

Nonexistence of Some Finite p -Groups of the Central Quotient Order of p^6

Min Hui

Baoji University of Arts and Sciences, Baoji Shaanxi
Email: bwlhuimin@163.com

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Abstract

Based on Rodney James' paper (The groups of order p^6 (p an odd prime), Mathematics of Computation, 1980, 34 (150): 613-637. [1]) and the idea of Schreier extension theory, act extended element on extended group, by the transposition substructure and the power structure we get the contradiction of $Z(G)$ that is cyclical group, and then we get a class finite p -groups that the central quotient are nonexistence, that is to say when H are the groups of Φ_{39} family of order p^6 and satisfied $G/Z(G) \cong H$, we get the nonexistence of G .

Keywords

Finite p -group, LA-Group, Central Quotient, Order

若干中心商的阶为 p^6 的有限 p -群的不存在性问题

惠 敏

宝鸡文理学院, 陕西 宝鸡
Email: bwlhuimin@163.com

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摘 要

基于Rodney James的文(The groups of order p^6 (p an odd prime). Mathematics of Computation, 1980, 34 (150): 613-637. [1])和Schreier扩张理论的思想, 将被扩元作用于被扩群, 通过换位子结构及

幂结构得到存在与 $Z(G)$ 为循环群的矛盾, 进而得到一类中心商不存在的有限 p -群, 即给出当 H 为 p^6 阶 Φ_{39} 家族中的群且满足条件 $G/Z(G) \cong H$ 时群 G 的不存在性问题。

关键词

有限 p -群, LA-群, 中心商, 阶

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1. 引言

在有限群自同构群的研究中, 值得一提的是悬而未决的著名的 LA-猜想: 设 G 是阶大于 p^2 的有限非循环 p -群, 则必有 $|G||\text{Aut}(G)|$, 满足 LA-猜想的群称为 LA-群。通过计算自同构群的阶来判断一个群是否是 LA-群是很困难的, 因为计算自同构群的阶比较复杂且能用到的工具也比较少[2] [3]。对于这个猜想的研究到目前已有半个多世纪, 但还未得到彻底解决。基于前人对满足条件 $|G/Z(G)| \leq p^4$ 和 $|G/Z(G)| \leq p^5$ 的 LA-群的研究[4] [5], 开始对满足条件 $|G/Z(G)| \leq p^6$ 的 LA-群进行研究, 近年来已产生了许多好的结果[6]-[12]。尽管如此, 中心商的阶为 p^6 的有限 p -群是否全是 LA-群还没有完全确立, 本文在 Rodney James 文章[1] Φ_{39} 家族群的基础上, 研究了这些群均是中心商不存在的有限 p -群, 进而这些群一定不是满足条件 $|G/Z(G)| = p^6$ 的 LA-群, 这对中心商的阶为 p^6 的 LA-猜想的完全解决具有一定的意义。

2. 基本引理

引理 2.1: 令 $Z \leq Z(G)$, $G/Z = \bar{H}$ 。如果群 H 包含元素 w 和生成子集 S , 使得 S 中每一个元素的某个相同次幂全都等于 w , 则有 $[w, H] = 1, w \in Z(G)$ 。

证明: 令 $s \in S, s^m = w$, 因为 $[w, s] = [s^m, s] = 1$, 所以 $[w, H] = 1$ 。又因为 $G = HZ$, 所以 $w \in Z(G)$ 。

引理 2.2: 令 $Z \leq Z(G)$, $G/Z = \bar{A}\bar{B}$, $[\bar{A}, \bar{B}] = \bar{1}$, 则 $[A', B] = 1$ 。

证明: 令 $a = [x, y], x, y \in A, b \in B$, 因为 $a[a, b] = a^b = [x, y]^b = [x^b, y^b] = [x, y] = a$, 所以 $[a, b] = 1$ 。因为 $A' = [A, A] = \langle [x, y] \mid x, y \in A \rangle$, 所以 $[A', B] = 1$ 。

根据引理 2.1 和引理 2.2, 得到如下推论:

