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The Construction and the *Mallat* Algorithm of Biorthogonal Two Dimensional Four-Direction Multi-Wavelet with Dilation Factor *a*

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Abstract

Through the concept of two dimensional four-direction multi-scaling function and two dimensional four-direction multi-wavelet, the two dimensional four-direction orthogonal multi-wavelet with dilation factor two is generalized to two dimensional biorthogonal multi-wavelet with dilation factor *a*. Furthermore, the construction algorithm of two dimensional biorthogonal multi-scaling function and multi-wavelet with dilation factor *a* is given. Finally, the *Mallat* algorithm of two dimensional four-direction multi-wavelet with dilation factor *a* is studied.

Keywords

Two Dimensional Four-Direction Multi-Scaling Function, Two Dimensional Four-Direction Multi-Wavelet, Biorthogonality, *Mallat* Algorithm

*a*尺度二维四向双正交多小波的构造和*Mallat*算法

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摘　要

通过二维四向多加细函数以及二维四向多小波的概念，推广二尺度二维四向正交多小波为*a*尺度二维四向双正交多小波，更进一步对于*a*尺度二维四向双正交多加细函数和多小波的构造算法做出了相应给出，最后研究了*a*尺度二维四向多小波的*Mallat*算法。

关键词

二维四向多加细函数，二维四向多小波，双正交性，*Mallat*算法

1. 引言

小波分析是这些年来发展起来的一门新兴数学理论以及方法，在信号处理，语音处理，图像处理，数据压缩，微分方程求解，地震勘探等各个领域有着广泛的应用。*Haar* 小波是同时具有正交性，对称性和紧支撑性的单小波，但是其它单小波并不能具有这样好的性质，所以人们引入了多小波。从带宽来看，二尺度小波高频端的带宽比较窄，那么从小波分析的效果来看二尺度小波效果相对比较差，所以人们提出了 a 尺度小波。双向小波的概念是杨守志等人首先提出的[1]，后来进一步得出了一系列好的理论和结果[2] [3]。本文在引入二维四向多小波基础上，建立了 a 尺度二维四向多加细函数和 a 尺度二维四向多小波，给出了 a 尺度二维四向多加细函数和 a 尺度二维四向多小波的正交和双正交准则，以及它们的构造算法，最后讨论了 a 尺度二维四向多小波的分解与重构的 *Mallat* 算法。

2. 预备知识

先给出文章要提到的记号：$\boldsymbol{C}^N$ 表示 N 维复欧几里德空间。$\boldsymbol{I}$ 表示 $r\times r$ 阶单位矩阵，$\boldsymbol{O}$ 表示 $r\times r$ 阶零矩阵，$\boldsymbol{T}$ 表示向量或矩阵的转置，向量值函数信号空间 $\boldsymbol{L}^2\left(\boldsymbol{R},\boldsymbol{C}^N\right)\triangleq\boldsymbol{L}^2$ 可以表示为 $\boldsymbol{L}^2\left(\boldsymbol{R},\boldsymbol{C}^N\right)\triangleq\left\{F(t)=\left(f_1(t),f_2(t),\cdots,f_N(t)\right)^{\mathrm{T}}:t\in\boldsymbol{R},f_k(t)\in\boldsymbol{L}^2(\boldsymbol{R}),k=1,2,\cdots,N\right\}$，(根据文献[4])对于 $F(t)\in\boldsymbol{L}^2\left(\boldsymbol{R},\boldsymbol{C}^N\right)$ 它的积分和 *Fourier* 变换分别定义为

$$\int_R F(t)\mathrm{d}t\triangleq\left[\int_R f_1(t)\mathrm{d}t,\int_R f_2(t)\mathrm{d}t,\cdots,\int_R f_N(t)\mathrm{d}t\right]^{\mathrm{T}}$$

和

$$F(\omega)\triangleq\left[\int_R f_1(t)\mathrm{e}^{-i\omega t}\mathrm{d}t,\int_R f_2(t)\mathrm{e}^{-i\omega t}\mathrm{d}t,\cdots,\int_R f_N(t)\mathrm{e}^{-i\omega t}\mathrm{d}t\right]^{\mathrm{T}}$$

若 F 和 G 都是一元函数空间，它们两个的基底分别为 $\left\{f_k(x)\right\}_{k\in\boldsymbol{Z}}$ 和 $\left\{g_k(x)\right\}_{k\in\boldsymbol{Z}}$，二元函数空间 H 表示为 $H=F\otimes G$ 是 F 和 G 的张量积空间，H 的基底可以表示为 $\left\{f_k(x)g_k(x)\right\}_{k\in\boldsymbol{Z}}$。

定义 $\left\{V_j\right\}$ 的张量积空间为 $V_j^2=V_j\otimes V_j=\left\{f_k(x)g_k(x)\right\}_{f(x),g(x)\in V_j}$，而 $\left\{V_j\right\}$ 表示为由一元尺度函数 φ 生成的一个正交多分辨分析。那么关于二元函数 $f(x,y)\in\boldsymbol{L}^2\left(\boldsymbol{R}^2\right)$，引入记号 $\varphi(x,y)=\varphi(x)\varphi(y)$。

二维四向加细尺度函数

基于双向尺度函数的概念，现在假设有 $r+1$ 个双向尺度函数 $\varphi_1(x),\varphi_2(x),\cdots,\varphi_r(x),\varphi(y)\in \boldsymbol{L}^2(\boldsymbol{R})$，记 $\boldsymbol{\Phi}(x)=\left[\varphi_1(x),\varphi_2(x),\cdots,\varphi_r(x)\right]^{\mathrm{T}}$，那么通过 $\boldsymbol{\Phi}(\boldsymbol{x})$ 和 $\varphi(y)$ 的张量积来构造二维四向多尺度函数。假设双向加细函数 $\varphi_i(x)$ 和 $\varphi(y)$ 都符合细分方程

$$\varphi_i(x)=\sum_k \boldsymbol{P}_{1,k}^{+}\varphi_i(ax-k)+\sum_k \boldsymbol{P}_{1,k}^{-}\varphi_i(k-ax),\quad i=1,\cdots,r$$

$$\varphi(y)=\sum_k \boldsymbol{P}_{2,k}^{+}\varphi(ay-k)+\sum_k \boldsymbol{P}_{2,k}^{-}\varphi(k-ay)$$

设 $\boldsymbol{\Phi}(x,y)=\varphi_i(x)\varphi(y)$，就能得出

$$\begin{aligned}\boldsymbol{\Phi}(x,y)&=\left\{\sum_k \boldsymbol{P}_{1,k}^{+}\varphi_i(ax-k)+\sum_k \boldsymbol{P}_{1,k}^{-}\varphi_i(k-ax)\right\}\otimes\left\{\sum_l \boldsymbol{P}_{2,l}^{+}\varphi(ay-l)+\sum_l \boldsymbol{P}_{2,l}^{-}\varphi(l-ay)\right\}\\&=\sum_{k,l}\boldsymbol{P}_{1,k}^{+}\boldsymbol{P}_{2,l}^{+}\varphi_i(ax-k)\varphi(ay-l)+\sum_{k,l}\boldsymbol{P}_{1,k}^{+}\boldsymbol{P}_{2,l}^{-}\varphi_i(ax-k)\varphi(l-ay)\\&\quad+\sum_{k,l}\boldsymbol{P}_{1,k}^{-}\boldsymbol{P}_{2,l}^{+}\varphi_i(k-ax)\varphi(ay-l)+\sum_{k,l}\boldsymbol{P}_{1,k}^{-}\boldsymbol{P}_{2,l}^{-}\varphi_i(k-ax)\varphi(l-ay)\\&=\sum_{k,l}\boldsymbol{P}_{1,k}^{+}\boldsymbol{P}_{2,l}^{+}\varphi(ax-k,ay-l)+\sum_{k,l}\boldsymbol{P}_{1,k}^{+}\boldsymbol{P}_{2,l}^{-}\varphi(ax-k,l-ay)\\&\quad+\sum_{k,l}\boldsymbol{P}_{1,k}^{-}\boldsymbol{P}_{2,l}^{+}\varphi(k-ax,ay-l)+\sum_{k,l}\boldsymbol{P}_{1,k}^{-}\boldsymbol{P}_{2,l}^{-}\varphi(k-ax,l-ay).\end{aligned}$$

从而根据适合的 $\left\{\boldsymbol{P}_{l,k}^{+,+}\right\},\left\{\boldsymbol{P}_{l,k}^{+,-}\right\},\left\{\boldsymbol{P}_{l,k}^{-,+}\right\},\left\{\boldsymbol{P}_{l,k}^{-,-}\right\}$，就有

$$\begin{aligned}\boldsymbol{\Phi}(x,y)&=\sum_{k,l}\boldsymbol{P}_{l,k}^{+,+}\varphi(ax-k,ay-l)+\sum_{k,l}\boldsymbol{P}_{l,k}^{+,-}\varphi(ax-k,l-ay)\\&\quad+\sum_{k,l}\boldsymbol{P}_{l,k}^{-,+}\varphi(k-ax,ay-l)+\sum_{k,l}\boldsymbol{P}_{l,k}^{-,-}\varphi(k-ax,l-ay).\end{aligned}$$

令 $\boldsymbol{\Phi}(x,y)=\boldsymbol{\Phi}(x)\cdot\varphi(y)$，则

$$\boldsymbol{\Phi}(x,y)=\left[\boldsymbol{\Phi}_1(x,y),\boldsymbol{\Phi}_2(x,y),\cdots,\boldsymbol{\Phi}_r(x,y)\right]^{\mathrm{T}}$$

那么就可以有 a 尺度多小波细分方程

$$\begin{aligned}\boldsymbol{\Phi}(x,y)&=\sum_{k,l}\boldsymbol{P}_{l,k}^{+,+}\boldsymbol{\Phi}(ax-k,ay-l)+\sum_{k,l}\boldsymbol{P}_{l,k}^{+,-}\boldsymbol{\Phi}(ax-k,l-ay)\\&\quad+\sum_{k,l}\boldsymbol{P}_{l,k}^{-,+}\boldsymbol{\Phi}(k-ax,ay-l)+\sum_{k,l}\boldsymbol{P}_{l,k}^{-,-}\boldsymbol{\Phi}(k-ax,l-ay).\end{aligned}\tag{1}$$

接下来对(1)式进行 *Fourier* 变换就可以有

$$\begin{aligned}\hat{\boldsymbol{\Phi}}(\omega_1,\omega_2)&=\boldsymbol{P}^{+,+}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right)\hat{\boldsymbol{\Phi}}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right)+\boldsymbol{P}^{+,-}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right)\hat{\boldsymbol{\Phi}}\left(\frac{\omega_1}{a},-\frac{\omega_2}{a}\right)\\&\quad+\boldsymbol{P}^{-,+}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right)\hat{\boldsymbol{\Phi}}\left(-\frac{\omega_1}{a},\frac{\omega_2}{a}\right)+\boldsymbol{P}^{-,-}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right)\hat{\boldsymbol{\Phi}}\left(-\frac{\omega_1}{a},-\frac{\omega_2}{a}\right),\end{aligned}\tag{2}$$

其中

$$\boldsymbol{P}^{+,+}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right)=\frac{1}{a^2}\sum_{k,l}\boldsymbol{P}_{k,l}^{+,+}\mathrm{e}^{-i\left(\frac{\omega_1 k}{a}+\frac{\omega_2 l}{a}\right)},$$

$$P^{+,-}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right)=\frac{1}{a^2}\sum_{k,l}P_{k,l}^{+,-}\mathrm{e}^{-i\left(\frac{\omega_1 k}{a}+\frac{\omega_2 l}{a}\right)},$$

$$P^{-,+}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right)=\frac{1}{a^2}\sum_{k,l}P_{k,l}^{-,+}\mathrm{e}^{-i\left(\frac{\omega_1 k}{a}+\frac{\omega_2 l}{a}\right)},$$

$$P^{-,-}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right)=\frac{1}{a^2}\sum_{k,l}P_{k,l}^{-,-}\mathrm{e}^{-i\left(\frac{\omega_1 k}{a}+\frac{\omega_2 l}{a}\right)},$$

令

$$P_{k,l}^{+,+}=\begin{bmatrix}P_{1,k,l}^{+,+} & P_{r,k,l}^{+,+}\\ 0 & P_{r^2,k,l}^{+,+}\end{bmatrix},P_{k,l}^{+,-}=\begin{bmatrix}P_{1,k,l}^{+,-} & P_{r,k,l}^{+,-}\\ 0 & P_{r^2,k,l}^{+,-}\end{bmatrix},$$

$$P_{k,l}^{-,+}=\begin{bmatrix}P_{1,k,l}^{-,+} & P_{r,k,l}^{-,+}\\ 0 & P_{r^2,k,l}^{-,+}\end{bmatrix},P_{k,l}^{-,-}=\begin{bmatrix}P_{1,k,l}^{-,-} & P_{r,k,l}^{-,-}\\ 0 & P_{r^2,k,l}^{-,-}\end{bmatrix},$$

为双正，正负和双负矩阵符号。下面对(1)进行变形有

$$\begin{aligned}\Phi(x,-y)=&\sum_{k,l}P_{l,k}^{+,+}\Phi(x)(ax-k,-ay-l)+\sum_{k,l}P_{l,k}^{+,-}\Phi(x)(ax-k,l+ay)\\&+\sum_{k,l}P_{l,k}^{-,+}\Phi(x)(k-ax,-ay-l)+\sum_{k,l}P_{l,k}^{-,-}\Phi(x)(k-ax,l+ay).\end{aligned}\tag{3}$$

$$\begin{aligned}\Phi(-x,y)=&\sum_{k,l}P_{l,k}^{+,+}\Phi(x)(-ax-k,ay-l)+\sum_{k,l}P_{l,k}^{+,-}\Phi(x)(-ax-k,l-ay)\\&+\sum_{k,l}P_{l,k}^{-,+}\Phi(x)(ax+k,ay-l)+\sum_{k,l}P_{l,k}^{-,-}\Phi(x)(ax+k,l-ay).\end{aligned}\tag{4}$$

$$\begin{aligned}\Phi(-x,-y)=&\sum_{k,l}P_{l,k}^{+,+}\Phi(x)(-ax-k,-ay-l)+\sum_{k,l}P_{l,k}^{+,-}\Phi(x)(-ax-k,l+ay)\\&+\sum_{k,l}P_{l,k}^{-,+}\Phi(x)(ax+k,-ay-l)+\sum_{k,l}P_{l,k}^{-,-}\Phi(x)(ax+k,l+ay).\end{aligned}\tag{5}$$

下面对(3)~(5)式都进行 *Fourier* 变换就有

$$\begin{aligned}\hat{\Phi}(\omega_1,-\omega_2)=&P^{+,+}\left(\frac{\omega_1}{a},-\frac{\omega_2}{a}\right)\hat{\Phi}\left(\frac{\omega_1}{a},-\frac{\omega_2}{a}\right)+P^{+,-}\left(\frac{\omega_1}{a},-\frac{\omega_2}{a}\right)\hat{\Phi}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right)\\&+P^{-,+}\left(\frac{\omega_1}{a},-\frac{\omega_2}{a}\right)\hat{\Phi}\left(-\frac{\omega_1}{a},-\frac{\omega_2}{a}\right)+P^{-,-}\left(\frac{\omega_1}{a},-\frac{\omega_2}{a}\right)\hat{\Phi}\left(-\frac{\omega_1}{a},\frac{\omega_2}{a}\right),\end{aligned}\tag{6}$$

$$\begin{aligned}\hat{\Phi}(-\omega_1,\omega_2)=&P^{+,+}\left(-\frac{\omega_1}{a},\frac{\omega_2}{a}\right)\hat{\Phi}\left(-\frac{\omega_1}{a},\frac{\omega_2}{a}\right)+P^{+,-}\left(-\frac{\omega_1}{a},\frac{\omega_2}{a}\right)\hat{\Phi}\left(-\frac{\omega_1}{a},-\frac{\omega_2}{a}\right)\\&+P^{-,+}\left(-\frac{\omega_1}{a},\frac{\omega_2}{a}\right)\hat{\Phi}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right)+P^{-,-}\left(-\frac{\omega_1}{a},\frac{\omega_2}{a}\right)\hat{\Phi}\left(\frac{\omega_1}{a},-\frac{\omega_2}{a}\right),\end{aligned}\tag{7}$$

$$\begin{aligned}\hat{\Phi}(-\omega_1,-\omega_2)=&P^{+,+}\left(-\frac{\omega_1}{a},-\frac{\omega_2}{a}\right)\hat{\Phi}\left(-\frac{\omega_1}{a},-\frac{\omega_2}{a}\right)+P^{+,-}\left(-\frac{\omega_1}{a},-\frac{\omega_2}{a}\right)\hat{\Phi}\left(-\frac{\omega_1}{a},\frac{\omega_2}{a}\right)\\&+P^{-,+}\left(-\frac{\omega_1}{a},-\frac{\omega_2}{a}\right)\hat{\Phi}\left(\frac{\omega_1}{a},-\frac{\omega_2}{a}\right)+P^{-,-}\left(-\frac{\omega_1}{a},-\frac{\omega_2}{a}\right)\hat{\Phi}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right),\end{aligned}\tag{8}$$

根据(2)(6)~(8)式可以有，令

$$\Gamma(x,y)=\left[\Phi(x,y),\Phi(x,-y),\Phi(-x,y),\Phi(-x,-y)\right]^{\mathrm{T}},$$

$$\hat{\Gamma}(\omega_1,\omega_2)=\begin{bmatrix} P^{+,+}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right) & P^{+,-}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right) & P^{-,+}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right) & P^{-,-}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right) \\ P^{+,-}\left(\frac{\omega_1}{a},-\frac{\omega_2}{a}\right) & P^{+,+}\left(\frac{\omega_1}{a},-\frac{\omega_2}{a}\right) & P^{-,-}\left(\frac{\omega_1}{a},-\frac{\omega_2}{a}\right) & P^{-,+}\left(\frac{\omega_1}{a},-\frac{\omega_2}{a}\right) \\ P^{-,+}\left(-\frac{\omega_1}{a},\frac{\omega_2}{a}\right) & P^{-,-}\left(-\frac{\omega_1}{a},\frac{\omega_2}{a}\right) & P^{+,+}\left(-\frac{\omega_1}{a},\frac{\omega_2}{a}\right) & P^{+,-}\left(-\frac{\omega_1}{a},\frac{\omega_2}{a}\right) \\ P^{-,-}\left(-\frac{\omega_1}{a},-\frac{\omega_2}{a}\right) & P^{-,+}\left(-\frac{\omega_1}{a},-\frac{\omega_2}{a}\right) & P^{+,-}\left(-\frac{\omega_1}{a},-\frac{\omega_2}{a}\right) & P^{+,+}\left(-\frac{\omega_1}{a},-\frac{\omega_2}{a}\right) \end{bmatrix}\hat{\Gamma}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right), \tag{9}$$

