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Representation of BiHom-Jordan Algebra

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Abstract

This article mainly introduces the definition, representation and $\mathcal{O}$-operator of BiHom-Jordan algebra. Firstly, we give the definition of BiHom-Jordan algebra and its representation, the conditions for judging the representation of BiHom-Jordan algebra and an example of representation. At the same time, it is given that the dual mapping of BiHom-Jordan algebra representation is the condition that the representation satisfies. Secondly, we give the definition of BiHom-pre-Jordan algebra. The relation between BiHom-pre-Jordan algebra and BiHom-Jordan algebra is found. Finally, we give the definition of $\mathcal{O}$-operator and Rota-Baxter operator on BiHom-Jordan algebra, and the relationship between $\mathcal{O}$-operator on BiHom-Jordan algebra and BiHom-pre-Jordan algebra.

Keywords

BiHom-Jordan Algebra, BiHom-pre-Jordan Algebra, $\mathcal{O}$-Operator, Rota-Baxter Operator

BiHom-Jordan代数的表示

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摘　要

本文主要介绍了**BiHom-Jordan**代数的定义、表示和$\mathcal{O}$-算子。首先，给出了**BiHom-Jordan**代数及其表示的定义和例子、**BiHom-Jordan**代数表示的等价条件，同时给出**BiHom-Jordan**代数表示的对偶映射是表示所满足的条件。其次，给出了**BiHom-pre-Jordan**代数的定义，找到**BiHom-pre-Jordan**代数与**BiHom-Jordan**代数之间的关系。最后，给出了**BiHom-Jordan**代数上的$\mathcal{O}$-算子、**Rota-Baxter**算子的定

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义，以及BiHom-Jordan代数上的$\mathcal{O}$-算子与BiHom-pre-Jordan代数之间的关系。

关键词

BiHom-Jordan代数，BiHom-pre-Jordan代数，$\mathcal{O}$-算子，Rota-Baxter算子

1. 引言

Jordan 代数是由物理学家 P. Jordan 在研究量子力学时所提出来的[1]，后来逐渐成为了一个独立的代数体系[2] [3]。Jordan 代数与李代数有密切的关系。在李代数中一个比较基本的问题是李代数上的经典 Yang-Baxter 方程，它对于构造李双代数有非常重要的作用。对于 Jordan 代数及 pre-Jordan 代数，许多学者也研究了它们的双代数结构以及与经典 Yang-Baxter 方程类似的方程[4] [5]。作为 Jordan 代数的推广，BiHom-Jordan 代数也是现在研究的热点，许多学者对它的结构作了研究[6] [7]，本文将进一步研究 BiHom-Jordan 代数的表示。

2. BiHom-Jordan 代数的表示

本文所说的线性空间都指域 F 上的线性空间。

定义 1.1 [6] 设 J 是线性空间，$\alpha,\beta\in End(J)$，J 中定义双线性代数运算 $(x,y)\to x*y$，若满足下面的条件

$$\alpha\beta=\beta\alpha, \tag{1.1}$$

$$\beta(x)*\alpha(y)=\beta(y)*\alpha(x), \tag{1.2}$$

$$\begin{aligned}&\left\{\left(\beta^2(x)*\alpha\beta(x)\right)*\alpha^2\beta(y)\right\}*\beta\alpha^3(x)\\&=\alpha\left(\beta^2(x)*\alpha\beta(x)\right)*\left(\alpha^2\beta(y)*\alpha^3(x)\right),\end{aligned} \tag{1.3}$$

其中 $\forall x,y\in J$，则称 $(J,*,\alpha,\beta)$ 是 BiHom-Jordan 代数。

注记 1.1 等式(1.3)有以下等价形式

$$\begin{aligned}&\left\{\left(\beta^2(x)*\alpha\beta(y)\right)*\alpha^2\beta(u)\right\}*\beta\alpha^3(z)+\left\{\left(\beta^2(x)*\alpha\beta(z)\right)*\alpha^2\beta(u)\right\}*\beta\alpha^3(y)\\&+\left\{\left(\beta^2(y)*\alpha\beta(z)\right)*\alpha^2\beta(u)\right\}*\beta\alpha^3(x)\\&=\alpha\left(\beta^2(x)*\alpha\beta(y)\right)*\left(\alpha^2\beta(u)*\alpha^3(z)\right)+\alpha\left(\beta^2(x)*\alpha\beta(z)\right)*\left(\alpha^2\beta(u)*\alpha^3(y)\right)\\&+\alpha\left(\beta^2(y)*\alpha\beta(z)\right)*\left(\alpha^2\beta(u)*\alpha^3(x)\right),\end{aligned} \tag{1.4}$$

其中 $\forall x,y,z,u\in J$。

证明：设 $(J,*,\alpha,\beta)$ 是 BiHom-Jordan 代数，$\forall x,y,z,u\in J$，由(1.3)得

$$\begin{aligned}&\left\{\left(\beta^2(x+y+z)*\alpha\beta(x+y+z)\right)*\alpha^2\beta(u)\right\}*\beta\alpha^3(x+y+z)\\&-\alpha\left(\beta^2(x+y+z)*\alpha\beta(x+y+z)\right)*\left(\alpha^2\beta(u)*\alpha^3(x+y+z)\right)\\&=\left\{\left(\beta^2(x)*\alpha\beta(y)\right)*\alpha^2\beta(u)\right\}*\beta\alpha^3(z)+\left\{\left(\beta^2(x)*\alpha\beta(z)\right)*\alpha^2\beta(u)\right\}*\beta\alpha^3(y)\\&\quad+\left\{\left(\beta^2(y)*\alpha\beta(z)\right)*\alpha^2\beta(u)\right\}*\beta\alpha^3(x)-\alpha\left(\beta^2(x)*\alpha\beta(y)\right)*\left(\alpha^2\beta(u)*\alpha^3(z)\right)\\&\quad-\alpha\left(\beta^2(x)*\alpha\beta(z)\right)*\left(\alpha^2\beta(u)*\alpha^3(y)\right)-\alpha\left(\beta^2(y)*\alpha\beta(z)\right)*\left(\alpha^2\beta(u)*\alpha^3(x)\right)\\&=0,\end{aligned}$$