推论 2.3: 令 $A = HK, [H, K] = 1, K$ 是 A 的子集, 并且 $H \leq A$, 群 A 包含非单位元 w 和生成子集 S , 使得 S 中的每个元素的某个相同次幂全都等于 w , 若 $w \in H'$, 则不存在群 G , 使得 $G/Z(G) \cong A$ 。

引理 2.4 [13]: 令 n, k 是非负整数, $x, y \in G, c_2 = [y, x], c_{31} = [y, 2x], c_{32} = [y, x, y], c_{41} = [y, 3x], c_{42} = [y, 2x, y], c_{44} = [y, x, 2y], c_{51} = [y, 4x], c_{52} = [y, 3x, y], c_{54} = [y, 2x, 2y], c_{58} = [y, x, 3y], d_1 = [y, 2x, [y, x]], d_2 = [y, x, y, [y, x]]$ 。如果 $c(G) \leq 5$,

则

$$(xy)^n = x^n y^n c_2^{\binom{n}{2}} c_{31}^{\binom{n}{3}} c_{32}^{\binom{n}{2} + 2\binom{n}{3}} c_{41}^{\binom{n}{4}} c_{42}^{\binom{n}{3} + 3\binom{n}{4}} c_{44}^{2\binom{n}{3} + 3\binom{n}{4}} cd,$$

其中

$$c = c_{51}^{\binom{n}{5}} c_{52}^{3\binom{n}{4}+4\binom{n}{5}} c_{54}^{\binom{n}{3}+6\binom{n}{4}+6\binom{n}{5}} c_{58}^{3\binom{n}{4}+4\binom{n}{5}}, d = d_1^{\binom{n}{3}+7\binom{n}{4}+6\binom{n}{5}} d_2^{6\binom{n}{3}+18\binom{n}{4}+12\binom{n}{5}},$$

$$[y^k, x^n] = c_2^{nk} c_{31}^{\binom{n}{2}k} c_{32}^{n\binom{k}{2}} c_{41}^{\binom{n}{3}k} c_{42}^{\binom{n}{2}\binom{k}{2}} c_{44}^{n\binom{k}{3}} c_{51}^{\binom{n}{4}k} c_{52}^{\binom{n}{3}\binom{k}{2}} c_{54}^{\binom{n}{2}\binom{k}{3}} c_{58}^{n\binom{k}{4}} e,$$

其中

$$e = d_1^{n\binom{n}{2}\binom{k}{2}+\binom{n}{3}k} d_2^{\binom{n}{2}\binom{k}{2}+n^2\binom{k}{3}}.$$

引理 2.5 [14]: 设 G 是群, $a, b \in G$ 且 $[a, b] \in Z(G)$, 又设 n 是正整数. 则有

- 1) $[a^n, b] = [a, b]^n$;
- 2) $[a, b^n] = [a, b]^n$;
- 3) $(ab)^n = a^n b^n [b, a]^{\binom{n}{2}}$.

引理 2.6 [14]: 设 G 是亚交换群, $x, y, z \in G$,

- 1) 若 $y \in G'$, 则 $[z, xy] = [z, x][z, y]$, $[xy, z] = [x, z][y, z]$;
- 2) 对 $\forall x, y, z \in G$, 有 $[x, y, z][y, z, x][z, x, y] = 1$.

引理 2.7 [14]: 设 G 是亚交换群, $a, b \in G, m \geq 2, i, j$ 为正整数,

$$[ia, jb] = \left[a, b, \underbrace{a, \dots, a}_{i-1}, \underbrace{b, \dots, b}_{j-1} \right],$$

则

$$(ab^{-1})^m = a^m \prod_{i+j \leq m} [ia, jb]^{\binom{m}{i+j}} b^{-m}.$$

3. 主要结果

定理: H 为 p^6 阶第三十九家族的群, 当 $H = \Phi_{39}(21^4)a_r, \Phi_{39}(21^4)b_r, \Phi_{39}(21^4)b_{p+r}, \Phi_{39}(1^6)$ 时, 不存在群 G 使得 $G/Z(G) \cong H$.