方程(2)有解，当且仅当式(9)有解。

设

$$\Gamma(x,y)=\sum_{k,l}\begin{bmatrix} P_{k,l}^{+,+} & P_{k,l}^{+,-} & P_{k,l}^{-,+} & P_{k,l}^{-,-} \\ P_{k,-l}^{+,-} & P_{k,-l}^{+,+} & P_{k,-l}^{-,-} & P_{k,-l}^{-,+} \\ P_{-k,l}^{-,+} & P_{-k,l}^{-,-} & P_{-k,l}^{+,+} & P_{-k,l}^{+,-} \\ P_{-k,-l}^{-,-} & P_{-k,-l}^{-,+} & P_{-k,-l}^{+,-} & P_{-k,-l}^{+,+} \end{bmatrix}\Gamma(ax-k,ay-l), \tag{10}$$

则(9)式为$\Gamma(x,y)$在频域里的a尺度加密方程，它的加细面具符号为

$$P(\omega_1,\omega_2)=\begin{bmatrix} P^{+,+}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right) & P^{+,-}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right) & P^{-,+}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right) & P^{-,-}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right) \\ P^{+,-}\left(\frac{\omega_1}{a},-\frac{\omega_2}{a}\right) & P^{+,+}\left(\frac{\omega_1}{a},-\frac{\omega_2}{a}\right) & P^{-,-}\left(\frac{\omega_1}{a},-\frac{\omega_2}{a}\right) & P^{-,+}\left(\frac{\omega_1}{a},-\frac{\omega_2}{a}\right) \\ P^{-,+}\left(-\frac{\omega_1}{a},\frac{\omega_2}{a}\right) & P^{-,-}\left(-\frac{\omega_1}{a},\frac{\omega_2}{a}\right) & P^{+,+}\left(-\frac{\omega_1}{a},\frac{\omega_2}{a}\right) & P^{+,-}\left(-\frac{\omega_1}{a},\frac{\omega_2}{a}\right) \\ P^{-,-}\left(-\frac{\omega_1}{a},-\frac{\omega_2}{a}\right) & P^{-,+}\left(-\frac{\omega_1}{a},-\frac{\omega_2}{a}\right) & P^{+,-}\left(-\frac{\omega_1}{a},-\frac{\omega_2}{a}\right) & P^{+,+}\left(-\frac{\omega_1}{a},-\frac{\omega_2}{a}\right) \end{bmatrix} \tag{11}$$

定义方程(1)的自相关矩阵符号

$$\Omega(x,y)=\sum_{k,l}\begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} & \Omega_{24} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} & \Omega_{34} \\ \Omega_{41} & \Omega_{42} & \Omega_{43} & \Omega_{44} \end{bmatrix}\mathrm{e}^{-i\left(\frac{\omega_1 k}{a}+\frac{\omega_2 l}{a}\right)},$$

其中

$$\Omega_{11}=\langle\Phi(x,y),\Phi(x-k,y-l)\rangle;\Omega_{12}=\langle\Phi(x,y),\Phi(x-k,l-y)\rangle;$$

$$\Omega_{13}=\langle\Phi(x,y),\Phi(k-x,y-l)\rangle;\Omega_{14}=\langle\Phi(x,y),\Phi(k-x,l-y)\rangle;$$

$$\Omega_{21}=\langle\Phi(x,-y),\Phi(x-k,y-l)\rangle;\Omega_{22}=\langle\Phi(x,-y),\Phi(x-k,l-y)\rangle;$$

$$\Omega_{23}=\langle\Phi(x,-y),\Phi(k-x,y-l)\rangle;\Omega_{24}=\langle\Phi(x,-y),\Phi(k-x,l-y)\rangle;$$

$$\Omega_{31}=\langle\Phi(-x,y),\Phi(x-k,y-l)\rangle;\Omega_{32}=\langle\Phi(-x,y),\Phi(x-k,l-y)\rangle;$$

$$\Omega_{33}=\langle\Phi(-x,y),\Phi(k-x,y-l)\rangle;\Omega_{34}=\langle\Phi(-x,y),\Phi(k-x,l-y)\rangle;$$

$$\Omega_{41}=\langle\Phi(-x,-y),\Phi(x-k,y-l)\rangle;\Omega_{42}=\langle\Phi(-x,-y),\Phi(x-k,l-y)\rangle;$$

$$\boldsymbol{\Omega}_{43}=\langle\boldsymbol{\Phi}(-x,-y),\boldsymbol{\Phi}(k-x,y-l)\rangle;\boldsymbol{\Omega}_{44}=\langle\boldsymbol{\Phi}(-x,-y),\boldsymbol{\Phi}(k-x,l-y)\rangle.$$

下面引入变换算子$\boldsymbol{\tau}$：

$$\boldsymbol{\tau}\boldsymbol{A}\left((\omega_1)^a,(\omega_2)^a\right)=\sum_{k,l=0}^{a-1}\boldsymbol{P}\left(\omega_1+\frac{2k\pi}{a},\omega_2+\frac{2l\pi}{a}\right)\boldsymbol{A}\left(\omega_1+\frac{2k\pi}{a},\omega_2+\frac{2l\pi}{a}\right)\boldsymbol{P}^*\left(\omega_1+\frac{2k\pi}{a},\omega_2+\frac{2l\pi}{a}\right), \tag{12}$$

其中$\boldsymbol{A}\left((\omega_1)^a,(\omega_2)^a\right)$是$\boldsymbol{P}(\omega_1,\omega_2)$的 *Laurent* 多项式，$\boldsymbol{P}(\omega_1,\omega_2)$由(12)式可知。

关于$\boldsymbol{\Omega}(\omega_1,\omega_2)$和$\boldsymbol{\tau}$可以有下面的引理。

引理 1：矩阵符号$\boldsymbol{\Omega}(\omega_1,\omega_2)$和变换算子$\boldsymbol{\tau}$的定义如上，那么由 *Poisson* 求和公式可得

$$\boldsymbol{\Omega}(\omega_1,\omega_2)=\sum_{k,l}\begin{bmatrix}\boldsymbol{\Phi}\left(\omega_1+\frac{2k\pi}{a},\omega_2+\frac{2l\pi}{a}\right)\\\boldsymbol{\Phi}\left(\omega_1+\frac{2k\pi}{a},-\omega_2-\frac{2l\pi}{a}\right)\\\boldsymbol{\Phi}\left(-\omega_1-\frac{2k\pi}{a},\omega_2+\frac{2l\pi}{a}\right)\\\boldsymbol{\Phi}\left(-\omega_1-\frac{2k\pi}{a},-\omega_2-\frac{2l\pi}{a}\right)\end{bmatrix}\begin{bmatrix}\boldsymbol{\Phi}\left(\omega_1+\frac{2k\pi}{a},\omega_2+\frac{2l\pi}{a}\right)\\\boldsymbol{\Phi}\left(\omega_1+\frac{2k\pi}{a},-\omega_2-\frac{2l\pi}{a}\right)\\\boldsymbol{\Phi}\left(-\omega_1-\frac{2k\pi}{a},\omega_2+\frac{2l\pi}{a}\right)\\\boldsymbol{\Phi}\left(-\omega_1-\frac{2k\pi}{a},-\omega_2-\frac{2l\pi}{a}\right)\end{bmatrix}^{\mathrm{T}}$$

进一步，我们可以得出$\boldsymbol{\Omega}(\omega_1,\omega_2)$是$\boldsymbol{\tau}$相应于特征值为 1 的特征矩阵。

定理 1：加细方程(1)有紧支撑解当且仅当它的面具符号满足下面情况之一

1) $$\begin{cases}\sum_{k,l}\left(\boldsymbol{P}_{k,l}^{+,+}+\boldsymbol{P}_{k,l}^{+,-}+\boldsymbol{P}_{k,l}^{-,+}+\boldsymbol{P}_{k,l}^{-,-}\right)=a^2\boldsymbol{I},\\\left|\sum_{k,l}\left(\boldsymbol{P}_{k,l}^{+,+}-\boldsymbol{P}_{k,l}^{+,-}-\boldsymbol{P}_{k,l}^{-,+}+\boldsymbol{P}_{k,l}^{-,-}\right)\right|\le a^2,\\\left|\sum_{k,l}\left(\boldsymbol{P}_{k,l}^{+,+}+\boldsymbol{P}_{k,l}^{+,-}-\boldsymbol{P}_{k,l}^{-,+}-\boldsymbol{P}_{k,l}^{-,-}\right)\right|\le a^2,\\\left|\sum_{k,l}\left(\boldsymbol{P}_{k,l}^{+,+}-\boldsymbol{P}_{k,l}^{+,-}+\boldsymbol{P}_{k,l}^{-,+}-\boldsymbol{P}_{k,l}^{-,-}\right)\right|\le a^2;\end{cases}$$

2) $$\begin{cases}\left|\sum_{k,l}\left(\boldsymbol{P}_{k,l}^{+,+}+\boldsymbol{P}_{k,l}^{+,-}+\boldsymbol{P}_{k,l}^{-,+}+\boldsymbol{P}_{k,l}^{-,-}\right)\right|\le a^2,\\\sum_{k,l}\left(\boldsymbol{P}_{k,l}^{+,+}-\boldsymbol{P}_{k,l}^{+,-}-\boldsymbol{P}_{k,l}^{-,+}+\boldsymbol{P}_{k,l}^{-,-}\right)=a^2\boldsymbol{I},\\\left|\sum_{k,l}\left(\boldsymbol{P}_{k,l}^{+,+}+\boldsymbol{P}_{k,l}^{+,-}-\boldsymbol{P}_{k,l}^{-,+}-\boldsymbol{P}_{k,l}^{-,-}\right)\right|\le a^2,\\\left|\sum_{k,l}\left(\boldsymbol{P}_{k,l}^{+,+}-\boldsymbol{P}_{k,l}^{+,-}+\boldsymbol{P}_{k,l}^{-,+}-\boldsymbol{P}_{k,l}^{-,-}\right)\right|\le a^2;\end{cases}$$

3) $$\begin{cases}\left|\sum_{k,l}\left(\boldsymbol{P}_{k,l}^{+,+}+\boldsymbol{P}_{k,l}^{+,-}+\boldsymbol{P}_{k,l}^{-,+}+\boldsymbol{P}_{k,l}^{-,-}\right)\right|\le a^2,\\\left|\sum_{k,l}\left(\boldsymbol{P}_{k,l}^{+,+}-\boldsymbol{P}_{k,l}^{+,-}-\boldsymbol{P}_{k,l}^{-,+}+\boldsymbol{P}_{k,l}^{-,-}\right)\right|\le a^2,\\\sum_{k,l}\left(\boldsymbol{P}_{k,l}^{+,+}+\boldsymbol{P}_{k,l}^{+,-}-\boldsymbol{P}_{k,l}^{-,+}-\boldsymbol{P}_{k,l}^{-,-}\right)=a^2\boldsymbol{I},\\\left|\sum_{k,l}\left(\boldsymbol{P}_{k,l}^{+,+}-\boldsymbol{P}_{k,l}^{+,-}+\boldsymbol{P}_{k,l}^{-,+}-\boldsymbol{P}_{k,l}^{-,-}\right)\right|\le a^2;\end{cases}$$

$$4)\ \begin{cases}\left|\sum_{k,l}\left(\boldsymbol{P}_{k,l}^{+,+}+\boldsymbol{P}_{k,l}^{+,-}+\boldsymbol{P}_{k,l}^{-,+}+\boldsymbol{P}_{k,l}^{-,-}\right)\right|\le a^2,\\ \left|\sum_{k,l}\left(\boldsymbol{P}_{k,l}^{+,+}-\boldsymbol{P}_{k,l}^{+,-}-\boldsymbol{P}_{k,l}^{-,+}+\boldsymbol{P}_{k,l}^{-,-}\right)\right|\le a^2,\\ \left|\sum_{k,l}\left(\boldsymbol{P}_{k,l}^{+,+}+\boldsymbol{P}_{k,l}^{+,-}-\boldsymbol{P}_{k,l}^{-,+}-\boldsymbol{P}_{k,l}^{-,-}\right)\right|\le a^2,\\ \sum_{k,l}\left(\boldsymbol{P}_{k,l}^{+,+}-\boldsymbol{P}_{k,l}^{+,-}+\boldsymbol{P}_{k,l}^{-,+}-\boldsymbol{P}_{k,l}^{-,-}\right)=a^2\boldsymbol{I};\end{cases}$$

证明：根据文献[5] [6] [7] [8]，方程(10)存在紧支撑分布解当且仅当 1 是(11)式定义矩阵 $\boldsymbol{P}(1,1)$ 的一个特征值，$\boldsymbol{P}(1,1)$ 的其他特征值的模都不大于 1。另外，$\boldsymbol{P}(1,1)$ 的 4 个特征值分别是 $\frac{1}{a^2}\sum_{k,l}\left(\boldsymbol{P}_{k,l}^{+,+}+\boldsymbol{P}_{k,l}^{+,-}+\boldsymbol{P}_{k,l}^{-,+}+\boldsymbol{P}_{k,l}^{-,-}\right)$，$\frac{1}{a^2}\sum_{k,l}\left(\boldsymbol{P}_{k,l}^{+,+}-\boldsymbol{P}_{k,l}^{+,-}-\boldsymbol{P}_{k,l}^{-,+}+\boldsymbol{P}_{k,l}^{-,-}\right)$，$\frac{1}{a^2}\sum_{k,l}\left(\boldsymbol{P}_{k,l}^{+,+}+\boldsymbol{P}_{k,l}^{+,-}-\boldsymbol{P}_{k,l}^{-,+}-\boldsymbol{P}_{k,l}^{-,-}\right)$ 以及 $\frac{1}{a^2}\sum_{k,l}\left(\boldsymbol{P}_{k,l}^{+,+}-\boldsymbol{P}_{k,l}^{+,-}+\boldsymbol{P}_{k,l}^{-,+}-\boldsymbol{P}_{k,l}^{-,-}\right)$。定理易证。

定理 2: 若要 $\Phi(x,y)$ 有紧支撑性，则要证明每个分量是紧支撑的。设 $\boldsymbol{\Phi}(x)=\left[\varphi_1(x),\varphi_2(x),\cdots,\varphi_r(x)\right]^{\mathrm{T}}$，其中 $\varphi_i(x)$ 和 $\varphi(y)$ 是双向细分函数满足

$$\varphi_i(x)=\sum_{k=0}^{N_1}\boldsymbol{P}_k^{+}\varphi_i(ax-k)+\sum_{k=-N_1}^{0}\boldsymbol{P}_k^{-}\varphi_i(k-ax)$$

$$\varphi(y)=\sum_{k=0}^{N_2}p_k^{+}\varphi(ay-k)+\sum_{k=-N_2}^{0}p_k^{-}\varphi(k-ay)$$

如果 $\Phi(x)$ 和 $\varphi(y)$ 是紧支撑的，那么可根据 $\Phi(x)$ 和 $\varphi(y)$ 的张量积生成二维四向加细函数 $Supp\ \Phi(x)\subseteq\left[-\frac{N_1}{a-1},\frac{N_1}{a-1}\right]$，$Supp\ \varphi(y)\subseteq\left[-\frac{N_2}{a-1},\frac{N_2}{a-1}\right]$，则有

$$Supp\ \Phi(x,y)=\Phi(x)\cdot\varphi(y)\subseteq\left[-\frac{N_1}{a-1},\frac{N_1}{a-1}\right]\times\left[-\frac{N_2}{a-1},\frac{N_2}{a-1}\right]$$

证明：根据文献[9]中定理 4 可知。

3. *a* 尺度二维四向多分辨分析

定义子空间序列 $\{V_j\}_{j\in\boldsymbol{Z}}\subset L^2(\boldsymbol{R}^2)$，

$$\begin{aligned}V_j=Close_{L^2(\boldsymbol{R}^2)}\Big\langle & a^j\boldsymbol{\Phi}(a^jx-k,a^jy-l),a^j\boldsymbol{\Phi}(a^jx-k,l-a^jy),\\ & a^j\boldsymbol{\Phi}(k-a^jx,a^jy-l),a^j\boldsymbol{\Phi}(k-a^jx,l-a^jy),k,l\in\boldsymbol{Z}\Big\rangle,\end{aligned}\tag{13}$$

那么要产生在 $L^2(\boldsymbol{R}^2)$ 中的一个多分辨分析 $\{V_j\}_{j\in\boldsymbol{Z}}$ 当且仅当(13)式里的 V_j 应当满足以下条件：

1) $\cdots\subset V_{-1}\subset V_0\subset V_1\subset\cdots$；

2) $Close_{L^2(\boldsymbol{R}^2)}\bigcup_{j\in\boldsymbol{Z}}V_j=L^2(\boldsymbol{R}^2)$；

3) $\bigcap_{j\in\boldsymbol{Z}}V_j=\{0\}$；

4) $f(x,y)\in V_j\Leftrightarrow f(ax,ay)\in V_{j+1}$；

5) 存在 $L^2(\boldsymbol{R}^2)$ 里的一个函数 $\boldsymbol{\Phi}(x,y)$，使得集合 $\{\boldsymbol{\Phi}(x-k,y-l),\boldsymbol{\Phi}(x-k,l-y),\boldsymbol{\Phi}(k-x,y-l),\boldsymbol{\Phi}(k-x,l-y):k,l\in\boldsymbol{Z}\}$ 是 V_0 的 *Riesz* 基，那么就有两个常数

$0 < A \le B < \infty$，则对于系数向量序列

$\left\{c^{k,l}\right\}_{k,l\in\mathbf{Z}} = \left\{\left[c_1^{k,l}, c_2^{k,l}, c_3^{k,l}, c_4^{k,l}\right]\right\}_{k,l\in\mathbf{Z}} \subset l^2\left(\mathbf{Z}^4\right)$ 有

$$A\left\|\sum_{k,l\in\mathbf{Z}} c_{k,l}c_{k,l}^*\right\|_2^2 \le \left\|\sum_{k,l\in\mathbf{Z}} c_1^{k,l}\Phi(x-k,y-l) + \sum_{k,l\in\mathbf{Z}} c_2^{k,l}\Phi(x-k,l-y) + \sum_{k,l\in\mathbf{Z}} c_3^{k,l}\Phi(k-x,y-l) + \sum_{k,l\in\mathbf{Z}} c_4^{k,l}\Phi(k-x,l-y)\right\|_2^2 \le B\left\|\sum_{k,l\in\mathbf{Z}} c_{k,l}c_{k,l}^*\right\|_2^2, \tag{14}$$