故有(1.4)成立。反之，在(1.4)中，令 $x=y=z$ 即可得(1.3)。

定义 1.2 设 $(J,*,\alpha,\beta)$ 是 BiHom-Jordan 代数，V 是一个线性空间，$\phi,\varphi\in End(V)$ 且 ϕ,φ 均可逆，$\rho:J\to End(V)$ 为线性映射，若满足

$$\phi\varphi=\varphi\phi, \tag{1.5}$$

$$\begin{aligned}&\rho\left(\left(\beta^2(x)*\alpha\beta(y)\right)*\alpha^2\beta(z)\right)\varphi\phi^2+\rho\left(\alpha^2\beta^2(y)\right)\phi\varphi^{-1}\rho\left(\alpha\beta^2(z)\right)\\&\phi\varphi^{-1}\rho\left(\beta^2(x)\right)\varphi+\rho\left(\alpha^2\beta^2(x)\right)\phi\varphi^{-1}\rho\left(\alpha\beta^2(z)\right)\phi\varphi^{-1}\rho\left(\beta^2(y)\right)\varphi\\&-\rho\left(\alpha\left(\beta^2(x)*\alpha\beta(y)\right)\right)\rho\left(\alpha^2\beta(z)\right)\phi^2-\rho\left(\alpha^{-1}\beta\left(\alpha^2\beta(z)*\alpha^3(y)\right)\right)\\&\phi^2\varphi^{-1}\rho\left(\beta^2(x)\right)\varphi-\rho\left(\alpha^{-1}\beta\left(\alpha^2\beta(z)*\alpha^3(x)\right)\right)\phi^2\varphi^{-1}\rho\left(\beta^2(y)\right)\varphi=0,\end{aligned} \tag{1.6}$$

$$\begin{aligned}&\rho\left(\alpha^2\beta^2(z)\right)\phi\varphi^{-1}\rho\left(\beta^2(x)*\alpha\beta(y)\right)\varphi+\rho\left(\alpha^2\beta^2(y)\right)\phi\varphi^{-1}\rho\left(\beta^2(x)*\alpha\beta(z)\right)\varphi\\&+\rho\left(\alpha^2\beta^2(x)\right)\phi\varphi^{-1}\rho\left(\beta^2(y)*\alpha\beta(z)\right)\varphi-\rho\left(\alpha\left(\beta^2(x)*\alpha\beta(y)\right)\right)\rho\left(\alpha^2\beta(z)\right)\phi\\&-\rho\left(\alpha\left(\beta^2(x)*\alpha\beta(z)\right)\right)\rho\left(\alpha^2\beta(y)\right)\phi-\rho\left(\alpha\left(\beta^2(y)*\alpha\beta(z)\right)\right)\rho\left(\alpha^2\beta(x)\right)\phi=0,\end{aligned} \tag{1.7}$$

其中 $\forall x,y,z\in J$，则称 (V,ρ,ϕ,φ) 为 $(J,*,\alpha,\beta)$ 的表示。

例 1.3 设 $(J,*,\alpha,\beta)$ 是 BiHom-Jordan 代数且 α,β 均可逆，定义 $rg:J\to End(J)$，其中 $rg(x)y=x*y$，$\forall x,y\in J$，则 (J,rg,α,β) 为 $(J,*,\alpha,\beta)$ 的表示，称为伴随表示。

证明：由 $(J,*,\alpha,\beta)$ 是 BiHom-Jordan 代数可知 $\alpha\beta=\beta\alpha$。由(1.4)和 rg 的定义知

$$\begin{aligned}&rg\left(\left(\beta^2(x)*\alpha\beta(y)\right)*\alpha^2\beta(z)\right)\beta\alpha^2(u)+rg\left(\alpha^2\beta^2(y)\right)\alpha\beta^{-1}rg\left(\alpha\beta^2(z)\right)\\&\alpha\beta^{-1}rg\left(\beta^2(x)\right)\beta(u)+rg\left(\alpha^2\beta^2(x)\right)\alpha\beta^{-1}rg\left(\alpha\beta^2(z)\right)\alpha\beta^{-1}rg\left(\beta^2(y)\right)\beta(u)\\&-rg\left(\alpha\left(\beta^2(x)*\alpha\beta(y)\right)\right)rg\left(\alpha^2\beta(z)\right)\alpha^2(u)-rg\left(\alpha^{-1}\beta\left(\alpha^2\beta(z)*\alpha^3(y)\right)\right)\\&\alpha^2\beta^{-1}rg\left(\beta^2(x)\right)\beta(u)-rg\left(\alpha^{-1}\beta\left(\alpha^2\beta(z)*\alpha^3(x)\right)\right)\alpha^2\beta^{-1}rg\left(\beta^2(y)\right)\beta(u)\\&=\left\{\left(\beta^2(x)*\alpha\beta(y)\right)*\alpha^2\beta(z)\right\}*\beta\alpha^3(u)+\left\{\left(\beta^2(x)*\alpha\beta(u)\right)*\alpha^2\beta(z)\right\}*\beta\alpha^3(y)\\&\quad+\left\{\left(\beta^2(y)*\alpha\beta(u)\right)*\alpha^2\beta(z)\right\}*\beta\alpha^3(x)-\alpha\left(\beta^2(x)*\alpha\beta(y)\right)*\left(\alpha^2\beta(z)*\alpha^3(u)\right)\\&\quad-\alpha\left(\beta^2(x)*\alpha\beta(u)\right)*\left(\alpha^2\beta(z)*\alpha^3(y)\right)-\alpha\left(\beta^2(y)*\alpha\beta(u)\right)*\left(\alpha^2\beta(z)*\alpha^3(x)\right)\\&=0,\end{aligned}$$

同理可得，

$$rg\left(\alpha^2\beta^2(z)\right)\alpha\beta^{-1}rg\left(\beta^2(x)*\alpha\beta(y)\right)\beta(u)+rg\left(\alpha^2\beta^2(y)\right)\alpha\beta^{-1}$$
$$rg\left(\beta^2(x)*\alpha\beta(z)\right)\beta(u)+rg\left(\alpha^2\beta^2(x)\right)\alpha\beta^{-1}rg\left(\beta^2(y)*\alpha\beta(z)\right)\beta(u)$$
$$-rg\left(\alpha\left(\beta^2(x)*\alpha\beta(y)\right)\right)rg\left(\alpha^2\beta(z)\right)\alpha(u)-rg\left(\alpha\left(\beta^2(x)*\alpha\beta(z)\right)\right)$$
$$rg\left(\alpha^2\beta(y)\right)\alpha(u)-rg\left(\alpha\left(\beta^2(y)*\alpha\beta(z)\right)\right)rg\left(\alpha^2\beta(x)\right)\alpha(u)$$
$$=\left\{\left(\beta^2(x)*\alpha\beta(y)\right)*\beta(u)\right\}*\beta\alpha^3(z)+\left\{\left(\beta^2(x)*\alpha\beta(z)\right)*\beta(u)\right\}*\beta\alpha^3(y)$$
$$+\left\{\left(\beta^2(y)*\alpha\beta(z)\right)*\beta(u)\right\}*\beta\alpha^3(x)-\alpha\left(\beta^2(x)*\alpha\beta(y)\right)*\left(\beta(u)*\alpha^3(z)\right)$$
$$-\alpha\left(\beta^2(x)*\alpha\beta(z)\right)*\left(\beta(u)*\alpha^3(y)\right)-\alpha\left(\beta^2(y)*\alpha\beta(z)\right)*\left(\beta(u)*\alpha^3(x)\right)$$
$$=0,$$