证明: 令 $H = \Phi_{39}$, 则我们有 $H = \langle \alpha, \alpha_1 \rangle$, $H_{i+1} = \langle \alpha_{i+1}, \dots, \alpha_5 \rangle$. 此时令 $S = \langle \alpha_2, \alpha \rangle$, $T = \langle \alpha_1, \alpha_2, \alpha_3 \rangle$, 则 $S_2 = H_3, T_2 = H_4$, 所以 S 和 T 是亚交换群. 令 $x = \alpha, y = \alpha_1^x$, 由引理 2.4, 知

$$c_2 = [\alpha_1^x, \alpha] = \alpha_1^{-x} (\alpha_1 \alpha_2)^x = \alpha_1^{-x} \alpha_1^x \prod_{i+j \leq p} [ia_1, ja_2^{-1}]^{\binom{p}{i+j}} \alpha_2^x = \alpha_4^{-\binom{x}{2}} [\alpha_4^{-1}, \alpha_1]^{\binom{x}{3}} \alpha_2^x = \alpha_2^x \alpha_4^{-\binom{x}{2}} \alpha_5^{-\binom{x}{3}}$$

$$c_{31} = [c_2, \alpha] = \left[\alpha_2^x \alpha_4^{-\binom{x}{2}} \alpha_5^{-\binom{x}{3}}, \alpha \right] = [\alpha_2^x, \alpha] = \alpha_3^x \alpha_5^{-\binom{x}{2}},$$

$$c_{32} = [c_2, \alpha_1^x] = \left[\alpha_2^x \alpha_4^{-\binom{x}{2}} \alpha_5^{-\binom{x}{3}}, \alpha_1^x \right] = [\alpha_2, \alpha_1^x]^x \alpha_5^{-x\binom{x}{2}} = \alpha_4^{-x^2} \alpha_5^{-2x\binom{x}{2}},$$

$$c_{41} = [c_{31}, \alpha] = \left[\alpha_3^x \alpha_5^{-\binom{x}{2}}, \alpha \right] = \alpha_4^x, c_{42} = [c_{31}, \alpha_1^x] = \left[\alpha_3^x \alpha_5^{-\binom{x}{2}}, \alpha_1^x \right] = \alpha_5^{x^2},$$

$$c_{44} = [c_{32}, \alpha_1^x] = \left[\alpha_4^{-x^2} \alpha_5^{-2x \binom{x}{2}}, \alpha_1^x \right] = \alpha_5^{-x^3}, c_{51} = [c_{41}, \alpha] = [\alpha_4^x, \alpha] = 1,$$

$$c_{52} = [\alpha_4^x, \alpha_1^x] = \alpha_5^{x^2}, c_{54} = [c_{42}, \alpha_1^x] = [\alpha_5^{x^2}, \alpha_1^x] = 1,$$

$$c_{58} = [c_{44}, \alpha_1^x] = [\alpha_5^{-x^3}, \alpha_1^x] = 1, d_1 = [c_{31}, c_2] = \alpha_5^{-x^2},$$

$$d_2 = [c_{32}, c_2] = \left[\alpha_4^{-x^2} \alpha_5^{-2x \binom{x}{2}}, \alpha_2^x \alpha_4^{-\binom{x}{2}} \alpha_5^{-\binom{x}{3}} \right] = 1, (\alpha \alpha_1^x)^p = \alpha^p \alpha_1^{xp} \alpha_5^{-2x^2 \binom{p}{5}}.$$

1) 令 $H = \Phi_{39}(21^4)_{a_r} \cong G/Z(G) = \langle \bar{\alpha}, \bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3, \bar{\alpha}_4, \bar{\alpha}_5 \mid [\bar{\alpha}_i, \bar{\alpha}] = \bar{\alpha}_{i+1}, [\bar{\alpha}_4, \bar{\alpha}_1]^k = \bar{\alpha}^p = [\bar{\alpha}_2, \bar{\alpha}_3]^k = [\bar{\alpha}_3, \bar{\alpha}_1]^k = \bar{\alpha}_5^{-k}, [\bar{\alpha}_1, \bar{\alpha}_2] = \bar{\alpha}_4, \bar{\alpha}_1^{-p} = \bar{\alpha}_{i+1}^{-p} = \bar{\alpha}_5^{-p} = \bar{1} (i=1,2,3) \rangle$, 则有 $(\alpha \alpha_1^x)^p = \alpha_5^{k-2x^2 \binom{p}{5}}$ 。如果 $p > 5$, 则不存在群 G 使得 $G/Z(G) \cong H$ 。此时若假设 $p = 5$, 因为 $k - 2x^2 \equiv 0$ 至多有两个根, 所以存在 x 且 $1 \leq x \leq 4$ 使得 $k - 2x^2 \neq 0$ 。因为 $H = \langle \alpha, \alpha \alpha_1^x \rangle$, 所以不存在群 G 使得 $G/Z(G) \cong H$ 。