称式(14)为稳定性条件。

根据多重多分辨分析的性质 $f(x,y)\in V_j \Leftrightarrow f\left(x+\frac{k}{a^j}, y+\frac{l}{a^j}\right)\in V_j$，可以定义：

$$\Phi_{j,k,l}^{+,+} = a^j\Phi\left(a^jx-k, a^jy-l\right),\quad \Phi_{j,k,l}^{+,-} = a^j\Phi\left(a^jx-k, l-a^jy\right),$$

$$\Phi_{j,k,l}^{-,+} = a^j\Phi\left(k-a^jx, a^jy-l\right),\quad \Phi_{j,k,l}^{-,-} = a^j\Phi\left(k-a^jx, l-a^jy\right).$$

则 $\left\{\Phi_{j,k,l}^{+,+}, \Phi_{j,k,l}^{+,-}, \Phi_{j,k,l}^{-,+}, \Phi_{j,k,l}^{-,-} : k,l\in\mathbf{Z}\right\}$ 同样可以构成 V_j 的 Riesz 基，有

$$A\left\|\sum_{k,l\in\mathbf{Z}} c_{k,l}c_{k,l}^*\right\|_2^2 \le \left\|\sum_{k,l\in\mathbf{Z}} c_1^{k,l}\Phi_{j,k,l}^{+,+} + \sum_{k,l\in\mathbf{Z}} c_2^{k,l}\Phi_{j,k,l}^{+,-} + \sum_{k,l\in\mathbf{Z}} c_3^{k,l}\Phi_{j,k,l}^{-,+} + \sum_{k,l\in\mathbf{Z}} c_4^{k,l}\Phi_{j,k,l}^{-,-}\right\|_2^2 \le B\left\|\sum_{k,l\in\mathbf{Z}} c_{k,l}c_{k,l}^*\right\|_2^2,$$

因为 $\Phi(x,y)\in V_0\subset V_1$，并且 $\left\{\Phi_{j,k,l}^{+,+}, \Phi_{j,k,l}^{+,-}, \Phi_{j,k,l}^{-,+}, \Phi_{j,k,l}^{-,-} : k,l\in\mathbf{Z}\right\}$ 同样也可以构成 V_1 的 *Riesz* 基，故有 $c_1^{k,l}, c_2^{k,l}, c_3^{k,l}, c_4^{k,l} \in l^2\left(\mathbf{Z}^4\right)$，从而使 $\Phi(x,y)$ 满足(1)式。

定理 3：如果尺度函数 $\Phi(x,y)\in L^2\left(\mathbf{R}^2\right)$ 符合多分辨分析，现定义 $V_j = \left\{\Phi_{j,k,l}^{+,+}, \Phi_{j,k,l}^{+,-}, \Phi_{j,k,l}^{-,+}, \Phi_{j,k,l}^{-,-} : k,l\in\mathbf{Z}\right\}$ 构成 V_0 的 *Riesz* 基。若存在函数集 $\left\{\Phi(x-k,y-l), \Phi(x-k,l-y), \Phi(k-x,y-l), \Phi(k-x,l-y) : k,l\in\mathbf{Z}\right\}$ 构成 V_0 的 *Riesz* 基，那么 $\bigcap_{j\in\mathbf{Z}} V_j = \{0\}$。

证明：由于 $\left\{\Phi_{j,k,l}^{+,+}, \Phi_{j,k,l}^{+,-}, \Phi_{j,k,l}^{-,+}, \Phi_{j,k,l}^{-,-} : k,l\in\mathbf{Z}\right\}$ 构成 V_0 的 *Riesz* 基，故存在两个常数 $0 < A \le B < \infty$，对任意的 $f(x,y)\in V_0$，有

$$A\|f\|_2^2 \le \sum_{k,l\in\mathbf{Z}}\left|\left\langle f, \Phi_{0,k,l}^{+,+}\right\rangle\right|^2 + \sum_{k,l\in\mathbf{Z}}\left|\left\langle f, \Phi_{0,k,l}^{+,-}\right\rangle\right|^2 + \sum_{k,l\in\mathbf{Z}}\left|\left\langle f, \Phi_{0,k,l}^{-,+}\right\rangle\right|^2 + \sum_{k,l\in\mathbf{Z}}\left|\left\langle f, \Phi_{0,k,l}^{-,-}\right\rangle\right|^2 \le B\|f\|_2^2,$$

所以对于所有的 $f(x,y)\in V_j$，就有

$$A\|f\|_2^2 \le \sum_{k,l\in\mathbf{Z}}\left|\left\langle f, \Phi_{j,k,l}^{+,+}\right\rangle\right|^2 + \sum_{k,l\in\mathbf{Z}}\left|\left\langle f, \Phi_{j,k,l}^{+,-}\right\rangle\right|^2 + \sum_{k,l\in\mathbf{Z}}\left|\left\langle f, \Phi_{j,k,l}^{-,+}\right\rangle\right|^2 + \sum_{k,l\in\mathbf{Z}}\left|\left\langle f, \Phi_{j,k,l}^{-,-}\right\rangle\right|^2 \le B\|f\|_2^2,$$

对于 $\forall\varepsilon > 0$，$f(x,y)\in\bigcap_{j\in\mathbf{Z}} V_j$，存在一个紧支撑连续函数 $\tilde{f}(x,y)$，使 $\left\|f-\tilde{f}\right\|_{L^2} \le \varepsilon$。

若 P_j 是 V_j 的正交投影算子，则 $\|f\| - \left\|P_j\tilde{f}\right\| \le \left\|f - P_j\tilde{f}\right\| = \left\|P_jf - P_j\tilde{f}\right\| \le \left\|f-\tilde{f}\right\| \le \varepsilon$，故 $\|f\| \le \varepsilon + \left\|P_j\tilde{f}\right\|, \forall j\in\mathbf{Z}$。

设 $Supp\ f(x,y) = [-r,r]\times[-r,r] = \mathbf{S}$，则

$$\sum_{k,l\in\mathbf{Z}}\left|\left\langle \tilde{f}, \Phi_{j,k,l}^{+,+}\right\rangle\right|^2 \le a^{2j}\sum_{k,l\in\mathbf{Z}}\left[\int_{(x,y)\in\mathbf{S}}\left|\tilde{f}(x,y)\right|\left|\Phi\left(a^jx-k, a^jy-l\right)\right|\mathrm{d}x\mathrm{d}y\right]^2$$

$$\le a^{2j}\left\|\tilde{f}(x,y)\right\|_{L^\infty}^2 \sum_{k,l\in\mathbf{Z}}\left[\int_{(x,y)\in\mathbf{S}}\left|\Phi\left(a^jx-k, a^jy-l\right)\right|\mathrm{d}x\mathrm{d}y\right]^2$$

$$\le a^{2j}\left\|\tilde{f}(x,y)\right\|_{L^\infty}^2 \sum_{k,l\in\mathbf{Z}}\int_{(x,y)\in\mathbf{S}}\left|\Phi\left(a^jx-k, a^jy-l\right)\right|^2\mathrm{d}x\mathrm{d}y$$

$$= \left\|\tilde{f}(x,y)\right\|_{L^\infty}^2 \int_{D^j}\left|\Phi(x,y)\right|^2\mathrm{d}x\mathrm{d}y$$

其中 $D^j=\bigcup_{k,l}\left[k-a^j r,k+a^j r\right]\times\left[l-a^j r,l+a^j r\right]$。所以，当 $j\to-\infty$ 时 $\sum_{k,l\in\boldsymbol{Z}}\left|\left\langle\tilde{f},\boldsymbol{\Phi}_{j,k,l}^{+,+}\right\rangle\right|^2\to 0$。同理，可以证明：$\sum_{k,l\in\boldsymbol{Z}}\left|\left\langle\tilde{f},\boldsymbol{\Phi}_{j,k,l}^{-,+}\right\rangle\right|^2\to 0$，$\sum_{k,l\in\boldsymbol{Z}}\left|\left\langle\tilde{f},\boldsymbol{\Phi}_{j,k,l}^{+,-}\right\rangle\right|^2\to 0$，$\sum_{k,l\in\boldsymbol{Z}}\left|\left\langle\tilde{f},\boldsymbol{\Phi}_{j,k,l}^{-,-}\right\rangle\right|^2\to 0$。又由于

$$\left\|P_j\tilde{f}\right\|^2\le\frac{1}{\boldsymbol{A}}\sum_{k,l\in\boldsymbol{Z}}\left[\left|\left\langle f,\boldsymbol{\Phi}_{0,k,l}^{+,+}\right\rangle\right|^2+\left|\left\langle f,\boldsymbol{\Phi}_{0,k,l}^{+,-}\right\rangle\right|^2+\left|\left\langle f,\boldsymbol{\Phi}_{0,k,l}^{-,+}\right\rangle\right|^2+\left|\left\langle f,\boldsymbol{\Phi}_{0,k,l}^{-,-}\right\rangle\right|^2\right]$$

所以 $\lim_{j\to-\infty}\left\|P_j\tilde{f}\right\|=0$。根据 $\|f\|\le\varepsilon+\left\|P_j\tilde{f}\right\|,\forall j\in\boldsymbol{Z}$，可以知道 $f(x,y)=0$。

定理 4： 若 $\Phi(x,y)\in L^2\left(\boldsymbol{R}^2\right)$ 符合定义式(1)，根据(13)式定义的 V_j，若 $\Phi(x,y)$ 满足 1) 集合 $\left\{\Phi(x-k,y-l),\Phi(x-k,l-y),\Phi(k-x,y-l),\Phi(k-x,l-y):k,l\in\boldsymbol{Z}\right\}$ 是 V_0 的 *Riesz* 基；2) 对所有的 $(\omega_1,\omega_2)\in\boldsymbol{R}^2$，$\hat{\Phi}(\omega_1,\omega_2)$ 有界；3) $\hat{\Phi}(\omega_1,\omega_2)$ 在 $(\omega_1,\omega_2)=(0,0)$ 附近连续，$\hat{\Phi}(\omega_1,\omega_2)\ne 0$，那么 $Close_{L^2\left(\boldsymbol{R}^2\right)}\bigcup_{j\in\boldsymbol{Z}}V_j=L^2\left(\boldsymbol{R}^2\right)$。

证明：由于 $\left\{\boldsymbol{\Phi}_{j,k,l}^{+,+},\boldsymbol{\Phi}_{j,k,l}^{+,-},\boldsymbol{\Phi}_{j,k,l}^{-,+},\boldsymbol{\Phi}_{j,k,l}^{-,-}:k,l\in\boldsymbol{Z}\right\}$ 构成 V_0 的 *Riesz* 基，故存在两个常数 $0<A\le B<\infty$，对任意的 $f(x,y)\in V_0$，有

$$A\|f\|_2^2\le\sum_{k,l\in\boldsymbol{Z}}\left|\left\langle f,\boldsymbol{\Phi}_{0,k,l}^{+,+}\right\rangle\right|^2+\sum_{k,l\in\boldsymbol{Z}}\left|\left\langle f,\boldsymbol{\Phi}_{0,k,l}^{+,-}\right\rangle\right|^2+\sum_{k,l\in\boldsymbol{Z}}\left|\left\langle f,\boldsymbol{\Phi}_{0,k,l}^{-,+}\right\rangle\right|^2+\sum_{k,l\in\boldsymbol{Z}}\left|\left\langle f,\boldsymbol{\Phi}_{0,k,l}^{-,-}\right\rangle\right|^2\le B\|f\|_2^2,$$

所以对于所有的 $f(x,y)\in V_j$，就有

$$A\|f\|_2^2\le\sum_{k,l\in\boldsymbol{Z}}\left|\left\langle f,\boldsymbol{\Phi}_{j,k,l}^{+,+}\right\rangle\right|^2+\sum_{k,l\in\boldsymbol{Z}}\left|\left\langle f,\boldsymbol{\Phi}_{j,k,l}^{+,-}\right\rangle\right|^2+\sum_{k,l\in\boldsymbol{Z}}\left|\left\langle f,\boldsymbol{\Phi}_{j,k,l}^{-,+}\right\rangle\right|^2+\sum_{k,l\in\boldsymbol{Z}}\left|\left\langle f,\boldsymbol{\Phi}_{j,k,l}^{-,-}\right\rangle\right|^2\le B\|f\|_2^2,$$

对于 $\forall\varepsilon>0$，$f(x,y)\in\left(\bigcup_{j\in\boldsymbol{Z}}V_j\right)^{\perp}$，那么就有紧支撑连续函数 $\tilde{f}(x,y)$，使 $\left\|f-\tilde{f}\right\|_{L^2}\le\varepsilon$。

若 P_j 是 V_j 的正交投影算子，则 $\left\|P_j\tilde{f}\right\|=\left\|P_j\left(\tilde{f}-f\right)\right\|_{L^2}\le\varepsilon$。又因为 $P_j\tilde{f}\in V_j$，则

$$B\left\|P_j\tilde{f}\right\|^2\ge\sum_{k,l\in\boldsymbol{Z}}\left|\left\langle P_j\tilde{f},\boldsymbol{\Phi}_{j,k,l}^{+,+}\right\rangle\right|^2+\sum_{k,l\in\boldsymbol{Z}}\left|\left\langle P_j\tilde{f},\boldsymbol{\Phi}_{j,k,l}^{+,-}\right\rangle\right|^2+\sum_{k,l\in\boldsymbol{Z}}\left|\left\langle P_j\tilde{f},\boldsymbol{\Phi}_{j,k,l}^{-,+}\right\rangle\right|^2+\sum_{k,l\in\boldsymbol{Z}}\left|\left\langle P_j\tilde{f},\boldsymbol{\Phi}_{j,k,l}^{-,-}\right\rangle\right|^2,$$

进一步可以得到

$$\begin{aligned}&\sum_{k,l\in\boldsymbol{Z}}\left|\left\langle P_j\tilde{f},\boldsymbol{\Phi}_{j,k,l}^{+,+}\right\rangle\right|^2+\sum_{k,l\in\boldsymbol{Z}}\left|\left\langle P_j\tilde{f},\boldsymbol{\Phi}_{j,k,l}^{+,-}\right\rangle\right|^2+\sum_{k,l\in\boldsymbol{Z}}\left|\left\langle P_j\tilde{f},\boldsymbol{\Phi}_{j,k,l}^{-,+}\right\rangle\right|^2+\sum_{k,l\in\boldsymbol{Z}}\left|\left\langle P_j\tilde{f},\boldsymbol{\Phi}_{j,k,l}^{-,-}\right\rangle\right|^2\\&=\frac{1}{(2\pi)^4}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(\left|\hat{\Phi}\left(a^{-j}\omega_1,a^{-j}\omega_2\right)\right|^2+\left|\hat{\Phi}\left(a^{-j}\omega_1,-a^{-j}\omega_2\right)\right|^2+\left|\hat{\Phi}\left(-a^{-j}\omega_1,a^{-j}\omega_2\right)\right|^2\right.\\&\quad\left.+\left|\hat{\Phi}\left(-a^{-j}\omega_1,-a^{-j}\omega_2\right)\right|^2\right)\widehat{P_j\tilde{f}}(\omega_1,\omega_2)\,\mathrm{d}\omega_1\mathrm{d}\omega_2+D\end{aligned}$$

其中

$$D\le\left(\left\|\hat{\Phi}(\omega_1,\omega_2)\right\|_{L^\infty}^2+\left\|\hat{\Phi}(\omega_1,-\omega_2)\right\|_{L^\infty}^2+\left\|\hat{\Phi}(-\omega_1,\omega_2)\right\|_{L^\infty}^2+\left\|\hat{\Phi}(-\omega_1,-\omega_2)\right\|_{L^\infty}^2\right)\cdot\sum_{k\ne0,l\ne0}\left|P_j\tilde{f}\left(a^jk,a^jl\right)\right|^2.$$

因为 $\tilde{f}(x,y)\in C^\infty$，所以存在一个常数 M，使 $\left|\tilde{f}(x,y)\right|\le M\left(1+|x+y|\right)^{-2}$ 故

$$\begin{aligned}D&\le M^2\left(\left\|\hat{\Phi}(\omega_1,\omega_2)\right\|_{L^\infty}^2+\left\|\hat{\Phi}(\omega_1,-\omega_2)\right\|_{L^\infty}^2+\left\|\hat{\Phi}(-\omega_1,\omega_2)\right\|_{L^\infty}^2+\left\|\hat{\Phi}(-\omega_1,-\omega_2)\right\|_{L^\infty}^2\right)\cdot\sum_{k\ne0,l\ne0}\left|P_j\tilde{f}\left(a^jk,a^jl\right)\right|^2\\&\le M^2\left(\left\|\hat{\Phi}(\omega_1,\omega_2)\right\|_{L^\infty}^2+\left\|\hat{\Phi}(\omega_1,-\omega_2)\right\|_{L^\infty}^2+\left\|\hat{\Phi}(-\omega_1,\omega_2)\right\|_{L^\infty}^2+\left\|\hat{\Phi}(-\omega_1,-\omega_2)\right\|_{L^\infty}^2\right)\cdot\sum_{k\ne0,l\ne0}\left(1+\left|a^jk+a^jl\right|\right)^{-2}\\&\le M'^2a^{-2j}\end{aligned}$$

由以上的推导，可以得到

$$
\begin{aligned}
&\frac{1}{(2\pi)^4}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Big(\left|\hat{\Phi}\left(a^{-j}\omega_1,a^{-j}\omega_2\right)\right|^2+\left|\hat{\Phi}\left(a^{-j}\omega_1,-a^{-j}\omega_2\right)\right|^2+\left|\hat{\Phi}\left(-a^{-j}\omega_1,a^{-j}\omega_2\right)\right|^2 \\
&+\left|\hat{\Phi}\left(-a^{-j}\omega_1,-a^{-j}\omega_2\right)\right|^2\Big)\widehat{P_j\tilde{f}}(\omega_1,\omega_2)\mathrm{d}\omega_1\mathrm{d}\omega_2\le\sum_{k,l\in\boldsymbol{Z}}\left|\left\langle P_j\tilde{f},\Phi_{j,k,l}^{+,+}\right\rangle\right|^2 \\
&+\sum_{k,l\in\boldsymbol{Z}}\left|\left\langle P_j\tilde{f},\Phi_{j,k,l}^{+,-}\right\rangle\right|^2+\sum_{k,l\in\boldsymbol{Z}}\left|\left\langle P_j\tilde{f},\Phi_{j,k,l}^{-,+}\right\rangle\right|^2+\sum_{k,l\in\boldsymbol{Z}}\left|\left\langle P_j\tilde{f},\Phi_{j,k,l}^{-,-}\right\rangle\right|^2+|D| \\
&\le B\varepsilon^2+M'^2a^{-2j}.
\end{aligned}
$$