因此，(J,rg,α,β) 为 $(J,*,\alpha,\beta)$ 的表示。

命题 1.4 设 $(J,*,\alpha,\beta)$ 是 BiHom-Jordan 代数，V 是一个线性空间，$\phi,\varphi\in End(V)$ 且 ϕ,φ 均可逆，$\rho:J\to End(V)$ 为线性映射，在 $J\oplus V$ 上定义

$$(x+a)*(y+b)=x*y+\rho(x)b+\rho\left(\alpha^{-1}\beta(y)\right)\phi\varphi^{-1}(a), \tag{1.8}$$

$$(\alpha+\phi)(x+a)=\alpha(x)+\phi(a),(\beta+\varphi)(y+b)=\beta(y)+\varphi(b), \tag{1.9}$$

其中 $\forall x,y\in J$，$a,b\in V$，则 $(J\oplus V,*,\alpha+\phi,\beta+\varphi)$ 为 BiHom-Jordan 代数的充分必要条件是 (V,ρ,ϕ,φ) 为 $(J,*,\alpha,\beta)$ 的表示。

证明：$\forall x+a,y+b,z+c,u+d\in J\oplus V$，则 $(J\oplus V,*,\alpha+\phi,\beta+\varphi)$ 为 BiHom-Jordan 代数当且仅当在 $(J\oplus V,*,\alpha+\phi,\beta+\varphi)$ 上(1.1)~(1.3)成立。

由于在 J 上(1.2)成立，所以有

$$\begin{aligned}&(\beta+\varphi)(x+a)*(\alpha+\phi)(y+b)-(\beta+\varphi)(y+b)*(\alpha+\phi)(x+a)\\&=\rho\left(\beta(x)\right)\phi(b)+\rho\left(\beta(y)\right)\phi(a)-\rho\left(\beta(y)\right)\phi(a)-\rho\left(\beta(x)\right)\phi(b),\end{aligned} \tag{1.10}$$

因此在 $J\oplus V$ 上(1.2)成立。

又由于

$$\begin{aligned}&(\alpha+\phi)(\beta+\varphi)(x+a)-(\beta+\varphi)(\alpha+\phi)(x+a)\\&=\alpha\beta(x)+\phi\varphi(a)-\beta\alpha(x)-\varphi\phi(a),\end{aligned} \tag{1.11}$$

$$\left\{\left((\beta+\varphi)^2(x+a)*(\alpha+\phi)(\beta+\varphi)(y+b)\right)*(\alpha+\phi)^2(\beta+\varphi)(u+d)\right\}$$
$$*(\beta+\varphi)(\alpha+\phi)^3(z+c)+\left\{\left((\beta+\varphi)^2(x+a)*(\alpha+\phi)(\beta+\varphi)(z+c)\right)\right.$$
$$\left.*(\alpha+\phi)^2(\beta+\varphi)(u+d)\right\}*(\beta+\varphi)(\alpha+\phi)^3(y+b)+\left\{\left((\beta+\varphi)^2(y+b)\right.\right.$$
$$\left.\left.*(\alpha+\phi)(\beta+\varphi)(z+c)\right)*(\alpha+\phi)^2(\beta+\varphi)(u+d)\right\}*(\beta+\varphi)(\alpha+\phi)^3$$
$$(x+a)-(\alpha+\phi)\left((\beta+\varphi)^2(x+a)*(\alpha+\phi)(\beta+\varphi)(y+b)\right)*\left((\alpha+\phi)^2\right.$$

$$
\begin{aligned}
&(\beta+\varphi)(u+d)*(\alpha+\phi)^3(z+c)\big)-(\alpha+\phi)\big((\beta+\varphi)^2(x+a)*(\alpha+\phi)\\
&(\beta+\varphi)(z+c)\big)*\big((\alpha+\phi)^2(\beta+\varphi)(u+d)*(\alpha+\phi)^3(y+b)\big)-(\alpha+\phi)\\
&\big((\beta+\varphi)^2(y+b)*(\alpha+\phi)(\beta+\varphi)(z+c)\big)*\big((\alpha+\phi)^2(\beta+\varphi)(u+d)*(\alpha+\phi)^3(x+a)\big)\\
&=\rho\big(\big(\beta^2(y)*\alpha\beta(z)\big)*\alpha^2\beta(u)\big)\varphi\phi^2(a)+\rho\big(\alpha^2\beta^2(y)\big)\phi\varphi^{-1}\rho\big(\alpha\beta^2(u)\big)\\
&\quad\phi\varphi^{-1}\rho\big(\beta^2(z)\big)\varphi(a)+\rho\big(\alpha^2\beta^2(z)\big)\phi\varphi^{-1}\rho\big(\alpha\beta^2(u)\big)\phi\varphi^{-1}\rho\big(\beta^2(y)\big)\varphi(a)\\
&\quad-\rho\big(\alpha\big(\beta^2(y)*\alpha\beta(z)\big)\big)\rho\big(\alpha^2\beta(u)\big)\phi^2(a)-\rho\big(\alpha^{-1}\beta\big(\alpha^2\beta(u)*\alpha^3(y)\big)\big)\\
&\quad\phi^2\varphi^{-1}\rho\big(\beta^2(z)\big)\varphi(a)-\rho\big(\alpha^{-1}\beta\big(\alpha^2\beta(u)*\alpha^3(z)\big)\big)\phi^2\varphi^{-1}\rho\big(\beta^2(y)\big)\varphi(a)\\
&\quad+\rho\big(\big(\beta^2(x)*\alpha\beta(z)\big)*\alpha^2\beta(u)\big)\varphi\phi^2(b)+\rho\big(\alpha^2\beta^2(x)\big)\phi\varphi^{-1}\rho\big(\alpha\beta^2(u)\big)\\
&\quad\phi\varphi^{-1}\rho\big(\beta^2(z)\big)\varphi(b)+\rho\big(\alpha^2\beta^2(z)\big)\phi\varphi^{-1}\rho\big(\alpha\beta^2(u)\big)\phi\varphi^{-1}\rho\big(\beta^2(x)\big)\varphi(b)\\
&\quad-\rho\big(\alpha\big(\beta^2(x)*\alpha\beta(z)\big)\big)\rho\big(\alpha^2\beta(u)\big)\phi^2(b)-\rho\big(\alpha^{-1}\beta\big(\alpha^2\beta(u)*\alpha^3(x)\big)\big)\\
&\quad\phi^2\varphi^{-1}\rho\big(\beta^2(z)\big)\varphi(b)-\rho\big(\alpha^{-1}\beta\big(\alpha^2\beta(u)*\alpha^3(z)\big)\big)\phi^2\varphi^{-1}\rho\big(\beta^2(x)\big)\varphi(b)\\
&\quad+\rho\big(\big(\beta^2(x)*\alpha\beta(y)\big)*\alpha^2\beta(u)\big)\varphi\phi^2(c)+\rho\big(\alpha^2\beta^2(y)\big)\phi\varphi^{-1}\rho\big(\alpha\beta^2(u)\big)\\
&\quad\phi\varphi^{-1}\rho\big(\beta^2(x)\big)\varphi(c)+\rho\big(\alpha^2\beta^2(x)\big)\phi\varphi^{-1}\rho\big(\alpha\beta^2(u)\big)\phi\varphi^{-1}\rho\big(\beta^2(y)\big)\varphi(c)\\
&\quad-\rho\big(\alpha\big(\beta^2(x)*\alpha\beta(y)\big)\big)\rho\big(\alpha^2\beta(u)\big)\phi^2(c)-\rho\big(\alpha^{-1}\beta\big(\alpha^2\beta(u)*\alpha^3(y)\big)\big)\\
&\quad\phi^2\varphi^{-1}\rho\big(\beta^2(x)\big)\varphi(c)-\rho\big(\alpha^{-1}\beta\big(\alpha^2\beta(u)*\alpha^3(x)\big)\big)\phi^2\varphi^{-1}\rho\big(\beta^2(y)\big)\varphi(c)\\
&\quad+\rho\big(\alpha^2\beta^2(z)\big)\phi\varphi^{-1}\rho\big(\beta^2(x)*\alpha\beta(y)\big)\varphi(d)+\rho\big(\alpha^2\beta^2(y)\big)\phi\varphi^{-1}\rho\big(\beta^2(x)\\
&\quad*\alpha\beta(z)\big)\varphi(d)+\rho\big(\alpha^2\beta^2(x)\big)\phi\varphi^{-1}\rho\big(\beta^2(y)*\alpha\beta(z)\big)\varphi(d)-\rho\big(\alpha\big(\beta^2(x)\\
&\quad*\alpha\beta(y)\big)\big)\rho\big(\alpha^2\beta(z)\big)\phi(d)-\rho\big(\alpha\big(\beta^2(x)*\alpha\beta(z)\big)\big)\rho\big(\alpha^2\beta(y)\big)\phi(d)\\
&\quad-\rho\big(\alpha\big(\beta^2(y)*\alpha\beta(z)\big)\big)\rho\big(\alpha^2\beta(x)\big)\phi(d),
\end{aligned}
\tag{1.12}
$$

