2}\right)\right|},\Delta\left(u'\left(\dfrac{1}{2}\right)\right)=Q_k\left(\dfrac{\left|u\left(\frac{1}{2}\right)\right|}{20+\left|u\left(\frac{1}{2}\right)\right|}\right), \\
u(0)-u'(0)=0,u(0)+{}^{C}D_{0+}^{\frac{1}{2}}u(1)=0
\end{cases}
$$

这里 $p=\frac{5}{3},\alpha=\frac{5}{4},a=b=c=d=m=1,\beta=\frac{1}{2},x_0=x_1=0$，取 $L_1=\frac{1}{50},L_2=\frac{1}{20},L_3=\frac{1}{40}$ 有 $\Lambda\le 1$ 满足条件，由**定理 3** 和**定理 4**，边值问题(3)至少存在一个解。

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