由于 $\Phi(\omega_1,\omega_2)$ 有界且在 $(\omega_1,\omega_2)=(0,0)$ 附近处连续且 $\Phi(0,0)\ne 0$，故当 $j\to+\infty$ 时，以上不等式收敛于

$$
\frac{1}{(2\pi)^4}\left(\left|\hat{\Phi}(0,0)\right|^2+\left|\hat{\Phi}(0,0)\right|^2+\left|\hat{\Phi}(0,0)\right|^2+\left|\hat{\Phi}(0,0)\right|^2\right)\cdot\left\|\widehat{P_j\tilde{f}}\right\|_{L^2}^2=\frac{1}{4\pi^4}\left|\hat{\Phi}(0,0)\right|^2\cdot\left\|\widehat{P_j\tilde{f}}\right\|_{L^2}^2
$$

从而 $\left\|\widehat{P_j\tilde{f}}\right\|_{L^2}^2\le\frac{(2\pi)^4B\varepsilon^2}{4\left|\hat{\Phi}(0,0)\right|^2}=\frac{4\pi^4B\varepsilon^2}{\left|\hat{\Phi}(0,0)\right|^2}$，所以 $\|f\|_{L^2}\le\varepsilon+\left\|\widehat{P_j\tilde{f}}\right\|_{L^2}\le\varepsilon+\frac{2\pi^2\sqrt{B\varepsilon}}{\sqrt{\left|\hat{\Phi}(0,0)\right|^2}}$，因为 ε 是任意小的，所以 $f=0$，则说明 $Close_{L^2(\boldsymbol{R}^2)}\bigcup_{j\in\boldsymbol{Z}}V_j=L^2(\boldsymbol{R}^2)$。

4. 双正交二维四向加细函数和小波

定理 5：如果二维四向加细函数是正交的，那么应该满足下列等式

$$
\begin{aligned}
&\langle\Phi(x,y),\Phi(x-k,y-l)\rangle=\delta_{0,k}\delta_{0,l}\boldsymbol{I}_r, &&\langle\Phi(x,y),\Phi(k-x,y-l)\rangle=\boldsymbol{O}_r,\\
&\langle\Phi(x,y),\Phi(x-k,l-y)\rangle=\boldsymbol{O}_r, &&\langle\Phi(x,y),\Phi(k-x,l-y)\rangle=\boldsymbol{O}_r.
\end{aligned}
$$

定理 6：如果二维四向加细函数 $\Phi(x,y)$ 和 $\tilde{\Phi}(x,y)$ 是双正交的，那么应该满足下列等式

$$
\begin{aligned}
&\langle\Phi(x,y),\tilde{\Phi}(x-k,y-l)\rangle=\delta_{0,k}\delta_{0,l}\boldsymbol{I}_r, &&\langle\Phi(x,y),\tilde{\Phi}(k-x,y-l)\rangle=\boldsymbol{O}_r,\\
&\langle\Phi(x,y),\tilde{\Phi}(x-k,l-y)\rangle=\boldsymbol{O}_r, &&\langle\Phi(x,y),\tilde{\Phi}(k-x,l-y)\rangle=\boldsymbol{O}_r.
\end{aligned}
$$

定理 7：如果二维四向加细函数 $\Phi(x,y)$ 和 $\tilde{\Phi}(x,y)$ 是双正交的尺度函数，那么它的双正面具，正负面具和双负面具符号都应该满足

$$
\begin{cases}
\sum_{k,l=0}^{a-1}\Big\{\boldsymbol{P}^{+,+}(p_1,p_2)\tilde{\boldsymbol{P}}^{+,+}(-p_1,-p_2)+\boldsymbol{P}^{-,+}(p_1,p_2)\tilde{\boldsymbol{P}}^{-,+}(-p_1,-p_2)\\
\quad+\boldsymbol{P}^{+,-}(p_1,p_2)\tilde{\boldsymbol{P}}^{+,-}(-p_1,-p_2)+\boldsymbol{P}^{-,-}(p_1,p_2)\tilde{\boldsymbol{P}}^{-,-}(-p_1,-p_2)\Big\}=\boldsymbol{I}_r\\
\sum_{k,l=0}^{a-1}\Big\{\boldsymbol{P}^{+,+}(p_1,p_2)\tilde{\boldsymbol{P}}^{+,-}(-p_1,p_2)+\boldsymbol{P}^{+,-}(p_1,p_2)\tilde{\boldsymbol{P}}^{+,+}(-p_1,p_2)\\
\quad+\boldsymbol{P}^{-,+}(p_1,p_2)\tilde{\boldsymbol{P}}^{-,-}(-p_1,p_2)+\boldsymbol{P}^{-,-}(p_1,p_2)\tilde{\boldsymbol{P}}^{-,+}(-p_1,p_2)\Big\}=\boldsymbol{O}_r\\
\sum_{k,l=0}^{a-1}\Big\{\boldsymbol{P}^{+,+}(p_1,p_2)\tilde{\boldsymbol{P}}^{-,+}(p_1,-p_2)+\boldsymbol{P}^{+,-}(p_1,p_2)\tilde{\boldsymbol{P}}^{-,-}(p_1,-p_2)\\
\quad+\boldsymbol{P}^{-,+}(p_1,p_2)\tilde{\boldsymbol{P}}^{+,+}(p_1,-p_2)+\boldsymbol{P}^{-,-}(p_1,p_2)\tilde{\boldsymbol{P}}^{+,-}(p_1,-p_2)\Big\}=\boldsymbol{O}_r\\
\sum_{k,l=0}^{a-1}\Big\{\boldsymbol{P}^{+,+}(p_1,p_2)\tilde{\boldsymbol{P}}^{-,-}(p_1,p_2)+\boldsymbol{P}^{+,-}(p_1,p_2)\tilde{\boldsymbol{P}}^{-,+}(p_1,p_2)\\
\quad+\boldsymbol{P}^{-,+}(p_1,p_2)\tilde{\boldsymbol{P}}^{+,-}(p_1,p_2)+\boldsymbol{P}^{-,-}(p_1,p_2)\tilde{\boldsymbol{P}}^{+,+}(p_1,p_2)\Big\}=\boldsymbol{O}_r
\end{cases}
\tag{15}
$$

其中： $p_1=\omega_1+\frac{2k\pi}{a}$ ； $p_2=\omega_2+\frac{2l\pi}{a}$ 。

证明：由文献[7]的定理 1 以及本文定理 5 定理 6 正交双正交定义易证。

假设 $\boldsymbol{\Phi}(x,y)$ 和 $\tilde{\boldsymbol{\Phi}}(x,y)$ 是双正交二维四向加细函数，对任意的 $j\in\boldsymbol{Z}$ ，定义 $V_{j+1}=V_j\oplus W_j$ ，其中 W_j 是 V_j 在 V_{j+1} 中的正交补。那么当 $j\neq k$ 时，就有 $W_j\perp W_k$ 并且 $L^2(R)=\oplus_{j\in\boldsymbol{Z}}W_j$ ，其中 $W_j=\overset{a-1}{\underset{h=1}{\oplus}}\left\{W_j^{h,1}\oplus W_j^{h,2}\oplus W_j^{h,3}\right\}$ ， $W_j^{h,1}\perp W_j^{h,2}$ ， $W_j^{h,1}\perp W_j^{h,1}$ ， $W_j^{h,2}\perp W_j^{h,3}$ 。

如果有 r 个小波函数 $\psi_1(x),\psi_2(x),\cdots,\psi_r(x)$ ，则记 $\boldsymbol{\Psi}^{h,i}(x)=\left[\psi_1(x),\psi_2(x),\cdots,\psi_r(x)\right]^{\mathrm{T}}$ ，则通过张量积的构造就有

$$\boldsymbol{\Psi}^{h,i}(x,y)=\boldsymbol{\Psi}^{h,i}(x)\psi(y)=\left[\boldsymbol{\Psi}_1(x,y),\boldsymbol{\Psi}_2(x,y),\cdots,\boldsymbol{\Psi}_r(x,y)\right]^{\mathrm{T}}$$

和

$$\tilde{\boldsymbol{\Psi}}^{h,i}(x,y)=\left[\tilde{\boldsymbol{\Psi}}_1(x,y),\tilde{\boldsymbol{\Psi}}_2(x,y),\cdots,\tilde{\boldsymbol{\Psi}}_r(x,y)\right]^{\mathrm{T}}$$

$(h=1,2,\cdots,a-1;i=1,2,3)$ ，使得集合 $\left\{\boldsymbol{\Psi}^{h,i}(x-k,y-l),\boldsymbol{\Psi}^{h,i}(x-k,l-y),\boldsymbol{\Psi}^{h,i}(k-x,y-l),\boldsymbol{\Psi}^{h,i}(k-x,l-y):k,l\in\boldsymbol{Z},h=1,2,\cdots,a-1;i=1,2,3\right\}$ 和集合 $\left\{\tilde{\boldsymbol{\Psi}}^{h,i}(x-k,y-l),\tilde{\boldsymbol{\Psi}}^{h,i}(x-k,l-y),\tilde{\boldsymbol{\Psi}}^{h,i}(k-x,y-l),\tilde{\boldsymbol{\Psi}}^{h,i}(k-x,l-y):k,l\in\boldsymbol{Z},h=1,2,\cdots,a-1;i=1,2,3\right\}$ 构成 W_0 的一组双正交基，则 $\boldsymbol{\Psi}^{h,i}(x,y)$ 和 $\tilde{\boldsymbol{\Psi}}^{h,i}(x,y)$ 是与 $\boldsymbol{\Phi}(x,y)$ 和 $\tilde{\boldsymbol{\Phi}}(x,y)$ 对应的双正交二维四向多小波函数，应该满足

$$\begin{gathered}
\tilde{\boldsymbol{\Phi}}(x,y),\boldsymbol{\Psi}^{h,i}(x-k,y-l)=\left\langle\tilde{\boldsymbol{\Phi}}(x,y),\boldsymbol{\Psi}^{h,i}(x-k,l-y)\right\rangle=\boldsymbol{O}_r,\\
\boldsymbol{\Phi}(x,y),\tilde{\boldsymbol{\Psi}}^{h,i}(k-x,y-l)=\left\langle\boldsymbol{\Phi}(x,y),\tilde{\boldsymbol{\Psi}}^{h,i}(k-x,l-y)\right\rangle=\boldsymbol{O}_r,\\
\tilde{\boldsymbol{\Phi}}(x,y),\boldsymbol{\Psi}^{h,i}(k-x,y-l)=\left\langle\tilde{\boldsymbol{\Phi}}(x,y),\boldsymbol{\Psi}^{h,i}(k-x,l-y)\right\rangle=\boldsymbol{O}_r,\\
\left\langle\boldsymbol{\Psi}^{n,m}(x,y),\tilde{\boldsymbol{\Psi}}^{s,t}(x-k,y-l)\right\rangle=\delta_{n,s}\delta_{m,t}\delta_{0,k}\delta_{0,l}\boldsymbol{I}_r,\\
\left\langle\boldsymbol{\Psi}^{n,m}(x,y),\tilde{\boldsymbol{\Psi}}^{s,t}(x-k,l-y)\right\rangle=\boldsymbol{O}_r,\\
\left\langle\tilde{\boldsymbol{\Psi}}^{n,m}(x,y),\tilde{\boldsymbol{\Psi}}^{s,t}(x-k,y-l)\right\rangle=\delta_{n,s}\delta_{m,t}\delta_{0,k}\delta_{0,l}\boldsymbol{I}_r,\\
\left\langle\tilde{\boldsymbol{\Psi}}^{n,m}(x,y),\tilde{\boldsymbol{\Psi}}^{s,t}(x-k,l-y)\right\rangle=\boldsymbol{O}_r,\\
\left\langle\boldsymbol{\Psi}^{n,m}(x,y),\boldsymbol{\Psi}^{s,t}(k-x,y-l)\right\rangle=\boldsymbol{O}_r,\\
\left\langle\boldsymbol{\Psi}^{n,m}(x,y),\boldsymbol{\Psi}^{s,t}(k-x,l-y)\right\rangle=\boldsymbol{O}_r,\\
\left\langle\tilde{\boldsymbol{\Psi}}^{n,m}(x,y),\tilde{\boldsymbol{\Psi}}^{s,t}(k-x,y-l)\right\rangle=\boldsymbol{O}_r,\\
\left\langle\tilde{\boldsymbol{\Psi}}^{n,m}(x,y),\tilde{\boldsymbol{\Psi}}^{s,t}(k-x,l-y)\right\rangle=\boldsymbol{O}_r,
\end{gathered}\tag{16}$$

其中： $h,n,s=1,2,\cdots,a-1;i,m,t=1,2,3$ 。

假如 $\boldsymbol{\Psi}^{h,i}(x,y)$ 是 $\boldsymbol{\Phi}(x,y)$ 对应的多小波函数，相应的 $\tilde{\boldsymbol{\Psi}}^{h,i}(x,y)$ 是 $\tilde{\boldsymbol{\Phi}}(x,y)$ 对应的多小波函数，那么就存在 $\left\{\boldsymbol{Q}_{k,l}^{h,i,+,+}\right\}_{k,l\in\boldsymbol{Z}}$ ， $\left\{\boldsymbol{Q}_{k,l}^{h,i,+,-}\right\}_{k,l\in\boldsymbol{Z}}$ ， $\left\{\boldsymbol{Q}_{k,l}^{h,i,-,+}\right\}_{k,l\in\boldsymbol{Z}}$ ， $\left\{\boldsymbol{Q}_{k,l}^{h,i,-,-}\right\}_{k,l\in\boldsymbol{Z}}$ 和 $\left\{\tilde{\boldsymbol{Q}}_{k,l}^{h,i,+,+}\right\}_{k,l\in\boldsymbol{Z}}$ ， $\left\{\tilde{\boldsymbol{Q}}_{k,l}^{h,i,+,-}\right\}_{k,l\in\boldsymbol{Z}}$ ， $\left\{\tilde{\boldsymbol{Q}}_{k,l}^{h,r,-,+}\right\}_{k,l\in\boldsymbol{Z}}$ ， $\left\{\tilde{\boldsymbol{Q}}_{k,l}^{h,i,-,-}\right\}_{k,l\in\boldsymbol{Z}}$ ，满足

$$\begin{aligned}
\boldsymbol{\Psi}^{h,i}(x,y)=&\sum_{k,l}\boldsymbol{Q}_{k,l}^{h,i,+,+}\boldsymbol{\Phi}(ax-k,ay-l)+\sum_{k,l}\boldsymbol{Q}_{k,l}^{h,i,+,-}\boldsymbol{\Phi}(ax-k,l-ay)\\
&+\sum_{k,l}\boldsymbol{Q}_{k,l}^{h,i,-,+}\boldsymbol{\Phi}(k-ax,ay-l)+\sum_{k,l}\boldsymbol{Q}_{k,l}^{h,i,-,-}\boldsymbol{\Phi}(k-ax,l-ay);
\end{aligned}\tag{17}$$

$$\tilde{\Psi}^{h,i}(x,y)=\sum_{k,l}\tilde{Q}_{k,l}^{h,i,+,+}\tilde{\Phi}(ax-k,ay-l)+\sum_{k,l}\tilde{Q}_{k,l}^{h,i,+,-}\tilde{\Phi}(ax-k,l-ay)$$
$$+\sum_{k,l}\tilde{Q}_{k,l}^{h,i,-,+}\tilde{\Phi}(k-ax,ay-l)+\sum_{k,l}\tilde{Q}_{k,l}^{h,i,-,-}\tilde{\Phi}(k-ax,l-ay). \quad (18)$$

对(17)式和(18)式两边做 *Fourier* 变换

$$\hat{\Psi}^{h,i}(\omega_1,\omega_2)=Q^{h,i,+,+}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right)\hat{\Phi}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right)+Q^{h,i,+,-}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right)\hat{\Phi}\left(\frac{\omega_1}{a},-\frac{\omega_2}{a}\right)$$
$$+Q^{h,i,-,+}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right)\hat{\Phi}\left(-\frac{\omega_1}{a},\frac{\omega_2}{a}\right)+Q^{h,i,-,-}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right)\hat{\Phi}\left(-\frac{\omega_1}{a},-\frac{\omega_2}{a}\right);$$

$$\hat{\tilde{\Psi}}^{h,i}(\omega_1,\omega_2)=\tilde{Q}^{h,i,+,+}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right)\hat{\Phi}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right)+\tilde{Q}^{h,i,+,-}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right)\hat{\Phi}\left(\frac{\omega_1}{a},-\frac{\omega_2}{a}\right)$$
$$+\tilde{Q}^{h,i,-,+}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right)\hat{\Phi}\left(-\frac{\omega_1}{a},\frac{\omega_2}{a}\right)+\tilde{Q}^{h,i,-,-}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right)\hat{\Phi}\left(-\frac{\omega_1}{a},-\frac{\omega_2}{a}\right);$$

其中

$$Q^{h,i,+,+}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right)=\frac{1}{a^2}\sum_{k,l}Q_{k,l}^{h,i,+,+}e^{-i\left(\frac{\omega_1 k}{a}+\frac{\omega_2 l}{a}\right)},Q^{h,i,+,-}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right)=\frac{1}{a^2}\sum_{k,l}Q_{k,l}^{h,i,+,-}e^{-i\left(\frac{\omega_1 k}{a}+\frac{\omega_2 l}{a}\right)},$$
$$Q^{h,i,-,+}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right)=\frac{1}{a^2}\sum_{k,l}Q_{k,l}^{h,i,-,+}e^{-i\left(\frac{\omega_1 k}{a}+\frac{\omega_2 l}{a}\right)},Q^{h,i,-,-}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right)=\frac{1}{a^2}\sum_{k,l}Q_{k,l}^{h,i,-,-}e^{-i\left(\frac{\omega_1 k}{a}+\frac{\omega_2 l}{a}\right)},$$

$(h=1,2,\cdots,a-1;i=1,2,3)$ 为 $\Psi^{h,i}(x,y)$ 的双正，正负，双负面具符号。

$$\tilde{Q}^{h,i,+,+}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right)=\frac{1}{a^2}\sum_{k,l}\tilde{Q}_{k,l}^{h,i,+,+}e^{-i\left(\frac{\omega_1 k}{a}+\frac{\omega_2 l}{a}\right)},\tilde{Q}^{h,i,+,-}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right)=\frac{1}{a^2}\sum_{k,l}\tilde{Q}_{k,l}^{h,i,+,-}e^{-i\left(\frac{\omega_1 k}{a}+\frac{\omega_2 l}{a}\right)},$$
$$\tilde{Q}^{h,i,-,+}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right)=\frac{1}{a^2}\sum_{k,l}\tilde{Q}_{k,l}^{h,i,-,+}e^{-i\left(\frac{\omega_1 k}{a}+\frac{\omega_2 l}{a}\right)},\tilde{Q}^{h,i,-,-}\left(\frac{\omega_1}{a},\frac{\omega_2}{a}\right)=\frac{1}{a^2}\sum_{k,l}\tilde{Q}_{k,l}^{h,i,-,-}e^{-i\left(\frac{\omega_1 k}{a}+\frac{\omega_2 l}{a}\right)},$$