因此，在 $J\oplus V$ 上(1.1)成立当且仅当(1.5)成立，(1.3)成立当且仅当(1.6)和(1.7)成立。因此，命题成立。

设 $\rho:J\to End(V)$，$\phi\in End(V)$，定义线性映射 $\rho^*:J\to End(V^*)$，$\phi^*\in End(V^*)$，其中

$$\langle\rho^*(x)f,v\rangle=\langle f,\rho(x)v\rangle\ \ (\forall x\in J,v\in V,f\in V^*),$$

$$\langle\phi^*(f),v\rangle=\langle f,\phi(v)\rangle\ \ (\forall v\in V,f\in V^*).$$

命题 1.5 设 $(J,*,\alpha,\beta)$ 是 BiHom-Jordan 代数，(V,ρ,ϕ,φ) 为 $(J,*,\alpha,\beta)$ 的表示，则 $(V^*,\rho^*,\phi^*,\varphi^*)$ 为 $(J,*,\alpha,\beta)$ 的表示当且仅当 (V,ρ,ϕ,φ) 满足下列条件

$$
\begin{aligned}
&\phi^2\varphi\rho\big(\big(\beta^2(x)*\alpha\beta(y)\big)*\alpha^2\beta(z)\big)+\varphi\rho\big(\beta^2(x)\big)\varphi^{-1}\phi\rho\big(\alpha\beta^2(z)\big)\varphi^{-1}\phi\rho\big(\alpha^2\beta^2(y)\big)\\
&+\varphi\rho\big(\beta^2(y)\big)\varphi^{-1}\phi\rho\big(\alpha\beta^2(z)\big)\varphi^{-1}\phi\rho\big(\alpha^2\beta^2(x)\big)-\phi^2\rho\big(\alpha^2\beta(z)\big)\rho\big(\alpha\big(\beta^2(x)*\alpha\beta(y)\big)\big)\\
&-\varphi\rho\big(\beta^2(x)\big)\varphi^{-1}\phi^2\rho\big(\alpha^{-1}\beta\big(\alpha^2\beta(z)*\alpha^3(y)\big)\big)-\varphi\rho\big(\beta^2(y)\big)\varphi^{-1}\phi^2\rho\big(\alpha^{-1}\beta\big(\alpha^2\beta(z)\\
&*\alpha^3(x)\big)\big)=0,
\end{aligned}
\tag{1.13}
$$

$$\begin{aligned}&\varphi\rho\left(\beta^2(x)*\alpha\beta(y)\right)\varphi^{-1}\phi\rho\left(\alpha^2\beta^2(z)\right)+\varphi\rho\left(\beta^2(x)*\alpha\beta(z)\right)\varphi^{-1}\phi\rho\left(\alpha^2\beta^2(y)\right)\\&+\varphi\rho\left(\beta^2(y)*\alpha\beta(z)\right)\varphi^{-1}\phi\rho\left(\alpha^2\beta^2(x)\right)-\phi\rho\left(\alpha^2\beta(z)\right)\rho\left(\alpha\left(\beta^2(x)*\alpha\beta(y)\right)\right)\\&-\phi\rho\left(\alpha^2\beta(y)\right)\rho\left(\alpha\left(\beta^2(x)*\alpha\beta(z)\right)\right)-\phi\rho\left(\alpha^2\beta(x)\right)\rho\left(\alpha\left(\beta^2(y)*\alpha\beta(z)\right)\right)=0,\end{aligned}\tag{1.14}$$