2) 令 $H = \Phi_{39}(21^4)_{b_r} \cong G/Z(G) = \langle \bar{\alpha}, \bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3, \bar{\alpha}_4, \bar{\alpha}_5 \mid [\bar{\alpha}_i, \bar{\alpha}] = \bar{\alpha}_{i+1}, [\bar{\alpha}_4, \bar{\alpha}_1]^r = \bar{\alpha}^p = [\bar{\alpha}_2, \bar{\alpha}_3]^r = \bar{\alpha}_1^{-p} = \bar{\alpha}_5^{-r}, [\bar{\alpha}_1, \bar{\alpha}_2] = \bar{\alpha}_4, \bar{\alpha}_{i+1}^{-p} = \bar{\alpha}_5^{-p} = \bar{1} (i=1,2,3) \rangle$, 则 $(\alpha \alpha_1^x)^p = \alpha_5^{r+rx-2x^2 \binom{p}{5}}$ 。如果 $p > 5$, 则不存在群 G 使得 $G/Z(G) \cong H$ 。此时假设 $p = 5$, 则 $(\alpha \alpha_1^x)^5 = \alpha_5^{r+rx-2x^2}$ 。因为 $r + rx - 2x^2 \equiv 0$ 至多有两个根, 所以存在 x 且 $1 \leq x \leq 4$ 使得 $r + rx - 2x^2 \neq 0$ 。因为 $H = \langle \alpha, \alpha \alpha_1^x \rangle$, 所以不存在群 G 使得 $G/Z(G) \cong H$ 。

3) 令 $H = \Phi_{39}(21^4)_{b_{p+r}} \cong G/Z(G) = \langle \bar{\alpha}, \bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3, \bar{\alpha}_4, \bar{\alpha}_5 \mid [\bar{\alpha}_i, \bar{\alpha}] = \bar{\alpha}_{i+1}, [\bar{\alpha}_1, \bar{\alpha}_2] = \bar{\alpha}_4, [\bar{\alpha}_2, \bar{\alpha}_3]^k = [\bar{\alpha}_3, \bar{\alpha}_1]^k = [\bar{\alpha}_4, \bar{\alpha}_1]^k = \bar{\alpha}_1^{-p} = \bar{\alpha}_5^{-k}, \bar{\alpha}^p = \bar{\alpha}_{i+1}^{-p} = \bar{\alpha}_5^{-p} = \bar{1} (i=1,2,3) \rangle$, 则 $(\alpha \alpha_1^x)^p = \alpha_5^{kx-2x^2 \binom{p}{5}}$ 。如果 $p > 5$, 则不存在群 G 使得 $G/Z(G) \cong H$ 。此时假设 $p = 5$, 则 $(\alpha \alpha_1^x)^5 = \alpha_5^{kx-2x^2}$ 。因为 $kx - 2x^2 \equiv 0$ 至多有两个根, 所以存在 x 使 $kx - 2x^2 \neq 0$ 。因为 $\langle \alpha_1, \alpha \alpha_1^x \rangle = H$, 所以不存在群 G 使得 $G/Z(G) \cong H$ 。