$(h=1,2,\cdots,a-1;i=1,2,3)$ 为 $\tilde{\Psi}^{h,i}(x,y)$ 的双正，正负，双负面具符号。

定理 8： 若 $\Phi(x,y)$ 和 $\tilde{\Phi}(x,y)$ 是双正交二维四向多加细函数，$\Psi^{h,i}(x,y)$ 和 $\tilde{\Psi}^{h,i}(x,y)$ 是相应的双正交二维四向多小波函数，则它们的面具符号满足

$$\begin{cases}\sum_{k,l=0}^{a-1}\left\{Q^{m,s,+,+}(p_1,p_2)\tilde{Q}^{n,t,+,+}(-p_1,-p_2)+Q^{m,s,-,+}(p_1,p_2)\tilde{Q}^{n,t,-,+}(-p_1,-p_2)\right.\\ \left.\quad+Q^{m,s,+,-}(p_1,p_2)\tilde{Q}^{n,t,+,-}(-p_1,-p_2)+Q^{-,-}(p_1,p_2)\tilde{Q}^{n,t,-,-}(-p_1,-p_2)\right\}=\delta_{m,s}\delta_{n,t}I_r\\ \sum_{k,l=0}^{a-1}\left\{Q^{m,s,+,+}(p_1,p_2)\tilde{Q}^{n,t,+,-}(-p_1,p_2)+Q^{m,s,+,-}(p_1,p_2)\tilde{Q}^{n,t,+,+}(-p_1,p_2)\right.\\ \left.\quad+Q^{m,s,-,+}(p_1,p_2)\tilde{Q}^{n,t,-,-}(-p_1,p_2)+Q^{m,s,-,-}(p_1,p_2)\tilde{Q}^{n,t,-,+}(-p_1,p_2)\right\}=O_r\\ \sum_{k,l=0}^{a-1}\left\{Q^{m,s,+,+}(p_1,p_2)\tilde{Q}^{n,t,-,+}(p_1,-p_2)+Q^{m,s,+,-}(p_1,p_2)\tilde{Q}^{n,t,-,-}(p_1,-p_2)\right.\\ \left.\quad+Q^{m,s,-,+}(p_1,p_2)\tilde{Q}^{n,t,+,+}(p_1,-p_2)+Q^{m,s,-,-}(p_1,p_2)\tilde{Q}^{n,t,+,-}(p_1,-p_2)\right\}=O_r\\ \sum_{k,l=0}^{a-1}\left\{Q^{m,s,+,+}(p_1,p_2)\tilde{Q}^{n,t,-,-}(p_1,p_2)+Q^{m,s,+,-}(p_1,p_2)\tilde{Q}^{n,t,-,+}(p_1,p_2)\right.\\ \left.\quad+Q^{m,s,-,+}(p_1,p_2)\tilde{Q}^{n,t,+,-}(p_1,p_2)+Q^{m,s,-,-}(p_1,p_2)\tilde{Q}^{n,t,+,+}(p_1,p_2)\right\}=O_r\end{cases} \quad (19)$$

$$\begin{cases}\sum_{k,l=0}^{a-1}\left\{\boldsymbol{P}^{+,+}(p_1,p_2)\tilde{\boldsymbol{Q}}^{h,i,+,+}(-p_1,-p_2)+\boldsymbol{P}^{-,+}(p_1,p_2)\tilde{\boldsymbol{Q}}^{h,i,-,+}(-p_1,-p_2)\right.\\ \quad\left.+\boldsymbol{P}^{+,-}(p_1,p_2)\tilde{\boldsymbol{Q}}^{h,i,+,-}(-p_1,-p_2)+\boldsymbol{P}^{-,-}(p_1,p_2)\tilde{\boldsymbol{Q}}^{h,i,-,-}(-p_1,-p_2)\right\}=\boldsymbol{O}_r\\ \sum_{k,l=0}^{a-1}\left\{\boldsymbol{P}^{+,+}(p_1,p_2)\tilde{\boldsymbol{Q}}^{h,i,+,-}(-p_1,p_2)+\boldsymbol{P}^{+,-}(p_1,p_2)\tilde{\boldsymbol{Q}}^{h,i,+,+}(-p_1,p_2)\right.\\ \quad\left.+\boldsymbol{P}^{-,+}(p_1,p_2)\tilde{\boldsymbol{Q}}^{h,i,-,-}(-p_1,p_2)+\boldsymbol{P}^{-,-}(p_1,p_2)\tilde{\boldsymbol{Q}}^{h,i,-,+}(-p_1,p_2)\right\}=\boldsymbol{O}_r\\ \sum_{k,l=0}^{a-1}\left\{\boldsymbol{P}^{+,+}(p_1,p_2)\tilde{\boldsymbol{Q}}^{h,i,-,+}(p_1,-p_2)+\boldsymbol{P}^{+,-}(p_1,p_2)\tilde{\boldsymbol{Q}}^{h,i,-,-}(p_1,-p_2)\right.\\ \quad\left.+\boldsymbol{P}^{-,+}(p_1,p_2)\tilde{\boldsymbol{Q}}^{h,i,+,+}(p_1,-p_2)+\boldsymbol{P}^{-,-}(p_1,p_2)\tilde{\boldsymbol{Q}}^{h,i,+,-}(p_1,-p_2)\right\}=\boldsymbol{O}_r\\ \sum_{k,l=0}^{a-1}\left\{\boldsymbol{P}^{+,+}(p_1,p_2)\tilde{\boldsymbol{Q}}^{h,i,-,-}(p_1,p_2)+\boldsymbol{P}^{+,-}(p_1,p_2)\tilde{\boldsymbol{Q}}^{h,i,-,+}(p_1,p_2)\right.\\ \quad\left.+\boldsymbol{P}^{-,+}(p_1,p_2)\tilde{\boldsymbol{Q}}^{h,i,+,-}(p_1,p_2)+\boldsymbol{P}^{-,-}(p_1,p_2)\tilde{\boldsymbol{Q}}^{h,i,+,+}(p_1,p_2)\right\}=\boldsymbol{O}_r\\ \sum_{k,l=0}^{a-1}\left\{\tilde{\boldsymbol{P}}^{+,+}(p_1,p_2)\boldsymbol{Q}^{h,i,+,+}(-p_1,-p_2)+\tilde{\boldsymbol{P}}^{-,+}(p_1,p_2)\boldsymbol{Q}^{h,i,-,+}(-p_1,-p_2)\right.\\ \quad\left.+\tilde{\boldsymbol{P}}^{+,-}(p_1,p_2)\boldsymbol{Q}^{h,i,+,-}(-p_1,-p_2)+\tilde{\boldsymbol{P}}^{-,-}(p_1,p_2)\boldsymbol{Q}^{h,i,-,-}(-p_1,-p_2)\right\}=\boldsymbol{O}_r\\ \sum_{k,l=0}^{a-1}\left\{\tilde{\boldsymbol{P}}^{+,+}(p_1,p_2)\boldsymbol{Q}^{h,i,+,-}(-p_1,p_2)+\tilde{\boldsymbol{P}}^{+,-}(p_1,p_2)\boldsymbol{Q}^{h,i,+,+}(-p_1,p_2)\right.\\ \quad\left.+\tilde{\boldsymbol{P}}^{-,+}(p_1,p_2)\boldsymbol{Q}^{h,i,-,-}(-p_1,p_2)+\tilde{\boldsymbol{P}}^{-,-}(p_1,p_2)\boldsymbol{Q}^{h,i,-,+}(-p_1,p_2)\right\}=\boldsymbol{O}_r\\ \sum_{k,l=0}^{a-1}\left\{\tilde{\boldsymbol{P}}^{+,+}(p_1,p_2)\boldsymbol{Q}^{h,i,-,+}(p_1,-p_2)+\tilde{\boldsymbol{P}}^{+,-}(p_1,p_2)\boldsymbol{Q}^{h,i,-,-}(p_1,-p_2)\right.\\ \quad\left.+\tilde{\boldsymbol{P}}^{-,+}(p_1,p_2)\boldsymbol{Q}^{h,i,+,+}(p_1,-p_2)+\tilde{\boldsymbol{P}}^{-,-}(p_1,p_2)\boldsymbol{Q}^{h,i,+,-}(p_1,-p_2)\right\}=\boldsymbol{O}_r\\ \sum_{k,l=0}^{a-1}\left\{\tilde{\boldsymbol{P}}^{+,+}(p_1,p_2)\boldsymbol{Q}^{h,i,-,-}(p_1,p_2)+\tilde{\boldsymbol{P}}^{+,-}(p_1,p_2)\boldsymbol{Q}^{h,i,-,+}(p_1,p_2)\right.\\ \quad\left.+\tilde{\boldsymbol{P}}^{-,+}(p_1,p_2)\boldsymbol{Q}^{h,i,+,-}(p_1,p_2)+\tilde{\boldsymbol{P}}^{-,-}(p_1,p_2)\boldsymbol{Q}^{h,i,+,+}(p_1,p_2)\right\}=\boldsymbol{O}_r\end{cases}\tag{20}$$

其中： $h,m,n=1,2,\cdots,a-1;\ i,s,t=1,2,3;\ p_1=\omega_1+\frac{2k\pi}{a};\ p_2=\omega_2+\frac{2l\pi}{a}$。

证明：根据(16)式的正交性易得。

5. 构造算法

定理 9：如果 $\boldsymbol{\Phi}(x,y)$ 和 $\tilde{\boldsymbol{\Phi}}(x,y)$ 是双正交二维四向多加细函数， $\boldsymbol{\Psi}^{h,i}(x,y)$ 和 $\tilde{\boldsymbol{\Psi}}^{h,i}(x,y)$ 是相应的双正交二维四向多小波函数， $\boldsymbol{P}(\omega_1,\omega_2),\tilde{\boldsymbol{P}}(\omega_1,\omega_2)$ 和 $\boldsymbol{Q}^{h,i}(\omega_1,\omega_2),\tilde{\boldsymbol{Q}}^{h,i}(\omega_1,\omega_2)$ 是矩阵符号，构造

$$\begin{cases}\boldsymbol{P}^{+,+}(\omega_1,\omega_2)=\lambda_0(\omega_1,\omega_2)\boldsymbol{P}(\omega_1,\omega_2)+\sum_{j=1}^{a-1}\lambda_j(\omega_1,\omega_2)\boldsymbol{Q}^{h,i}(\omega_1,\omega_2),\\ \boldsymbol{P}^{+,-}(\omega_1,\omega_2)=\lambda_a(\omega_1,\omega_2)\boldsymbol{P}(\omega_1,-\omega_2)+\sum_{j=a+1}^{2a-1}\lambda_j(\omega_1,\omega_2)\boldsymbol{Q}^{h,i}(\omega_1,-\omega_2),\\ \boldsymbol{P}^{-,+}(\omega_1,\omega_2)=\lambda_{2a}(\omega_1,\omega_2)\boldsymbol{P}(-\omega_1,\omega_2)+\sum_{j=2a+1}^{3a-1}\lambda_j(\omega_1,\omega_2)\boldsymbol{Q}^{h,i}(-\omega_1,\omega_2),\\ \boldsymbol{P}^{-,-}(\omega_1,\omega_2)=\lambda_{3a}(\omega_1,\omega_2)\boldsymbol{P}(-\omega_1,-\omega_2)+\sum_{j=3a+1}^{4a-1}\lambda_j(\omega_1,\omega_2)\boldsymbol{Q}^{h,i}(-\omega_1,-\omega_2);\end{cases}$$

$$\begin{cases}\tilde{\boldsymbol{P}}^{+,+}(\omega_1,\omega_2)=\tilde{\lambda}_0(\omega_1,\omega_2)\tilde{\boldsymbol{P}}(\omega_1,\omega_2)+\sum_{j=1}^{a-1}\tilde{\lambda}_j(\omega_1,\omega_2)\tilde{\boldsymbol{Q}}^{h,i}(\omega_1,\omega_2),\\ \tilde{\boldsymbol{P}}^{+,-}(\omega_1,\omega_2)=\tilde{\lambda}_a(\omega_1,\omega_2)\tilde{\boldsymbol{P}}(\omega_1,-\omega_2)+\sum_{j=a+1}^{2a-1}\tilde{\lambda}_j(\omega_1,\omega_2)\tilde{\boldsymbol{Q}}^{h,i}(\omega_1,-\omega_2),\\ \tilde{\boldsymbol{P}}^{-,+}(\omega_1,\omega_2)=\tilde{\lambda}_{2a}(\omega_1,\omega_2)\tilde{\boldsymbol{P}}(-\omega_1,\omega_2)+\sum_{j=2a+1}^{3a-1}\tilde{\lambda}_j(\omega_1,\omega_2)\tilde{\boldsymbol{Q}}^{h,i}(-\omega_1,\omega_2),\\ \tilde{\boldsymbol{P}}^{-,-}(\omega_1,\omega_2)=\tilde{\lambda}_{3a}(\omega_1,\omega_2)\tilde{\boldsymbol{P}}(-\omega_1,-\omega_2)+\sum_{j=3a+1}^{4a-1}\tilde{\lambda}_j(\omega_1,\omega_2)\tilde{\boldsymbol{Q}}^{h,i}(-\omega_1,-\omega_2).\end{cases}$$

$(h:0\le h\le a-10\le h\le 4a-1;i:1\le i\le 3;i,h\in Z^+)$。函数 $\lambda_j(\omega_1,\omega_2)$ 和 $\tilde{\lambda}_j(\omega_1,\omega_2)$ 以 $\frac{2\pi}{a}$ 为周期，且满足

$$1)\begin{cases}\sum_{j=0}^{4a-1}\lambda_j(\omega_1,\omega_2)\tilde{\lambda}_j(-\omega_1,-\omega_2)=1,\\ \sum_{j=0}^{a-1}\left\{\lambda_j(\omega_1,\omega_2)\tilde{\lambda}_{a+j}(-\omega_1,-\omega_2)+\lambda_{2a+j}(\omega_1,\omega_2)\tilde{\lambda}_{3a+j}(-\omega_1,-\omega_2)\right\}=0,\\ \sum_{j=0}^{a-1}\left\{\lambda_j(\omega_1,\omega_2)\tilde{\lambda}_{2a+j}(-\omega_1,-\omega_2)+\lambda_{a+j}(\omega_1,\omega_2)\tilde{\lambda}_{3a+j}(-\omega_1,-\omega_2)\right\}=0,\\ \sum_{j=0}^{a-1}\left\{\lambda_j(\omega_1,\omega_2)\tilde{\lambda}_{3a+j}(-\omega_1,-\omega_2)+\lambda_{a+j}(\omega_1,\omega_2)\tilde{\lambda}_{2a+j}(-\omega_1,-\omega_2)\right\}=0,\\ \lambda_0(0,0)+\lambda_a(0,0)+\lambda_{2a}(0,0)+\lambda_{3a}(0,0)=1,\\ \tilde{\lambda}_0(0,0)+\tilde{\lambda}_a(0,0)+\tilde{\lambda}_{2a}(0,0)+\tilde{\lambda}_{3a}(0,0)=1,\\ \left|\lambda_0(0,0)-\lambda_a(0,0)-\lambda_{2a}(0,0)+\lambda_{3a}(0,0)\right|<1,\\ \left|\tilde{\lambda}_0(0,0)-\tilde{\lambda}_a(0,0)-\tilde{\lambda}_{2a}(0,0)+\tilde{\lambda}_{3a}(0,0)\right|<1,\\ \left|\lambda_0(0,0)+\lambda_a(0,0)-\lambda_{2a}(0,0)-\lambda_{3a}(0,0)\right|<1,\\ \left|\tilde{\lambda}_0(0,0)+\tilde{\lambda}_a(0,0)-\tilde{\lambda}_{2a}(0,0)-\tilde{\lambda}_{3a}(0,0)\right|<1,\\ \left|\lambda_0(0,0)-\lambda_a(0,0)+\lambda_{2a}(0,0)-\lambda_{3a}(0,0)\right|<1,\\ \left|\tilde{\lambda}_0(0,0)-\tilde{\lambda}_a(0,0)+\tilde{\lambda}_{2a}(0,0)-\tilde{\lambda}_{3a}(0,0)\right|<1;\end{cases}$$

$$2)\begin{cases}\sum_{j=0}^{4a-1}\lambda_j(\omega_1,\omega_2)\tilde{\lambda}_j(-\omega_1,-\omega_2)=1,\\ \sum_{j=0}^{a-1}\left\{\lambda_j(\omega_1,\omega_2)\tilde{\lambda}_{a+j}(-\omega_1,-\omega_2)+\lambda_{2a+j}(\omega_1,\omega_2)\tilde{\lambda}_{3a+j}(-\omega_1,-\omega_2)\right\}=0,\\ \sum_{j=0}^{a-1}\left\{\lambda_j(\omega_1,\omega_2)\tilde{\lambda}_{2a+j}(-\omega_1,-\omega_2)+\lambda_{a+j}(\omega_1,\omega_2)\tilde{\lambda}_{3a+j}(-\omega_1,-\omega_2)\right\}=0,\\ \sum_{j=0}^{a-1}\left\{\lambda_j(\omega_1,\omega_2)\tilde{\lambda}_{3a+j}(-\omega_1,-\omega_2)+\lambda_{a+j}(\omega_1,\omega_2)\tilde{\lambda}_{2a+j}(-\omega_1,-\omega_2)\right\}=0,\\ \left|\lambda_0(0,0)+\lambda_a(0,0)+\lambda_{2a}(0,0)+\lambda_{3a}(0,0)\right|<1,\\ \left|\tilde{\lambda}_0(0,0)+\tilde{\lambda}_a(0,0)+\tilde{\lambda}_{2a}(0,0)+\tilde{\lambda}_{3a}(0,0)\right|<1,\\ \lambda_0(0,0)-\lambda_a(0,0)-\lambda_{2a}(0,0)+\lambda_{3a}(0,0)=1,\\ \tilde{\lambda}_0(0,0)-\tilde{\lambda}_a(0,0)-\tilde{\lambda}_{2a}(0,0)+\tilde{\lambda}_{3a}(0,0)=1,\\ \left|\lambda_0(0,0)+\lambda_a(0,0)-\lambda_{2a}(0,0)-\lambda_{3a}(0,0)\right|<1,\\ \left|\tilde{\lambda}_0(0,0)+\tilde{\lambda}_a(0,0)-\tilde{\lambda}_{2a}(0,0)-\tilde{\lambda}_{3a}(0,0)\right|<1,\\ \left|\lambda_0(0,0)-\lambda_a(0,0)+\lambda_{2a}(0,0)-\lambda_{3a}(0,0)\right|<1,\\ \left|\tilde{\lambda}_0(0,0)-\tilde{\lambda}_a(0,0)+\tilde{\lambda}_{2a}(0,0)-\tilde{\lambda}_{3a}(0,0)\right|<1;\end{cases}$$