其中 $\forall x,y,z\in J$ 。

证明：$\left(V^*,\rho^*,\phi^*,\varphi^*\right)$ 为 $(J,*,\alpha,\beta)$ 的表示当且仅当对于 $\left(V^*,\rho^*,\phi^*,\varphi^*\right)$ 有(1.5)~(1.7)成立。$\forall x,y,z\in J$，$v\in V$， $f\in V^*$，因为

$$\left\langle\left(\phi^*\varphi^*-\varphi^*\phi^*\right)f,v\right\rangle=\left\langle f,(\varphi\phi-\phi\varphi)v\right\rangle,$$

$$\begin{aligned}&\Big\langle\Big(\rho^*\left(\left(\beta^2(x)*\alpha\beta(y)\right)*\alpha^2\beta(z)\right)\varphi^*\phi^{*2}+\rho^*\left(\alpha^2\beta^2(y)\right)\phi^*\left(\varphi^{-1}\right)^*\rho^*\left(\alpha\beta^2(z)\right)\phi^*\left(\varphi^{-1}\right)^*\\&\rho^*\left(\beta^2(x)\right)\varphi^*+\rho^*\left(\alpha^2\beta^2(x)\right)\phi^*\left(\varphi^{-1}\right)^*\rho^*\left(\alpha\beta^2(z)\right)\phi^*\left(\varphi^{-1}\right)^*\rho^*\left(\beta^2(y)\right)\varphi^*\\&-\rho^*\left(\alpha\left(\beta^2(x)*\alpha\beta(y)\right)\right)\rho^*\left(\alpha^2\beta(z)\right)\phi^{*2}-\rho^*\left(\alpha^{-1}\beta\left(\alpha^2\beta(z)*\alpha^3(y)\right)\right)\phi^{*2}\left(\varphi^{-1}\right)^*\rho^*\\&\left(\beta^2(x)\right)\varphi^*-\rho^*\left(\alpha^{-1}\beta\left(\alpha^2\beta(z)*\alpha^3(x)\right)\right)\phi^{*2}\left(\varphi^{-1}\right)^*\rho^*\left(\beta^2(y)\right)\varphi^*\Big)f,v\Big\rangle\\&=\Big\langle f,\Big(\phi^2\varphi\rho\left(\left(\beta^2(x)*\alpha\beta(y)\right)*\alpha^2\beta(z)\right)+\varphi\rho\left(\beta^2(x)\right)\left(\varphi^{-1}\right)\phi\rho\left(\alpha\beta^2(z)\right)\varphi^{-1}\phi\rho\big(\alpha^2\\&\beta^2(y)\big)+\varphi\rho\left(\beta^2(y)\right)\varphi^{-1}\phi\rho\left(\alpha\beta^2(z)\right)\varphi^{-1}\phi\rho\left(\alpha^2\beta^2(x)\right)-\phi^2\rho\left(\alpha^2\beta(z)\right)\rho\big(\alpha\big(\beta^2(x)\\&*\alpha\beta(y)\big)\big)-\varphi\rho\left(\beta^2(x)\right)\varphi^{-1}\phi^2\rho\left(\alpha^{-1}\beta\left(\alpha^2\beta(z)*\alpha^3(y)\right)\right)-\varphi\rho\left(\beta^2(y)\right)\varphi^{-1}\phi^2\\&\rho\left(\alpha^{-1}\beta\left(\alpha^2\beta(z)*\alpha^3(x)\right)\right)\Big)v\Big\rangle,\end{aligned}$$

$$\begin{aligned}&\Big\langle\Big(\rho^*\left(\alpha^2\beta^2(z)\right)\phi^*\left(\varphi^{-1}\right)^*\rho^*\left(\beta^2(x)*\alpha\beta(y)\right)\varphi^*+\rho^*\left(\alpha^2\beta^2(y)\right)\phi^*\left(\varphi^{-1}\right)^*\rho^*\big(\beta^2(x)\\&*\alpha\beta(z)\big)\varphi^*+\rho^*\left(\alpha^2\beta^2(x)\right)\phi^*\left(\varphi^{-1}\right)^*\rho^*\left(\beta^2(y)*\alpha\beta(z)\right)\varphi^*-\rho^*\big(\alpha\big(\beta^2(x)\\&*\alpha\beta(y)\big)\big)\rho^*\left(\alpha^2\beta(z)\right)\phi^*-\rho^*\left(\alpha\left(\beta^2(x)*\alpha\beta(z)\right)\right)\rho^*\left(\alpha^2\beta(y)\right)\phi^*-\rho^*\big(\alpha\big(\beta^2(y)\\&*\alpha\beta(z)\big)\big)\rho^*\left(\alpha^2\beta(x)\right)\phi^*\Big)f,v\Big\rangle\\&=\Big\langle f,\Big(\varphi\rho\left(\beta^2(x)*\alpha\beta(y)\right)\varphi^{-1}\phi\rho\left(\alpha^2\beta^2(z)\right)+\varphi\rho\left(\beta^2(x)*\alpha\beta(z)\right)\varphi^{-1}\phi\rho\left(\alpha^2\beta^2(y)\right)\\&+\varphi\rho\left(\beta^2(y)*\alpha\beta(z)\right)\varphi^{-1}\phi\rho\left(\alpha^2\beta^2(x)\right)-\phi\rho\left(\alpha^2\beta(z)\right)\rho\left(\alpha\left(\beta^2(x)*\alpha\beta(y)\right)\right)\\&-\phi\rho\left(\alpha^2\beta(y)\right)\rho\left(\alpha\left(\beta^2(x)*\alpha\beta(z)\right)\right)-\phi\rho\left(\alpha^2\beta(x)\right)\rho\left(\alpha\left(\beta^2(y)*\alpha\beta(z)\right)\right)\Big)v\Big\rangle,\end{aligned}$$

由此可知(1.6)成立当且仅当(1.13)成立，(1.7)成立当且仅当(1.14)成立。因此，命题成立。

推论 1.6 设 $(J,*,\alpha,\beta)$ 是 BiHom-Jordan 代数且 α,β 均可逆，则 $\left(rg^*,J^*,\alpha^*,\beta^*\right)$ 是 $(J,*,\alpha,\beta)$ 的表示当且仅当下面的等式成立

$$\begin{aligned}&\alpha^2\beta\left(\left(\left(\beta^2(x)*\alpha\beta(y)\right)*\alpha^2\beta(z)\right)*u\right)+\beta\left(\beta^2(x)*\beta^{-1}\alpha\left(\alpha\beta^2(z)*\beta^{-1}\alpha\left(\alpha^2\beta^2(y)*u\right)\right)\right)\\&+\beta\left(\beta^2(y)*\beta^{-1}\alpha\left(\alpha\beta^2(z)*\beta^{-1}\alpha\left(\alpha^2\beta^2(x)*u\right)\right)\right)-\alpha^2\left(\alpha^2\beta(z)*\left(\alpha\left(\beta^2(x)*\alpha\beta(y)\right)*u\right)\right)\\&-\beta\left(\beta^2(x)*\beta^{-1}\alpha^2\left(\alpha^{-1}\beta\left(\alpha^2\beta(z)*\alpha^3(y)\right)*u\right)\right)-\beta\big(\beta^2(y)*\beta^{-1}\alpha^2\big(\alpha^{-1}\beta\big(\alpha^2\beta(z)\\&*\alpha^3(x)\big)*u\big)\big)=0,\end{aligned}\tag{1.15}$$

$$\beta\left(\left(\beta^2(x)*\alpha\beta(y)\right)*\beta^{-1}\alpha\left(\alpha^2\beta^2(z)*u\right)\right)+\beta\left(\left(\beta^2(x)*\alpha\beta(z)\right)*\beta^{-1}\alpha\left(\alpha^2\beta^2(y)*u\right)\right)$$
$$+\beta\left(\left(\beta^2(y)*\alpha\beta(z)\right)*\beta^{-1}\alpha\left(\alpha^2\beta^2(x)*u\right)\right)-\alpha\left(\alpha^2\beta(z)*\left(\alpha\left(\beta^2(x)*\alpha\beta(y)\right)*u\right)\right) \tag{1.16}$$
$$-\alpha\left(\alpha^2\beta(y)*\left(\alpha\left(\beta^2(x)*\alpha\beta(z)\right)*u\right)\right)-\alpha\left(\alpha^2\beta(x)*\left(\alpha\left(\beta^2(y)*\alpha\beta(z)\right)*u\right)\right)=0,$$