4) 令 $H = \Phi_{39}(1^6) \cong G/Z(G) = \langle \bar{\alpha}, \bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3, \bar{\alpha}_4, \bar{\alpha}_5 \mid [\bar{\alpha}_i, \bar{\alpha}] = \bar{\alpha}_{i+1}, [\bar{\alpha}_1, \bar{\alpha}_2] = \bar{\alpha}_4, \bar{\alpha}_1^{-p} = \bar{\alpha}^p = \bar{\alpha}_{i+1}^{-p} = \bar{\alpha}_5^{-p} = \bar{1}, [\bar{\alpha}_2, \bar{\alpha}_3] = [\bar{\alpha}_3, \bar{\alpha}_1] = [\bar{\alpha}_4, \bar{\alpha}_1] = \bar{\alpha}_5, (i=1,2,3) \rangle$, 则 $(\alpha \alpha_1^x)^5 = \alpha_5^{-2x^2}$ 。因为 $H = \langle \alpha \alpha_1, \alpha \alpha_1^2 \rangle$, 如果 $p = 5$, 则不存在群 G 使得 $G/Z(G) \cong H$ 。此时假设 $p > 5$ 。不妨设 $[\alpha_i, \alpha] = \alpha_{i+1}$, $[\alpha_4, \alpha_1] = \alpha_5$, 由于 $[\alpha_i, \alpha] = \alpha_{i+1} a_i$, $[\alpha_4, \alpha_1] = \alpha_5 b$, $a_i, b \in Z(G)$, 令 $\alpha'_{i+1} = \alpha_{i+1} a_i$, $\alpha'_5 = \alpha_5 b$, 则 $\alpha_{i+1} = \alpha'_{i+1} a_i^{-1}$, $\alpha_5 = \alpha'_5 b^{-1}$ 。此时 $[\alpha, \alpha_{i+3}], [\alpha_1, \alpha_5], [\alpha_4, \alpha_5], [\alpha_2, \alpha_{i+3}], [\alpha_3, \alpha_{i+3}] \in Z(G)$ 。因为 $[\alpha_5, \alpha_i] = [\alpha_5, \alpha_i]^\alpha = [\alpha_5, \alpha_i \alpha_{i+1}] = [\alpha_5, \alpha_{i+1}] [\alpha_5, \alpha_i]$, 所以 $[\alpha_5, \alpha_{i+1}] = 1$ 。因为 $[\alpha_2, \alpha_4] = [\alpha_2, \alpha_4]^\alpha = [\alpha_2^\alpha, \alpha_4^\alpha] = [\alpha_2 \alpha_3, \alpha_4] = [\alpha_2, \alpha_4] [\alpha_3, \alpha_4]$, 所以 $[\alpha_3, \alpha_4] = 1$ 。因为 $\alpha_5^\alpha = [\alpha_4^\alpha, \alpha_1^\alpha] = [\alpha_4, \alpha_1 \alpha_2] = [\alpha_4, \alpha_2] \alpha_5$, 所以 $[\alpha_5, \alpha] = [\alpha_4, \alpha_2]$ 。因为 $[\alpha_3, \alpha_1] = [\alpha_3, \alpha_1]^{\alpha^2} = [\alpha_3^{\alpha^2}, \alpha_1^{\alpha^2}] = [\alpha_3 \alpha_5^{-1}, \alpha_1 \alpha_4] = [\alpha_3, \alpha_1] [\alpha_5^{-1}, \alpha_1]$, 所以 $[\alpha_5, \alpha_1] = 1$ 。令 $[\alpha_2, \alpha_3] = \alpha_5 z$,

其中 $z \in Z(G)$, 则因为有 $\alpha_5[\alpha_5, \alpha] = \alpha_5^\alpha = [\alpha_2, \alpha_3]^\alpha z^{-1} = [\alpha_2\alpha_3, \alpha_3\alpha_4] z^{-1} = [\alpha_2, \alpha_4][\alpha_2, \alpha_3] z^{-1} = [\alpha_2, \alpha_4]\alpha_5$, 所以 $[\alpha_5, \alpha] = [\alpha_2, \alpha_4]$, 这意味着 $[\alpha_5, \alpha] = 1$. 又因为 $G = \langle \alpha, \alpha_1, Z(G) \rangle$, $\alpha_5 \in Z(G)$, 矛盾, 所以不存在群 G 使得 $G/Z(G) \cong H$. 至此, 定理证明完毕。

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