$$
3)\begin{cases}
\sum_{j=0}^{4a-1}\lambda_j(\omega_1,\omega_2)\tilde{\lambda}_j(-\omega_1,-\omega_2)=1,\\
\sum_{j=0}^{a-1}\left\{\lambda_j(\omega_1,\omega_2)\tilde{\lambda}_{a+j}(-\omega_1,-\omega_2)+\lambda_{2a+j}(\omega_1,\omega_2)\tilde{\lambda}_{3a+j}(-\omega_1,-\omega_2)\right\}=0,\\
\sum_{j=0}^{a-1}\left\{\lambda_j(\omega_1,\omega_2)\tilde{\lambda}_{2a+j}(-\omega_1,-\omega_2)+\lambda_{a+j}(\omega_1,\omega_2)\tilde{\lambda}_{3a+j}(-\omega_1,-\omega_2)\right\}=0,\\
\sum_{j=0}^{a-1}\left\{\lambda_j(\omega_1,\omega_2)\tilde{\lambda}_{3a+j}(-\omega_1,-\omega_2)+\lambda_{a+j}(\omega_1,\omega_2)\tilde{\lambda}_{2a+j}(-\omega_1,-\omega_2)\right\}=0,\\
\left|\lambda_0(0,0)+\lambda_a(0,0)+\lambda_{2a}(0,0)+\lambda_{3a}(0,0)\right|<1,\\
\left|\tilde{\lambda}_0(0,0)+\tilde{\lambda}_a(0,0)+\tilde{\lambda}_{2a}(0,0)+\tilde{\lambda}_{3a}(0,0)\right|<1,\\
\left|\lambda_0(0,0)-\lambda_a(0,0)-\lambda_{2a}(0,0)+\lambda_{3a}(0,0)\right|<1,\\
\left|\tilde{\lambda}_0(0,0)-\tilde{\lambda}_a(0,0)-\tilde{\lambda}_{2a}(0,0)+\tilde{\lambda}_{3a}(0,0)\right|<1,\\
\lambda_0(0,0)+\lambda_a(0,0)-\lambda_{2a}(0,0)-\lambda_{3a}(0,0)=1,\\
\tilde{\lambda}_0(0,0)+\tilde{\lambda}_a(0,0)-\tilde{\lambda}_{2a}(0,0)-\tilde{\lambda}_{3a}(0,0)=1,\\
\left|\lambda_0(0,0)-\lambda_a(0,0)+\lambda_{2a}(0,0)-\lambda_{3a}(0,0)\right|<1,\\
\left|\tilde{\lambda}_0(0,0)-\tilde{\lambda}_a(0,0)+\tilde{\lambda}_{2a}(0,0)-\tilde{\lambda}_{3a}(0,0)\right|<1;
\end{cases}
$$

$$
4)\begin{cases}
\sum_{j=0}^{4a-1}\lambda_j(\omega_1,\omega_2)\tilde{\lambda}_j(-\omega_1,-\omega_2)=1,\\
\sum_{j=0}^{a-1}\left\{\lambda_j(\omega_1,\omega_2)\tilde{\lambda}_{a+j}(-\omega_1,-\omega_2)+\lambda_{2a+j}(\omega_1,\omega_2)\tilde{\lambda}_{3a+j}(-\omega_1,-\omega_2)\right\}=0,\\
\sum_{j=0}^{a-1}\left\{\lambda_j(\omega_1,\omega_2)\tilde{\lambda}_{2a+j}(-\omega_1,-\omega_2)+\lambda_{a+j}(\omega_1,\omega_2)\tilde{\lambda}_{3a+j}(-\omega_1,-\omega_2)\right\}=0,\\
\sum_{j=0}^{a-1}\left\{\lambda_j(\omega_1,\omega_2)\tilde{\lambda}_{3a+j}(-\omega_1,-\omega_2)+\lambda_{a+j}(\omega_1,\omega_2)\tilde{\lambda}_{2a+j}(-\omega_1,-\omega_2)\right\}=0,\\
\left|\lambda_0(0,0)+\lambda_a(0,0)+\lambda_{2a}(0,0)+\lambda_{3a}(0,0)\right|<1,\\
\left|\tilde{\lambda}_0(0,0)+\tilde{\lambda}_a(0,0)+\tilde{\lambda}_{2a}(0,0)+\tilde{\lambda}_{3a}(0,0)\right|<1,\\
\left|\lambda_0(0,0)-\lambda_a(0,0)-\lambda_{2a}(0,0)+\lambda_{3a}(0,0)\right|<1,\\
\left|\tilde{\lambda}_0(0,0)-\tilde{\lambda}_a(0,0)-\tilde{\lambda}_{2a}(0,0)+\tilde{\lambda}_{3a}(0,0)\right|<1,\\
\left|\lambda_0(0,0)+\lambda_a(0,0)-\lambda_{2a}(0,0)-\lambda_{3a}(0,0)\right|<1,\\
\left|\tilde{\lambda}_0(0,0)+\tilde{\lambda}_a(0,0)-\tilde{\lambda}_{2a}(0,0)-\tilde{\lambda}_{3a}(0,0)\right|<1,\\
\lambda_0(0,0)-\lambda_a(0,0)+\lambda_{2a}(0,0)-\lambda_{3a}(0,0)=1,\\
\tilde{\lambda}_0(0,0)-\tilde{\lambda}_a(0,0)+\tilde{\lambda}_{2a}(0,0)-\tilde{\lambda}_{3a}(0,0)=1;
\end{cases}
$$

则 $\boldsymbol{P}^{+,+}(\omega_1,\omega_2),\boldsymbol{P}^{+,-}(\omega_1,\omega_2),\boldsymbol{P}^{-,+}(\omega_1,\omega_2),\boldsymbol{P}^{-,-}(\omega_1,\omega_2)$ 和 $\tilde{\boldsymbol{P}}^{+,+}(\omega_1,\omega_2),\tilde{\boldsymbol{P}}^{+,-}(\omega_1,\omega_2),\tilde{\boldsymbol{P}}^{-,+}(\omega_1,\omega_2),\tilde{\boldsymbol{P}}^{-,-}(\omega_1,\omega_2)$，产生一个双正交二维四向多细分函数 $\boldsymbol{\Phi}(x,y)$ 并满足

$$
\boldsymbol{\Phi}(x,y)=\sum_{k,l}\boldsymbol{P}_{l,k}^{+,+}\boldsymbol{\Phi}(ax-k,ay-l)+\sum_{k,l}\boldsymbol{P}_{l,k}^{+,-}\boldsymbol{\Phi}(ax-k,l-ay)+\sum_{k,l}\boldsymbol{P}_{l,k}^{-,+}\boldsymbol{\Phi}(k-ax,ay-l)+\sum_{k,l}\boldsymbol{P}_{l,k}^{-,-}\boldsymbol{\Phi}(k-ax,l-ay).
$$

$$
\tilde{\boldsymbol{\Phi}}(x,y)=\sum_{k,l}\tilde{\boldsymbol{P}}_{l,k}^{+,+}\tilde{\boldsymbol{\Phi}}(ax-k,ay-l)+\sum_{k,l}\tilde{\boldsymbol{P}}_{l,k}^{+,-}\tilde{\boldsymbol{\Phi}}(ax-k,l-ay)+\sum_{k,l}\tilde{\boldsymbol{P}}_{l,k}^{-,+}\tilde{\boldsymbol{\Phi}}(k-ax,ay-l)+\sum_{k,l}\tilde{\boldsymbol{P}}_{l,k}^{-,-}\tilde{\boldsymbol{\Phi}}(k-ax,l-ay).
$$

证明：将上面已经构造的 $\boldsymbol{P}^{+,+}(\omega_1,\omega_2),\boldsymbol{P}^{+,-}(\omega_1,\omega_2),\boldsymbol{P}^{-,+}(\omega_1,\omega_2),\boldsymbol{P}^{-,-}(\omega_1,\omega_2)$ 和 $\tilde{\boldsymbol{P}}^{+,+}(\omega_1,\omega_2),\tilde{\boldsymbol{P}}^{+,-}(\omega_1,\omega_2),\tilde{\boldsymbol{P}}^{-,+}(\omega_1,\omega_2),\tilde{\boldsymbol{P}}^{-,-}(\omega_1,\omega_2)$，代入(15)式化简可得。再令(11)式 $\boldsymbol{P}(\omega_1,\omega_2)=\boldsymbol{P}(0,0)$，则化简 $\boldsymbol{P}(0,0)$ 并求特征值可证得定理。

定理 10： 如果 $\boldsymbol{\Phi}(x,y)$ 和 $\tilde{\boldsymbol{\Phi}}(x,y)$ 是双正交二维四向多加细函数，$\boldsymbol{\Psi}^{h,i}(x,y)$ 和 $\tilde{\boldsymbol{\Psi}}^{h,i}(x,y)$ 是相应的双正交二维四向多小波函数，$\boldsymbol{P}(\omega_1,\omega_2),\tilde{\boldsymbol{P}}(\omega_1,\omega_2)$ 和 $\boldsymbol{Q}^{j,s}(\omega_1,\omega_2),\tilde{\boldsymbol{Q}}^{j,s}(\omega_1,\omega_2)$ 是面具符号，构造

$$\begin{cases}\boldsymbol{Q}^{h,i,+,+}(\omega_1,\omega_2)=\mu_0^{h,i}(\omega_1,\omega_2)\boldsymbol{P}(\omega_1,\omega_2)+\sum_{j=1}^{a-1}\mu_j^{h,i}(\omega_1,\omega_2)\boldsymbol{Q}^{j,s}(\omega_1,\omega_2),\\ \boldsymbol{Q}^{h,i,+,-}(\omega_1,\omega_2)=\mu_a^{h,i}(\omega_1,\omega_2)\boldsymbol{P}(\omega_1,-\omega_2)+\sum_{j=a+1}^{2a-1}\mu_j^{h,i}(\omega_1,\omega_2)\boldsymbol{Q}^{j,s}(\omega_1,-\omega_2),\\ \boldsymbol{Q}^{h,i,-,+}(\omega_1,\omega_2)=\mu_{2a}^{h,i}(\omega_1,\omega_2)\boldsymbol{P}(-\omega_1,\omega_2)+\sum_{j=2a+1}^{3a-1}\mu_j^{h,i}(\omega_1,\omega_2)\boldsymbol{Q}^{j,s}(-\omega_1,\omega_2),\\ \boldsymbol{Q}^{h,i,-,-}(\omega_1,\omega_2)=\mu_{3a}^{h,i}(\omega_1,\omega_2)\boldsymbol{P}(-\omega_1,-\omega_2)+\sum_{j=3a+1}^{4a-1}\mu_j^{h,i}(\omega_1,\omega_2)\boldsymbol{Q}^{j,s}(-\omega_1,-\omega_2);\end{cases}$$

$$\begin{cases}\tilde{\boldsymbol{Q}}^{h,i,+,+}(\omega_1,\omega_2)=\tilde{\mu}_0^{h,i}(\omega_1,\omega_2)\tilde{\boldsymbol{P}}(\omega_1,\omega_2)+\sum_{j=1}^{a-1}\tilde{\mu}_j^{h,i}(\omega_1,\omega_2)\tilde{\boldsymbol{Q}}^{j,s}(\omega_1,\omega_2),\\ \tilde{\boldsymbol{Q}}^{h,i,+,-}(\omega_1,\omega_2)=\tilde{\mu}_a^{h,i}(\omega_1,\omega_2)\tilde{\boldsymbol{P}}(\omega_1,-\omega_2)+\sum_{j=a+1}^{2a-1}\tilde{\mu}_j^{h,i}(\omega_1,\omega_2)\tilde{\boldsymbol{Q}}^{j,s}(\omega_1,-\omega_2),\\ \tilde{\boldsymbol{Q}}^{h,i,-,+}(\omega_1,\omega_2)=\tilde{\mu}_{2a}^{h,i}(\omega_1,\omega_2)\tilde{\boldsymbol{P}}(-\omega_1,\omega_2)+\sum_{j=2a+1}^{3a-1}\tilde{\mu}_j^{h,i}(\omega_1,\omega_2)\tilde{\boldsymbol{Q}}^{j,s}(-\omega_1,\omega_2),\\ \tilde{\boldsymbol{Q}}^{h,i,-,-}(\omega_1,\omega_2)=\tilde{\mu}_{3a}^{h,i}(\omega_1,\omega_2)\tilde{\boldsymbol{P}}(-\omega_1,-\omega_2)+\sum_{j=3a+1}^{4a-1}\tilde{\mu}_j^{h,i}(\omega_1,\omega_2)\tilde{\boldsymbol{Q}}^{j,s}(-\omega_1,-\omega_2).\end{cases}$$

$(h:0\le h\le a-10\le h\le 4a-1;i,s:1\le i,s\le 3;i,h,s\in Z^+)$。函数 $\lambda_j(\omega_1,\omega_2)$，$\mu_j^{h,i}(\omega_1,\omega_2)$ 和 $\tilde{\mu}_j^{h,i}(\omega_1,\omega_2)$ 以 $\frac{2\pi}{a}$ 为周期，且满足

1) $$\begin{cases}\sum_{j=0}^{4a-1}\lambda_j(\omega_1,\omega_2)\tilde{\mu}_j^{h,i}(-\omega_1,-\omega_2)=0,\\ \sum_{j=0}^{4a-1}\tilde{\lambda}_j(\omega_1,\omega_2)\mu_j^{h,i}(-\omega_1,-\omega_2)=0,\\ \sum_{j=0}^{a-1}\lambda_j(\omega_1,\omega_2)\tilde{\mu}_{j+a}^{h,i}(-\omega_1,-\omega_2)+\lambda_{j+a}(\omega_1,\omega_2)\tilde{\mu}_j^{h,i}(-\omega_1,-\omega_2)\\ \quad+\lambda_{j+2a}(\omega_1,\omega_2)\tilde{\mu}_{j+3a}^{h,i}(-\omega_1,-\omega_2)+\lambda_{j+3a}(\omega_1,\omega_2)\tilde{\mu}_{j+2a}^{h,i}(-\omega_1,-\omega_2)=0,\\ \sum_{j=0}^{a-1}\tilde{\lambda}_j(\omega_1,\omega_2)\mu_{j+a}^{h,i}(-\omega_1,-\omega_2)+\tilde{\lambda}_{j+a}(\omega_1,\omega_2)\mu_j^{h,i}(-\omega_1,-\omega_2)\\ \quad+\tilde{\lambda}_{j+2a}(\omega_1,\omega_2)\mu_{j+3a}^{h,i}(-\omega_1,-\omega_2)+\tilde{\lambda}_{j+3a}(\omega_1,\omega_2)\mu_{j+2a}^{h,i}(-\omega_1,-\omega_2)=0,\\ \sum_{j=0}^{a-1}\lambda_j(\omega_1,\omega_2)\tilde{\mu}_{j+2a}^{h,i}(-\omega_1,-\omega_2)+\lambda_{j+2a}(\omega_1,\omega_2)\tilde{\mu}_j^{h,i}(-\omega_1,-\omega_2)\\ \quad+\lambda_{j+a}(\omega_1,\omega_2)\tilde{\mu}_{j+3a}^{h,i}(-\omega_1,-\omega_2)+\lambda_{j+3a}(\omega_1,\omega_2)\tilde{\mu}_{j+a}^{h,i}(-\omega_1,-\omega_2)=0,\\ \sum_{j=0}^{a-1}\tilde{\lambda}_j(\omega_1,\omega_2)\mu_{j+2a}^{h,i}(-\omega_1,-\omega_2)+\tilde{\lambda}_{j+2a}(\omega_1,\omega_2)\mu_j^{h,i}(-\omega_1,-\omega_2)\\ \quad+\tilde{\lambda}_{j+a}(\omega_1,\omega_2)\mu_{j+3a}^{h,i}(-\omega_1,-\omega_2)+\tilde{\lambda}_{j+3a}(\omega_1,\omega_2)\mu_{j+a}^{h,i}(-\omega_1,-\omega_2)=0,\\ \sum_{j=0}^{a-1}\lambda_j(\omega_1,\omega_2)\tilde{\mu}_{j+3a}^{h,i}(-\omega_1,-\omega_2)+\lambda_{j+3a}(\omega_1,\omega_2)\tilde{\mu}_j^{h,i}(-\omega_1,-\omega_2)\\ \quad+\lambda_{j+a}(\omega_1,\omega_2)\tilde{\mu}_{j+2a}^{h,i}(-\omega_1,-\omega_2)+\lambda_{j+2a}((\omega_1,\omega_2)\tilde{\mu}_{j+a}^{h,i}(-\omega_1,-\omega_2)=0,\\ \sum_{j=0}^{a-1}\tilde{\lambda}_j(\omega_1,\omega_2)\mu_{j+3a}^{h,i}(-\omega_1,-\omega_2)+\tilde{\lambda}_{j+3a}(\omega_1,\omega_2)\mu_j^{h,i}(-\omega_1,-\omega_2)\\ \quad+\tilde{\lambda}_{j+a}(\omega_1,\omega_2)\mu_{j+2a}^{h,i}(-\omega_1,-\omega_2)+\tilde{\lambda}_{j+2a}(\omega_1,\omega_2)\mu_{j+a}^{h,i}(-\omega_1,-\omega_2)=0,\end{cases}$$

$$2)\ \begin{cases} \sum_{j=0}^{4a-1} \mu_j^{h,k}(\omega_1,\omega_2)\tilde{\mu}_j^{s,t}(-\omega_1,-\omega_2) = \delta_{h,s}\delta_{k,t}, \\ \sum_{j=0}^{a-1}\left\{\mu_j^{h,i}(\omega_1,\omega_2)\tilde{\mu}_{a+j}^{h,i}(-\omega_1,-\omega_2)+\mu_{2a+j}^{h,i}(\omega_1,\omega_2)\tilde{\mu}_{3a+j}^{h,i}(-\omega_1,-\omega_2)\right\}=0, \\ \sum_{j=0}^{a-1}\left\{\mu_j^{h,i}(\omega_1,\omega_2)\tilde{\mu}_{2a+j}^{h,i}(-\omega_1,-\omega_2)+\mu_{a+j}^{h,i}(\omega_1,\omega_2)\tilde{\mu}_{3a+j}^{h,i}(-\omega_1,-\omega_2)\right\}=0, \\ \sum_{j=0}^{a-1}\left\{\mu_j^{h,i}(\omega_1,\omega_2)\tilde{\mu}_{3a+j}^{h,i}(-\omega_1,-\omega_2)+\mu_{a+j}^{h,i}(\omega_1,\omega_2)\tilde{\mu}_{2a+j}^{h,i}(-\omega_1,-\omega_2)\right\}=0; \end{cases}$$