其中 $\forall x,y,z,u\in J$ 。

证明：由命题 1.5，取 $\rho=rg$ ，利用 rg 的定义可直接通过计算得出。

3. BiHom-pre-Jordan 代数

定义 2.1 设 J 是线性空间，J 上有双线性的代数运算 $(x,y)\to x\cdot y$， $\alpha,\beta\in End(J)$ 且 α,β 均可逆，若满足下面的条件

$$\alpha\beta=\beta\alpha, \tag{2.1}$$

$$\left\{\left(\beta^2(x)\cdot\alpha\beta(y)+\beta^2(y)\cdot\alpha\beta(x)\right)\cdot\alpha^2\beta(u)\right\}\cdot\beta\alpha^3(z)+\left\{\alpha\beta^2(u)\cdot\alpha\beta^{-1}\left(\beta^2(x)\cdot\alpha\beta(y)\right.\right.$$
$$\left.\left.+\beta^2(y)\cdot\alpha\beta(x)\right)\right\}\cdot\beta\alpha^3(z)+\alpha^2\beta^2(y)\cdot\alpha\beta^{-1}\left\{\alpha\beta^2(u)\cdot\alpha\beta^{-1}\left(\beta^2(x)\cdot\alpha\beta(z)\right)\right\}$$
$$+\alpha^2\beta^2(x)\cdot\alpha\beta^{-1}\left\{\alpha\beta^2(u)\cdot\alpha\beta^{-1}\left(\beta^2(y)\cdot\alpha\beta(z)\right)\right\}-\alpha\left(\beta^2(x)\cdot\alpha\beta(y)+\beta^2(y)\cdot\alpha\beta(x)\right) \tag{2.2}$$
$$\cdot\left(\alpha^2\beta(u)\cdot\alpha^3(z)\right)-\alpha^{-1}\beta\left(\alpha^2\beta(u)\cdot\alpha^3(y)+\alpha^2\beta(y)\cdot\alpha^3(u)\right)\cdot\alpha^2\beta^{-1}\left(\beta^2(x)\cdot\alpha\beta(z)\right)$$
$$-\alpha^{-1}\beta\left(\alpha^2\beta(u)\cdot\alpha^3(x)+\alpha^2\beta(x)\cdot\alpha^3(u)\right)\cdot\alpha^2\beta^{-1}\left(\beta^2(y)\cdot\alpha\beta(z)\right)=0,$$

$$\alpha^2\beta^2(z)\cdot\alpha\beta^{-1}\left\{\left[\beta^2(x)\cdot\alpha\beta(y)+\beta^2(y)\cdot\alpha\beta(x)\right]\cdot\alpha^2\beta(u)\right\}+\alpha^2\beta^2(y)\cdot\alpha\beta^{-1}\left\{\left[\beta^2(x)\right.\right.$$
$$\left.\left.\cdot\alpha\beta(z)+\beta^2(z)\cdot\alpha\beta(x)\right]\cdot\alpha^2\beta(u)\right\}+\alpha^2\beta^2(x)\cdot\alpha\beta^{-1}\left\{\left[\beta^2(y)\cdot\alpha\beta(z)+\beta^2(z)\right.\right.$$
$$\left.\left.\cdot\alpha\beta(y)\right]\cdot\alpha^2\beta(u)\right\}-\alpha\left(\beta^2(x)\cdot\alpha\beta(y)+\beta^2(y)\cdot\alpha\beta(x)\right)\cdot\left(\alpha^2\beta(z)\cdot\alpha^3(u)\right)-\alpha\left(\beta^2(x)\right. \tag{2.3}$$
$$\left.\cdot\alpha\beta(z)+\beta^2(z)\cdot\alpha\beta(x)\right)\cdot\left(\alpha^2\beta(y)\cdot\alpha^3(u)\right)-\alpha\left(\beta^2(y)\cdot\alpha\beta(z)+\beta^2(z)\cdot\alpha\beta(y)\right)$$
$$\cdot\left(\alpha^2\beta(x)\cdot\alpha^3(u)\right)=0,$$

其中 $\forall x,y,z,u\in J$ ，则称 $(J,\cdot,\alpha,\beta)$ 是 BiHom-pre-Jordan 代数。

定理 2.2 设 $(J,\cdot,\alpha,\beta)$ 是 BiHom-pre-Jordan 代数，在 J 上定义

$$x*y=x\cdot y+\alpha^{-1}\beta(y)\cdot\alpha\beta^{-1}(x),\ \forall x,y\in J, \tag{2.4}$$

则 $(J,*,\alpha,\beta)$ 是 BiHom-Jordan 代数。

证明：由(2.4)可知

$$\begin{aligned}&\beta(x)*\alpha(y)-\beta(y)*\alpha(x)\\&=\beta(x)\cdot\alpha(y)+\beta(y)\cdot\alpha(x)-\beta(y)\cdot\alpha(x)-\beta(x)\cdot\alpha(y)\\&=0,\end{aligned}$$

经直接计算，可得

$$
\begin{aligned}
&\{(\beta^2(x)*\alpha\beta(y))*\alpha^2\beta(u)\}*\beta\alpha^3(z)+\{(\beta^2(x)*\alpha\beta(z))*\alpha^2\beta(u)\}*\beta\alpha^3(y)\\
&+\{(\beta^2(y)*\alpha\beta(z))*\alpha^2\beta(u)\}*\beta\alpha^3(x)-\alpha(\beta^2(x)*\alpha\beta(y))*(\alpha^2\beta(u)*\alpha^3(z))\\
&-\alpha(\beta^2(x)*\alpha\beta(z))*(\alpha^2\beta(u)*\alpha^3(y))-\alpha(\beta^2(y)*\alpha\beta(z))*(\alpha^2\beta(u)*\alpha^3(x))\\
&=\{(\beta^2(x)\cdot\alpha\beta(y)+\beta^2(y)\cdot\alpha\beta(x))\cdot\alpha^2\beta(u)\}\cdot\beta\alpha^3(z)+\{\alpha\beta^2(u)\cdot\alpha\beta^{-1}(\beta^2(x)\\
&\cdot\alpha\beta(y)+\beta^2(y)\cdot\alpha\beta(x))\}\cdot\beta\alpha^3(z)+\alpha^2\beta^2(y)\cdot\alpha\beta^{-1}\{\alpha\beta^2(u)\cdot\alpha\beta^{-1}(\beta^2(x)\cdot\alpha\beta(z))\}\\
&+\alpha^2\beta^2(x)\cdot\alpha\beta^{-1}\{\alpha\beta^2(u)\cdot\alpha\beta^{-1}(\beta^2(y)\cdot\alpha\beta(z))\}-\alpha(\beta^2(x)\cdot\alpha\beta(y)+\beta^2(y)\\
&\cdot\alpha\beta(x))\cdot(\alpha^2\beta(u)\cdot\alpha^3(z))-\alpha^{-1}\beta(\alpha^2\beta(u)\cdot\alpha^3(y)+\alpha^2\beta(y)\cdot\alpha^3(u))\cdot\alpha^2\beta^{-1}(\beta^2(x)\\
&\cdot\alpha\beta(z))-\alpha^{-1}\beta(\alpha^2\beta(u)\cdot\alpha^3(x)+\alpha^2\beta(x)\cdot\alpha^3(u))\cdot\alpha^2\beta^{-1}(\beta^2(y)\cdot\alpha\beta(z))+\alpha^2\beta^2(z)\\
&\cdot\alpha\beta^{-1}\{(\beta^2(x)\cdot\alpha\beta(y)+\beta^2(y)\cdot\alpha\beta(x))\cdot\alpha^2\beta(u)\}+\alpha^2\beta^2(y)\cdot\alpha\beta^{-1}\{(\beta^2(x)\\
&\cdot\alpha\beta(z)+\beta^2(z)\cdot\alpha\beta(x))\cdot\alpha^2\beta(u)\}+\alpha^2\beta^2(x)\cdot\alpha\beta^{-1}\{(\beta^2(y)\cdot\alpha\beta(z)+\beta^2(z)\cdot\alpha\beta(y))\\
&\cdot\alpha^2\beta(u)\}-\alpha(\beta^2(x)\cdot\alpha\beta(y)+\beta^2(y)\cdot\alpha\beta(x))\cdot(\alpha^2\beta(z)\cdot\alpha^3(u))-\alpha(\beta^2(x)\cdot\alpha\beta(z)\\
&+\beta^2(z)\cdot\alpha\beta(x))\cdot(\alpha^2\beta(y)\cdot\alpha^3(u))-\alpha(\beta^2(y)\cdot\alpha\beta(z)+\beta^2(z)\cdot\alpha\beta(y))\cdot(\alpha^2\beta(u)\\
&\cdot\alpha^3(x))-\alpha(\beta^2(y)\cdot\alpha\beta(z)+\beta^2(z)\cdot\alpha\beta(y))\cdot(\alpha^2\beta(x)\cdot\alpha^3(u)))+\{(\beta^2(x)\cdot\alpha\beta(z)\\
&+\beta^2(z)\cdot\alpha\beta(x))\cdot\alpha^2\beta(u)\}\cdot\beta\alpha^3(y)+\{\alpha\beta^2(u)\cdot\alpha\beta^{-1}(\beta^2(x)\cdot\alpha\beta(z)+\beta^2(z)\cdot\alpha\beta(x))\}\\
&\cdot\beta\alpha^3(y)+\alpha^2\beta^2(z)\cdot\alpha\beta^{-1}\{\alpha\beta^2(u)\cdot\alpha\beta^{-1}(\beta^2(x)\cdot\alpha\beta(y))\}+\alpha^2\beta^2(x)\cdot\alpha\beta^{-1}\{\alpha\beta^2(u)\\
&\cdot\alpha\beta^{-1}(\beta^2(z)\cdot\alpha\beta(y))\}-\alpha(\beta^2(x)\cdot\alpha\beta(z)+\beta^2(z)\cdot\alpha\beta(x))\cdot(\alpha^2\beta(u)\cdot\alpha^3(y))-\alpha^{-1}\beta\\
&(\alpha^2\beta(u)\cdot\alpha^3(z)+\alpha^2\beta(z)\cdot\alpha^3(u))\cdot\alpha^2\beta^{-1}(\beta^2(x)\cdot\alpha\beta(y))-\alpha^{-1}\beta(\alpha^2\beta(u)\cdot\alpha^3(x)\\
&+\alpha^2\beta(x)\cdot\alpha^3(u))\cdot\alpha^2\beta^{-1}(\beta^2(z)\cdot\alpha\beta(y)))+\{(\beta^2(y)\cdot\alpha\beta(z)+\beta^2(z)\cdot\alpha\beta(y))\cdot\alpha^2\beta(u)\}\\
&\cdot\beta\alpha^3(x)+\{\alpha\beta^2(u)\cdot\alpha\beta^{-1}(\beta^2(y)\cdot\alpha\beta(z)+\beta^2(z)\cdot\alpha\beta(y))\}\cdot\beta\alpha^3(x)+\alpha^2\beta^2(z)\cdot\alpha\beta^{-1}\\
&\{\alpha\beta^2(u)\cdot\alpha\beta^{-1}(\beta^2(y)\cdot\alpha\beta(x))\}+\alpha^2\beta^2(y)\cdot\alpha\beta^{-1}\{\alpha\beta^2(u)\cdot\alpha\beta^{-1}(\beta^2(z)\cdot\alpha\beta(x))\}\\
&-\alpha(\beta^2(y)\cdot\alpha\beta(z)+\beta^2(z)\cdot\alpha\beta(y))\cdot(\alpha^2\beta(u)\cdot\alpha^3(x))-\alpha^{-1}\beta(\alpha^2\beta(u)\cdot\alpha^3(z)\\
&+\alpha^2\beta(z)\cdot\alpha^3(u))\cdot\alpha^2\beta^{-1}(\beta^2(y)\cdot\alpha\beta(x))-\alpha^{-1}\beta(\alpha^2\beta(u)\cdot\alpha^3(y)+\alpha^2\beta(y)\cdot\alpha^3(u))\\
&\cdot\alpha^2\beta^{-1}(\beta^2(z)\cdot\alpha\beta(x))\\
&=0,
\end{aligned}
$$