那么就可以得出 $\boldsymbol{Q}^{h,i,+,+}(\omega_1,\omega_2),\boldsymbol{Q}^{h,i,+,-}(\omega_1,\omega_2),\boldsymbol{Q}^{h,i,-,+}(\omega_1,\omega_2),\boldsymbol{Q}^{h,i,-,-}(\omega_1,\omega_2)$ 和 $\tilde{\boldsymbol{Q}}^{h,i,+,+}(\omega_1,\omega_2),\tilde{\boldsymbol{Q}}^{h,i,+,-}(\omega_1,\omega_2),\tilde{\boldsymbol{Q}}^{h,i,-,+}(\omega_1,\omega_2),\tilde{\boldsymbol{Q}}^{h,i,-,-}(\omega_1,\omega_2)$ 产生一个双正交二维四向多小波函数 $\boldsymbol{\Phi}(x,y)$ 并满足

$$\begin{aligned}\boldsymbol{\Psi}^{h,i}(x,y) &= \sum_{k,l}\boldsymbol{Q}_{k,l}^{h,i,+,+}\boldsymbol{\Phi}(ax-k,ay-l)+\sum_{k,l}\boldsymbol{Q}_{k,l}^{h,i,+,-}\boldsymbol{\Phi}(ax-k,l-ay)\\ &\quad+\sum_{k,l}\boldsymbol{Q}_{k,l}^{h,i,-,+}\boldsymbol{\Phi}(k-ax,ay-l)+\sum_{k,l}\boldsymbol{Q}_{k,l}^{h,i,-,-}\boldsymbol{\Phi}(k-ax,l-ay);\\ \tilde{\boldsymbol{\Psi}}^{h,i}(x,y) &= \sum_{k,l}\tilde{\boldsymbol{Q}}_{k,l}^{h,i,+,+}\tilde{\boldsymbol{\Phi}}(ax-k,ay-l)+\sum_{k,l}\tilde{\boldsymbol{Q}}_{k,l}^{h,i,+,-}\tilde{\boldsymbol{\Phi}}(ax-k,l-ay)\\ &\quad+\sum_{k,l}\tilde{\boldsymbol{Q}}_{k,l}^{h,i,-,+}\tilde{\boldsymbol{\Phi}}(k-ax,ay-l)+\sum_{k,l}\tilde{\boldsymbol{Q}}_{k,l}^{h,i,-,-}\tilde{\boldsymbol{\Phi}}(k-ax,l-ay).\end{aligned}$$

证明：由于 $\boldsymbol{\Phi}(x,y)$，$\tilde{\boldsymbol{\Phi}}(x,y)$ 和 $\boldsymbol{\Psi}^{h,i}(x,y)$，$\tilde{\boldsymbol{\Psi}}^{h,i}(x,y)$ 的双正交性，可以将已构造的 $\boldsymbol{Q}^{h,i,+,+}(\omega_1,\omega_2),\boldsymbol{Q}^{h,i,+,-}(\omega_1,\omega_2),\boldsymbol{Q}^{h,i,-,+}(\omega_1,\omega_2),\boldsymbol{Q}^{h,i,-,-}(\omega_1,\omega_2)$ 和 $\tilde{\boldsymbol{Q}}^{h,i,+,+}(\omega_1,\omega_2),\tilde{\boldsymbol{Q}}^{h,i,+,-}(\omega_1,\omega_2),\tilde{\boldsymbol{Q}}^{h,i,-,+}(\omega_1,\omega_2),\tilde{\boldsymbol{Q}}^{h,i,-,-}(\omega_1,\omega_2)$，代入(16)式和(19)式化简可证得定理。

6. *Mallat* 算法

如果 $\boldsymbol{\Phi}(x,y)$ 和 $\tilde{\boldsymbol{\Phi}}(x,y)$ 是双正交二维四向多加细函数，$\boldsymbol{\Psi}^{h,i}(x,y)$ 和 $\tilde{\boldsymbol{\Psi}}^{h,i}(x,y)$ 是相应的双正交二维四向多小波函数，对于能量有限信号 $f(x,y)\in L(\boldsymbol{R}^2)$ 在分辨率 a_j 下的近似函数 $f_j(x,y)\in V_j$，则 V_{j+1} 可分解为 $V_{j+1}=V_j\oplus W_j=V_j\bigoplus_{h=1}^{a-1}\left\{W_j^{h,1}\oplus W_j^{h,2}\oplus W_j^{h,3}\right\}$，则 $\forall f_{j+1}(x,y)\in V_{j+1}$，记 $\boldsymbol{c}_{j,k_1,k_2}=\left[c_{j,k_1,k_2}^1,c_{j,k_1,k_2}^2,\cdots,c_{j,k_1,k_2}^r\right]^{\mathrm{T}}$，$\boldsymbol{d}_{j,k_1,k_2}=\left[d_{j,k_1,k_2}^1,d_{j,k_1,k_2}^2,\cdots,d_{j,k_1,k_2}^r\right]^{\mathrm{T}}$ 有

$$\begin{aligned}f_{j+1}(x,y) &= \sum_{k_1,k_2}\boldsymbol{c}_{j+1,k_1,k_2}^{+,+}\boldsymbol{\Phi}_{j+1,k_1,k_2}^{+,+}+\sum_{k_1,k_2}\boldsymbol{c}_{j+1,k_1,k_2}^{+,-}\boldsymbol{\Phi}_{j+1,k_1,k_2}^{+,-}\\ &\quad+\sum_{k_1,k_2}\boldsymbol{c}_{j+1,k_1,k_2}^{-,+}\boldsymbol{\Phi}_{j+1,k_1,k_2}^{-,+}+\sum_{k_1,k_2}\boldsymbol{c}_{j+1,k_1,k_2}^{-,-}\boldsymbol{\Phi}_{j+1,k_1,k_2}^{-,-}\\ &= \sum_{k_1,k_2}\boldsymbol{c}_{j,k_1,k_2}^{+,+}\boldsymbol{\Phi}_{j,k_1,k_2}^{+,+}+\sum_{k_1,k_2}\boldsymbol{c}_{j,k_1,k_2}^{+,-}\boldsymbol{\Phi}_{j,k_1,k_2}^{+,-}+\sum_{k_1,k_2}\boldsymbol{c}_{j,k_1,k_2}^{-,+}\boldsymbol{\Phi}_{j,k_1,k_2}^{-,+}\\ &\quad+\sum_{k_1,k_2}\boldsymbol{c}_{j,k_1,k_2}^{-,-}\boldsymbol{\Phi}_{j,k_1,k_2}^{-,-}+\sum_{h=1}^{a-1}\sum_{r=1}^{3}\Bigg[\sum_{k_1,k_2}\boldsymbol{d}_{j,k_1,k_2}^{h,i,+,+}\boldsymbol{\Psi}_{j,k_1,k_2}^{h,i,+,+}+\sum_{k_1,k_2}\boldsymbol{d}_{j,k_1,k_2}^{h,i,+,-}\boldsymbol{\Psi}_{j,k_1,k_2}^{h,i,+,-}\\ &\quad+\sum_{k_1,k_2}\boldsymbol{d}_{j,k_1,k_2}^{h,i,-,+}\boldsymbol{\Psi}_{j,k_1,k_2}^{h,i,-,+}+\sum_{k_1,k_2}\boldsymbol{d}_{j,k_1,k_2}^{h,i,-,-}\boldsymbol{\Psi}_{j,k_1,k_2}^{h,i,-,-}\Bigg]\end{aligned}$$

6.1. 分解算法

如果 $f(x,y)$ 在分辨率 a_{j+1} 下的近似函数为 $f_{j+1}(x,y)$，那么它可以进一步分解为 $f(x,y)$ 在 a_j 下的

主要部分(通过低通滤波器得到)和细节部分(通过高通滤波器得到)。即分解算法要实现的目标就是：已知 $\boldsymbol{c}^{+,+}_{j+1,k_1,k_2},\boldsymbol{c}^{+,-}_{j+1,k_1,k_2},\boldsymbol{c}^{-,+}_{j+1,k_1,k_2},\boldsymbol{c}^{-,-}_{j+1,k_1,k_2}$，求 $\boldsymbol{c}^{+,+}_{j,k_1,k_2},\boldsymbol{c}^{+,-}_{j,k_1,k_2},\boldsymbol{c}^{-,+}_{j,k_1,k_2},\boldsymbol{c}^{-,-}_{j,k_1,k_2}$ 和 $\boldsymbol{d}^{+,+}_{j,k_1,k_2},\boldsymbol{d}^{+,-}_{j,k_1,k_2},\boldsymbol{d}^{-,+}_{j,k_1,k_2},\boldsymbol{d}^{-,-}_{j,k_1,k_2}$。

定理 11： 若 $f_{j+1}(x,y)=\sum_{k_1,k_2}\boldsymbol{c}^{+,+}_{j+1,k_1,k_2}\boldsymbol{\Phi}^{+,+}_{j+1,k_1,k_2}+\sum_{k_1,k_2}\boldsymbol{c}^{+,-}_{j+1,k_1,k_2}\boldsymbol{\Phi}^{+,-}_{j+1,k_1,k_2}+\sum_{k_1,k_2}\boldsymbol{c}^{-,+}_{j+1,k_1,k_2}\boldsymbol{\Phi}^{-,+}_{j+1,k_1,k_2}+\sum_{k_1,k_2}\boldsymbol{c}^{-,-}_{j+1,k_1,k_2}\boldsymbol{\Phi}^{-,-}_{j+1,k_1,k_2}$ 并且由(1)式和(17)式，我们可以得到下列分解公式

$$\boldsymbol{c}^{+,+}_{j,n_1,n_2}=\frac{1}{a^2}\Bigg[\sum_{k_1,k_2}\boldsymbol{c}^{+,+}_{j+1,k_1,k_2}\overline{\boldsymbol{P}^{+,+}_{k_1-an_1,k_2-an_2}}+\sum_{k_1,k_2}\boldsymbol{c}^{+,-}_{j+1,k_1,k_2}\overline{\boldsymbol{P}^{+,-}_{k_1-an_1,k_2-an_2}}$$
$$+\sum_{k_1,k_2}\boldsymbol{c}^{-,+}_{j+1,k_1,k_2}\overline{\boldsymbol{P}^{-,+}_{k_1-an_1,k_2-an_2}}+\sum_{k_1,k_2}\boldsymbol{c}^{-,-}_{j+1,k_1,k_2}\overline{\boldsymbol{P}^{-,-}_{k_1-an_1,k_2-an_2}}\Bigg];$$

$$\boldsymbol{c}^{+,-}_{j,n_1,n_2}=\frac{1}{a^2}\Bigg[\sum_{k_1,k_2}\boldsymbol{c}^{+,+}_{j+1,k_1,k_2}\overline{\boldsymbol{P}^{+,-}_{k_1-an_1,an_2-k_2}}+\sum_{k_1,k_2}\boldsymbol{c}^{+,-}_{j+1,k_1,k_2}\overline{\boldsymbol{P}^{+,+}_{k_1-an_1,an_2-k_2}}$$
$$+\sum_{k_1,k_2}\boldsymbol{c}^{-,+}_{j+1,k_1,k_2}\overline{\boldsymbol{P}^{-,-}_{k_1-an_1,an_2-k_2}}+\sum_{k_1,k_2}\boldsymbol{c}^{-,-}_{j+1,k_1,k_2}\overline{\boldsymbol{P}^{-,+}_{k_1-an_1,an_2-k_2}}\Bigg];$$

$$\boldsymbol{c}^{-,+}_{j,n_1,n_2}=\frac{1}{a^2}\Bigg[\sum_{k_1,k_2}\boldsymbol{c}^{+,+}_{j+1,k_1,k_2}\overline{\boldsymbol{P}^{-,+}_{an_1-k_1,k_2-an_2}}+\sum_{k_1,k_2}\boldsymbol{c}^{+,-}_{j+1,k_1,k_2}\overline{\boldsymbol{P}^{-,-}_{an_1-k_1,k_2-an_2}}$$
$$+\sum_{k_1,k_2}\boldsymbol{c}^{-,+}_{j+1,k_1,k_2}\overline{\boldsymbol{P}^{+,+}_{an_1-k_1,k_2-an_2}}+\sum_{k_1,k_2}\boldsymbol{c}^{-,-}_{j+1,k_1,k_2}\overline{\boldsymbol{P}^{+,-}_{an_1-k_1,k_2-an_2}}\Bigg];$$

$$\boldsymbol{c}^{-,-}_{j,n_1,n_2}=\frac{1}{a^2}\Bigg[\sum_{k_1,k_2}\boldsymbol{c}^{+,+}_{j+1,k_1,k_2}\overline{\boldsymbol{P}^{-,-}_{an_1-k_1,an_2-k_2}}+\sum_{k_1,k_2}\boldsymbol{c}^{+,-}_{j+1,k_1,k_2}\overline{\boldsymbol{P}^{-,+}_{an_1-k_1,an_2-k_2}}$$
$$+\sum_{k_1,k_2}\boldsymbol{c}^{-,+}_{j+1,k_1,k_2}\overline{\boldsymbol{P}^{-,+}_{an_1-k_1,an_2-k_2}}+\sum_{k_1,k_2}\boldsymbol{c}^{-,-}_{j+1,k_1,k_2}\overline{\boldsymbol{P}^{+,+}_{an_1-k_1,an_2-k_2}}\Bigg];$$

$$\boldsymbol{d}^{h,i,+,+}_{j,n_1,n_2}=\frac{1}{a^2}\Bigg[\sum_{k_1,k_2}\boldsymbol{c}^{+,+}_{j+1,k_1,k_2}\overline{\boldsymbol{Q}^{h,i,+,+}_{k_1-an_1,k_2-an_2}}+\sum_{k_1,k_2}\boldsymbol{c}^{+,-}_{j+1,k_1,k_2}\overline{\boldsymbol{Q}^{h,i,+,-}_{k_1-an_1,k_2-an_2}}$$
$$+\sum_{k_1,k_2}\boldsymbol{c}^{-,+}_{j+1,k_1,k_2}\overline{\boldsymbol{Q}^{h,i,-,+}_{k_1-an_1,k_2-an_2}}+\sum_{k_1,k_2}\boldsymbol{c}^{-,-}_{j+1,k_1,k_2}\overline{\boldsymbol{Q}^{h,i,-,-}_{k_1-an_1,k_2-an_2}}\Bigg];$$

$$\boldsymbol{d}^{h,i,+,-}_{j,n_1,n_2}=\frac{1}{a^2}\Bigg[\sum_{k_1,k_2}\boldsymbol{c}^{+,+}_{j+1,k_1,k_2}\overline{\boldsymbol{Q}^{h,i,+,-}_{k_1-an_1,an_2-k_2}}+\sum_{k_1,k_2}\boldsymbol{c}^{+,-}_{j+1,k_1,k_2}\overline{\boldsymbol{Q}^{h,i,+,+}_{k_1-an_1,an_2-k_2}}$$
$$+\sum_{k_1,k_2}\boldsymbol{c}^{-,+}_{j+1,k_1,k_2}\overline{\boldsymbol{Q}^{h,i,-,-}_{k_1-an_1,an_2-k_2}}+\sum_{k_1,k_2}\boldsymbol{c}^{-,-}_{j+1,k_1,k_2}\overline{\boldsymbol{Q}^{h,i,-,+}_{k_1-an_1,an_2-k_2}}\Bigg];$$

$$\boldsymbol{d}^{h,i,-,+}_{j,n_1,n_2}=\frac{1}{a^2}\Bigg[\sum_{k_1,k_2}\boldsymbol{c}^{+,+}_{j+1,k_1,k_2}\overline{\boldsymbol{Q}^{h,i,-,+}_{an_1-k_1,k_2-an_2}}+\sum_{k_1,k_2}\boldsymbol{c}^{+,-}_{j+1,k_1,k_2}\overline{\boldsymbol{Q}^{h,i,-,-}_{an_1-k_1,k_2-an_2}}$$
$$+\sum_{k_1,k_2}\boldsymbol{c}^{-,+}_{j+1,k_1,k_2}\overline{\boldsymbol{Q}^{h,i,+,+}_{an_1-k_1,k_2-an_2}}+\sum_{k_1,k_2}\boldsymbol{c}^{-,-}_{j+1,k_1,k_2}\overline{\boldsymbol{Q}^{h,i,+,-}_{an_1-k_1,k_2-an_2}}\Bigg];$$

$$\boldsymbol{d}^{h,i,-,-}_{j,n_1,n_2}=\frac{1}{a^2}\Bigg[\sum_{k_1,k_2}\boldsymbol{c}^{+,+}_{j+1,k_1,k_2}\overline{\boldsymbol{Q}^{h,i,-,-}_{an_1-k_1,an_2-k_2}}+\sum_{k_1,k_2}\boldsymbol{c}^{+,-}_{j+1,k_1,k_2}\overline{\boldsymbol{Q}^{h,i,-,+}_{an_1-k_1,an_2-k_2}}$$
$$+\sum_{k_1,k_2}\boldsymbol{c}^{-,+}_{j+1,k_1,k_2}\overline{\boldsymbol{Q}^{h,i,+,-}_{an_1-k_1,an_2-k_2}}+\sum_{k_1,k_2}\boldsymbol{c}^{-,-}_{j+1,k_1,k_2}\overline{\boldsymbol{Q}^{h,i,+,+}_{an_1-k_1,an_2-k_2}}\Bigg];$$

其中： $h=1,2,\cdots,a-1; i=1,2,3$ 。

证明：因为 $\{\Phi(x-k,y-l),\Phi(x-k,l-y),\Phi(k-x,y-l),\Phi(k-x,l-y):k,l\in \boldsymbol{Z}\}$ 是 V_0 的标准正交基，所以就有

$$\begin{aligned}\boldsymbol{c}_{j,n_1,n_2}^{+,+} &= \left\langle f_{j+1},\boldsymbol{\Phi}_{j,n_1,n_2}^{+,+}\right\rangle \\ &= \sum_{k_1,k_2}\boldsymbol{c}_{j+1,k_1,k_2}^{+,+}\left\langle \boldsymbol{\Phi}_{j+1,n_1,n_2}^{+,+},\boldsymbol{\Phi}_{j,n_1,n_2}^{+,+}\right\rangle \\ &\quad+\sum_{k_1,k_2}\boldsymbol{c}_{j+1,k_1,k_2}^{+,-}\left\langle \boldsymbol{\Phi}_{j+1,n_1,n_2}^{+,-},\boldsymbol{\Phi}_{j,n_1,n_2}^{+,+}\right\rangle \\ &\quad+\sum_{k_1,k_2}\boldsymbol{c}_{j+1,k_1,k_2}^{-,+}\left\langle \boldsymbol{\Phi}_{j+1,n_1,n_2}^{-,+},\boldsymbol{\Phi}_{j,n_1,n_2}^{+,+}\right\rangle \\ &\quad+\sum_{k_1,k_2}\boldsymbol{c}_{j+1,k_1,k_2}^{-,-}\left\langle \boldsymbol{\Phi}_{j+1,n_1,n_2}^{-,-},\boldsymbol{\Phi}_{j,n_1,n_2}^{+,+}\right\rangle,\end{aligned}$$

则根据(1)式和(20)式可以知道， $\left\langle \boldsymbol{\Phi}_{j+1,n_1,n_2}^{+,+},\boldsymbol{\Phi}_{j,n_1,n_2}^{+,+}\right\rangle=\frac{1}{a^2}\overline{\boldsymbol{P}_{k_1-an_1,k_2-an_2}^{+,+}}$ 。同理可以证得：

$$\left\langle \boldsymbol{\Phi}_{j+1,n_1,n_2}^{+,-},\boldsymbol{\Phi}_{j,n_1,n_2}^{+,+}\right\rangle=\frac{1}{a^2}\overline{\boldsymbol{P}_{k_1-an_1,k_2-an_2}^{+,-}};$$

$$\left\langle \boldsymbol{\Phi}_{j+1,n_1,n_2}^{-,+},\boldsymbol{\Phi}_{j,n_1,n_2}^{+,+}\right\rangle=\frac{1}{a^2}\overline{\boldsymbol{P}_{k_1-an_1,k_2-an_2}^{-,+}};$$

$$\left\langle \boldsymbol{\Phi}_{j+1,n_1,n_2}^{-,-},\boldsymbol{\Phi}_{j,n_1,n_2}^{+,+}\right\rangle=\frac{1}{a^2}\overline{\boldsymbol{P}_{k_1-an_1,k_2-an_2}^{-,-}},$$

故我们就可以得到

$$\begin{aligned}\boldsymbol{c}_{j,n_1,n_2}^{+,+}=\frac{1}{a^2}\Bigg[&\sum_{k_1,k_2}\boldsymbol{c}_{j+1,k_1,k_2}^{+,+}\overline{\boldsymbol{P}_{k_1-an_1,k_2-an_2}^{+,+}}+\sum_{k_1,k_2}\boldsymbol{c}_{j+1,k_1,k_2}^{+,-}\overline{\boldsymbol{P}_{k_1-an_1,k_2-an_2}^{+,-}}\\ &+\sum_{k_1,k_2}\boldsymbol{c}_{j+1,k_1,k_2}^{-,+}\overline{\boldsymbol{P}_{k_1-an_1,k_2-an_2}^{-,+}}+\sum_{k_1,k_2}\boldsymbol{c}_{j+1,k_1,k_2}^{-,-}\overline{\boldsymbol{P}_{k_1-an_1,k_2-an_2}^{-,-}}\Bigg],\end{aligned}$$

其它等式同理可证。

6.2. 重构算法

通过 $f(x,y)$ 在分辨率 a_j 下的主要部分和细节部分，来重构 $f(x,y)$ 在分辨率 a_{j+1} 下的主要部分，也即重构算法是分解算法的逆过程，那么要实现的目标就是：已知 $\boldsymbol{c}_{j,k_1,k_2}^{+,+},\boldsymbol{c}_{j,k_1,k_2}^{+,-},\boldsymbol{c}_{j,k_1,k_2}^{-,+},\boldsymbol{c}_{j,k_1,k_2}^{-,-}$ 和 $\boldsymbol{d}_{j,k_1,k_2}^{+,+},\boldsymbol{d}_{j,k_1,k_2}^{+,-},\boldsymbol{d}_{j,k_1,k_2}^{-,+},\boldsymbol{d}_{j,k_1,k_2}^{-,-}$ 求 $\boldsymbol{c}_{j+1,k_1,k_2}^{+,+},\boldsymbol{c}_{j+1,k_1,k_2}^{+,-},\boldsymbol{c}_{j+1,k_1,k_2}^{-,+},\boldsymbol{c}_{j+1,k_1,k_2}^{-,-}$ 。

定理 12： 若 $f_{j+1}(x,y)=\sum_{k_1,k_2}\boldsymbol{c}_{j+1,k_1,k_2}^{+,+}\boldsymbol{\Phi}_{j+1,k_1,k_2}^{+,+}+\sum_{k_1,k_2}\boldsymbol{c}_{j+1,k_1,k_2}^{+,-}\boldsymbol{\Phi}_{j+1,k_1,k_2}^{+,-}+\sum_{k_1,k_2}\boldsymbol{c}_{j+1,k_1,k_2}^{-,+}\boldsymbol{\Phi}_{j+1,k_1,k_2}^{-,+}+\sum_{k_1,k_2}\boldsymbol{c}_{j+1,k_1,k_2}^{-,-}\boldsymbol{\Phi}_{j+1,k_1,k_2}^{-,-}$ 并且由(1)式和(17)式，我们可以得到下列重构公式

$$\begin{aligned}\boldsymbol{c}_{j+1,n_1,n_2}^{+,+}=\frac{1}{a^2}\Bigg[&\sum_{k_1,k_2}\boldsymbol{c}_{j,n_1,n_2}^{+,+}\boldsymbol{P}_{n_1-ak_1,n_2-ak_2}^{+,+}+\sum_{k_1,k_2}\boldsymbol{c}_{j,n_1,n_2}^{+,-}\boldsymbol{P}_{n_1-ak_1,n_2-ak_2}^{+,-}\\ &+\sum_{k_1,k_2}\boldsymbol{c}_{j,n_1,n_2}^{-,+}\boldsymbol{P}_{n_1-ak_1,n_2-ak_2}^{-,+}+\sum_{k_1,k_2}\boldsymbol{c}_{j,n_1,n_2}^{-,-}\boldsymbol{P}_{n_1-ak_1,n_2-ak_2}^{-,-}\\ &+\sum_{h=1}^{a-1}\sum_{r=1}^{3}\Bigg(\sum_{k_1,k_2}\boldsymbol{d}_{j,n_1,n_2}^{h,i,+,+}\boldsymbol{Q}_{n_1-ak_1,n_2-ak_2}^{h,i,+,+}+\sum_{k_1,k_2}\boldsymbol{d}_{j,n_1,n_2}^{h,i,+,-}\boldsymbol{Q}_{n_1-ak_1,n_2-ak_2}^{h,i,+,-}\\ &+\sum_{k_1,k_2}\boldsymbol{d}_{j,n_1,n_2}^{h,i,-,+}\boldsymbol{Q}_{n_1-ak_1,n_2-ak_2}^{h,i,+-,+}+\sum_{k_1,k_2}\boldsymbol{d}_{j,n_1,n_2}^{h,i,-,-}\boldsymbol{Q}_{n_1-ak_1,n_2-ak_2}^{h,i,-,-}\Bigg)\Bigg];\end{aligned}$$

$$
\begin{aligned}
\boldsymbol{c}_{j+1,n_1,n_2}^{+,-} = \frac{1}{a^2}\Bigg[& \sum_{k_1,k_2} \boldsymbol{c}_{j,n_1,n_2}^{+,+} \boldsymbol{P}_{n_1-ak_1,ak_2-n_2}^{+,-} + \sum_{k_1,k_2} \boldsymbol{c}_{j,n_1,n_2}^{+,-} \boldsymbol{P}_{n_1-ak_1,ak_2-n_2}^{+,+} \\
& + \sum_{k_1,k_2} \boldsymbol{c}_{j,n_1,n_2}^{-,+} \boldsymbol{P}_{n_1-ak_1,ak_2-n_2}^{-,-} + \sum_{k_1,k_2} \boldsymbol{c}_{j,n_1,n_2}^{-,-} \boldsymbol{P}_{n_1-ak_1,ak_2-n_2}^{-,+} \\
& + \sum_{h=1}^{a-1}\sum_{r=1}^{3}\Bigg(\sum_{k_1,k_2} \boldsymbol{d}_{j,n_1,n_2}^{h,i,+,+} \boldsymbol{Q}_{n_1-ak_1,ak_2-n_2}^{h,i,+,-} + \sum_{k_1,k_2} \boldsymbol{d}_{j,n_1,n_2}^{h,i,+,-} \boldsymbol{Q}_{n_1-ak_1,ak_2-n_2}^{h,i,+,+} \\
& + \sum_{k_1,k_2} \boldsymbol{d}_{j,n_1,n_2}^{h,i,-,+} \boldsymbol{Q}_{n_1-ak_1,ak_2-n_2}^{h,i,-,-} + \sum_{k_1,k_2} \boldsymbol{d}_{j,n_1,n_2}^{h,i,-,-} \boldsymbol{Q}_{n_1-ak_1,ak_2-n_2}^{h,i,-,+} \Bigg)\Bigg];
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{c}_{j+1,n_1,n_2}^{-,+} = \frac{1}{a^2}\Bigg[& \sum_{k_1,k_2} \boldsymbol{c}_{j,n_1,n_2}^{+,+} \boldsymbol{P}_{ak_1-n_1,n_2-ak_2}^{-,+} + \sum_{k_1,k_2} \boldsymbol{c}_{j,n_1,n_2}^{+,-} \boldsymbol{P}_{ak_1-n_1,n_2-ak_2}^{-,-} \\
& + \sum_{k_1,k_2} \boldsymbol{c}_{j,n_1,n_2}^{-,+} \boldsymbol{P}_{ak_1-n_1,n_2-ak_2}^{+,+} + \sum_{k_1,k_2} \boldsymbol{c}_{j,n_1,n_2}^{-,-} \boldsymbol{P}_{ak_1-n_1,n_2-ak_2}^{+,-} \\
& + \sum_{h=1}^{a-1}\sum_{r=1}^{3}\Bigg(\sum_{k_1,k_2} \boldsymbol{d}_{j,n_1,n_2}^{h,i,+,+} \boldsymbol{Q}_{ak_1-n_1,n_2-ak_2}^{h,i,-,+} + \sum_{k_1,k_2} \boldsymbol{d}_{j,n_1,n_2}^{h,i,+,-} \boldsymbol{Q}_{ak_1-n_1,n_2-ak_2}^{h,i,-,-} \\
& + \sum_{k_1,k_2} \boldsymbol{d}_{j,n_1,n_2}^{h,i,-,+} \boldsymbol{Q}_{ak_1-n_1,n_2-ak_2}^{h,i,+,+} + \sum_{k_1,k_2} \boldsymbol{d}_{j,n_1,n_2}^{h,i,-,-} \boldsymbol{Q}_{ak_1-n_1,n_2-ak_2}^{h,i,+,-} \Bigg)\Bigg];
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{c}_{j+1,n_1,n_2}^{-,-} = \frac{1}{a^2}\Bigg[& \sum_{k_1,k_2} \boldsymbol{c}_{j,n_1,n_2}^{+,+} \boldsymbol{P}_{ak_1-n_1,ak_2-n_2}^{-,-} + \sum_{k_1,k_2} \boldsymbol{c}_{j,n_1,n_2}^{+,-} \boldsymbol{P}_{ak_1-n_1,ak_2-n_2}^{-,+} \\
& + \sum_{k_1,k_2} \boldsymbol{c}_{j,n_1,n_2}^{-,+} \boldsymbol{P}_{ak_1-n_1,ak_2-n_2}^{+,-} + \sum_{k_1,k_2} \boldsymbol{c}_{j,n_1,n_2}^{-,-} \boldsymbol{P}_{ak_1-n_1,ak_2-n_2}^{+,+} \\
& + \sum_{h=1}^{a-1}\sum_{r=1}^{3}\Bigg(\sum_{k_1,k_2} \boldsymbol{d}_{j,n_1,n_2}^{h,i,+,+} \boldsymbol{Q}_{ak_1-n_1,ak_2-n_2}^{h,i,-,-} + \sum_{k_1,k_2} \boldsymbol{d}_{j,n_1,n_2}^{h,i,+,-} \boldsymbol{Q}_{ak_1-n_1,ak_2-n_2}^{h,i,-,+} \\
& + \sum_{k_1,k_2} \boldsymbol{d}_{j,n_1,n_2}^{h,i,-,+} \boldsymbol{Q}_{ak_1-n_1,ak_2-n_2}^{h,i,+,-} + \sum_{k_1,k_2} \boldsymbol{d}_{j,n_1,n_2}^{h,i,-,-} \boldsymbol{Q}_{ak_1-n_1,ak_2-n_2}^{h,i,+,+} \Bigg)\Bigg].
\end{aligned}
$$

证明：由于

$$
\begin{aligned}
\boldsymbol{c}_{j+1,n_1,n_2}^{+,+} &= \left\langle f_{j+1}, \boldsymbol{\Phi}_{j+1,n_1,n_2}^{+,+} \right\rangle \\
&= \sum_{k_1,k_2} \boldsymbol{c}_{j,k_1,k_2}^{+,+} \left\langle \boldsymbol{\Phi}_{j,n_1,n_2}^{+,+}, \boldsymbol{\Phi}_{j+1,n_1,n_2}^{+,+} \right\rangle + \sum_{k_1,k_2} \boldsymbol{c}_{j,k_1,k_2}^{+,-} \left\langle \boldsymbol{\Phi}_{j,n_1,n_2}^{+,-}, \boldsymbol{\Phi}_{j+1,n_1,n_2}^{+,+} \right\rangle \\
&\quad + \sum_{k_1,k_2} \boldsymbol{c}_{j,k_1,k_2}^{-,+} \left\langle \boldsymbol{\Phi}_{j,n_1,n_2}^{-,+}, \boldsymbol{\Phi}_{j+1,n_1,n_2}^{+,+} \right\rangle + \sum_{k_1,k_2} \boldsymbol{c}_{j+1,k_1,k_2}^{-,-} \left\langle \boldsymbol{\Phi}_{j,n_1,n_2}^{-,-}, \boldsymbol{\Phi}_{j+1,n_1,n_2}^{+,+} \right\rangle \\
&\quad + \sum_{h=1}^{a-1}\sum_{r=1}^{3}\Bigg[\sum_{k_1,k_2} \boldsymbol{d}_{j,n_1,n_2}^{h,i,+,+} \left\langle \boldsymbol{\Psi}_{j,k_1,k_2}^{h,i,+,+}, \boldsymbol{\Phi}_{j+1,n_1,n_2}^{+,+} \right\rangle + \sum_{k_1,k_2} \boldsymbol{d}_{j,n_1,n_2}^{h,i,+,-} \left\langle \boldsymbol{\Psi}_{j,k_1,k_2}^{h,i,+,-}, \boldsymbol{\Phi}_{j+1,n_1,n_2}^{+,+} \right\rangle \\
&\quad + \sum_{k_1,k_2} \boldsymbol{d}_{j,n_1,n_2}^{h,i,-,+} \left\langle \boldsymbol{\Psi}_{j,k_1,k_2}^{h,i,-,+}, \boldsymbol{\Phi}_{j+1,n_1,n_2}^{+,+} \right\rangle + \sum_{k_1,k_2} \boldsymbol{d}_{j,n_1,n_2}^{h,i,-,-} \left\langle \boldsymbol{\Psi}_{j,k_1,k_2}^{h,i,-,-}, \boldsymbol{\Phi}_{j+1,n_1,n_2}^{+,+} \right\rangle \Bigg],
\end{aligned}
$$

而且 $\left\langle \boldsymbol{\Phi}_{j,n_1,n_2}^{+,+}, \boldsymbol{\Phi}_{j+1,n_1,n_2}^{+,+} \right\rangle = \frac{1}{a^2} \boldsymbol{P}_{n_1-ak_1,n_2-ak_2}^{+,+}$ ，类似可得

$$\left\langle \boldsymbol{\Phi}_{j,n_1,n_2}^{+,-}, \boldsymbol{\Phi}_{j+1,n_1,n_2}^{+,+} \right\rangle = \frac{1}{a^2} \boldsymbol{P}_{n_1-ak_1,n_2-ak_2}^{+,-};$$

$$\left\langle \boldsymbol{\Phi}_{j,n_1,n_2}^{-,+}, \boldsymbol{\Phi}_{j+1,n_1,n_2}^{+,+} \right\rangle = \frac{1}{a^2} \boldsymbol{P}_{n_1-ak_1,n_2-ak_2}^{-,+};$$

$$\left\langle \boldsymbol{\Phi}_{j,n_1,n_2}^{-,-}, \boldsymbol{\Phi}_{j+1,n_1,n_2}^{+,+} \right\rangle = \frac{1}{a^2} \boldsymbol{P}_{n_1-ak_1,n_2-ak_2}^{-,-}.$$

故我们可以证得

$$\begin{aligned}\boldsymbol{c}_{j+1,n_1,n_2}^{+,+}=\frac{1}{a^2}\Bigg[&\sum_{k_1,k_2}\boldsymbol{c}_{j,n_1,n_2}^{+,+}\boldsymbol{P}_{n_1-ak_1,n_2-ak_2}^{+,+}+\sum_{k_1,k_2}\boldsymbol{c}_{j,n_1,n_2}^{+,-}\boldsymbol{P}_{n_1-ak_1,n_2-ak_2}^{+,-}\\&+\sum_{k_1,k_2}\boldsymbol{c}_{j,n_1,n_2}^{-,+}\boldsymbol{P}_{n_1-ak_1,n_2-ak_2}^{-,+}+\sum_{k_1,k_2}\boldsymbol{c}_{j,n_1,n_2}^{-,-}\boldsymbol{P}_{n_1-ak_1,n_2-ak_2}^{-,-}\\&+\sum_{h=1}^{a-1}\sum_{r=1}^{3}\Bigg(\sum_{k_1,k_2}\boldsymbol{d}_{j,n_1,n_2}^{h,i,+,+}\boldsymbol{Q}_{n_1-ak_1,n_2-ak_2}^{h,i,+,+}+\sum_{k_1,k_2}\boldsymbol{d}_{j,n_1,n_2}^{h,i,+,-}\boldsymbol{Q}_{n_1-ak_1,n_2-ak_2}^{h,i,+,-}\\&+\sum_{k_1,k_2}\boldsymbol{d}_{j,n_1,n_2}^{h,i,-,+}\boldsymbol{Q}_{n_1-ak_1,n_2-ak_2}^{h,i,+-,+}+\sum_{k_1,k_2}\boldsymbol{d}_{j,n_1,n_2}^{h,i,-,-}\boldsymbol{Q}_{n_1-ak_1,n_2-ak_2}^{h,i,-,-}\Bigg)\Bigg];\end{aligned}$$

其它等式同理可证。

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