所以，$(J,*,\alpha,\beta)$ 是 BiHom-Jordan 代数。

4. BiHom-Jordan 代数的$\mathcal{O}$-算子

定义 3.1 设 $(J,*,\alpha,\beta)$ 是 BiHom-Jordan 代数，(V,ρ,ϕ,φ) 为 $(J,*,\alpha,\beta)$ 的表示，如果线性映射 $T:V\to J$ 满足下面的条件

$$T(a)*T(b)=T\left(\rho(T(a))b+\rho\left(T\left(\phi^{-1}\varphi(b)\right)\right)\phi\varphi^{-1}(a)\right),\ \forall a,b\in V, \tag{3.1}$$

$$T\phi=\alpha T,\ T\varphi=\beta T, \tag{3.2}$$

则称 T 为 $(J,*,\alpha,\beta)$ 上与表示 (V,ρ,ϕ,φ) 相关的一个 $\mathcal{O}$-算子。

定义 3.2 设 $(J,*,\alpha,\beta)$ 是 BiHom-Jordan 代数且 α,β 均可逆，R 是 J 上的线性变换，若满足下面的条件

$$R(x)*R(y)=R\Big(R(x)*y+R\big(\alpha^{-1}\beta(y)\big)*\alpha\beta^{-1}(x)\Big),\ \forall x,y\in J, \tag{3.3}$$

$$R\alpha=\alpha R,\ R\beta=\beta R, \tag{3.4}$$

则称 R 为 $(J,*,\alpha,\beta)$ 上的 Rota-Baxter 算子。

定理 3.3 设 $(J,*,\alpha,\beta)$ 是 BiHom-Jordan 代数，(V,ρ,ϕ,φ) 是它的表示，如果 T 是 $(J,*,\alpha,\beta)$ 上的与表示 (V,ρ,ϕ,φ) 相关的 $\mathcal{O}$-算子，在 V 上定义

$$a\cdot b=\rho\big(T(a)\big)b,\ \forall a,b\in V, \tag{3.5}$$

则 $(V,\cdot,\phi,\varphi)$ 是 BiHom-pre-Jordan 代数。

证明：由 $(V,\cdot,\phi,\varphi)$ 为 $(J,*,\alpha,\beta)$ 的表示知 $\phi\varphi=\varphi\phi$ 并且有(1.6)、(1.7)成立。对任意的 $a,b,c,d\in V$，令 $x=T(a)$，$y=T(b)$，$z=T(c)$，$u=T(d)$，经直接计算，可得

$$\begin{aligned}
&\Big\{\big(\varphi^2(a)\cdot\phi\varphi(b)+\varphi^2(b)\cdot\phi\varphi(a)\big)\cdot\phi^2\varphi(d)\Big\}\cdot\varphi\phi^3(c)+\Big\{\phi\varphi^2(d)\cdot\phi\varphi^{-1}\big(\varphi^2(a)\cdot\phi\varphi(b)\\
&+\varphi^2(b)\cdot\phi\varphi(a)\big)\Big\}\cdot\varphi\phi^3(c)+\phi^2\varphi^2(b)\cdot\phi\varphi^{-1}\Big\{\phi\varphi^2(d)\cdot\phi\varphi^{-1}\big(\varphi^2(a)\cdot\phi\varphi(c)\big)\Big\}+\phi^2\varphi^2(a)\\
&\cdot\phi\varphi^{-1}\Big\{\phi\varphi^2(d)\cdot\phi\varphi^{-1}\big(\varphi^2(b)\cdot\phi\varphi(c)\big)\Big\}-\phi\big(\varphi^2(a)\cdot\phi\varphi(b)+\varphi^2(b)\cdot\phi\varphi(a)\big)\cdot\big(\phi^2\varphi(d)\\
&\cdot\phi^3(c)\big)-\phi^{-1}\varphi\big(\phi^2\varphi(d)\cdot\phi^3(b)+\phi^2\varphi(b)\cdot\phi^3(d)\big)\cdot\phi^2\varphi^{-1}\big(\varphi^2(a)\cdot\phi\varphi(c)\big)-\phi^{-1}\varphi\big(\phi^2\varphi(d)\\
&\cdot\phi^3(a)+\phi^2\varphi(a)\cdot\phi^3(d)\big)\cdot\phi^2\varphi^{-1}\big(\varphi^2(b)\cdot\phi\varphi(c)\big)\\
&=\rho\Big(\big(\beta^2(x)*\alpha\beta(y)\big)*\alpha^2\beta(u)\Big)\varphi\phi^3(c)+\rho\big(\alpha^2\beta^2(y)\big)\phi\varphi^{-1}\rho\big(\alpha\beta^2(u)\big)\phi\varphi^{-1}\rho\big(\beta^2(x)\big)\\
&\quad\phi\varphi(c)+\rho\big(\alpha^2\beta^2(x)\big)\phi\varphi^{-1}\rho\big(\alpha\beta^2(u)\big)\phi\varphi^{-1}\rho\big(\beta^2(y)\big)\phi\varphi(c)-\rho\Big(\alpha\big(\beta^2(x)*\alpha\beta(y)\big)\Big)\\
&\quad\rho\big(\alpha^2\beta(u)\big)\phi^3(c)-\rho\Big(\alpha^{-1}\beta\big(\alpha^2\beta(u)*\alpha^3(y)\big)\Big)\phi^2\varphi^{-1}\rho\big(\beta^2(x)\big)\phi\varphi(c)-\rho\Big(\alpha^{-1}\beta\\
&\quad\big(\alpha^2\beta(u)*\alpha^3(x)\big)\Big)\phi^2\varphi^{-1}\rho\big(\beta^2(y)\big)\phi\varphi(c)\\
&=0,
\end{aligned}$$

$$\begin{aligned}
&\phi^2\varphi^2(c)\cdot\phi\varphi^{-1}\Big\{\big(\varphi^2(a)\cdot\phi\varphi(b)+\varphi^2(b)\cdot\phi\varphi(a)\big)\cdot\phi^2\varphi(d)\Big\}+\phi^2\varphi^2(b)\cdot\phi\varphi^{-1}\Big\{\big(\varphi^2(a)\\
&\cdot\phi\varphi(c)+\varphi^2(c)\cdot\phi\varphi(a)\big)\cdot\phi^2\varphi(d)\Big\}+\phi^2\varphi^2(a)\cdot\phi\varphi^{-1}\Big\{\big(\varphi^2(b)\cdot\phi\varphi(c)+\varphi^2(c)\cdot\phi\varphi(b)\big)\\
&\cdot\phi^2\varphi(d)\Big\}-\phi\big(\varphi^2(a)\cdot\phi\varphi(b)+\varphi^2(b)\cdot\phi\varphi(a)\big)\cdot\big(\phi^2\varphi(c)\cdot\phi^3(d)\big)-\phi\big(\varphi^2(a)\cdot\phi\varphi(c)\\
&+\varphi^2(c)\cdot\phi\varphi(a)\big)\cdot\big(\phi^2\varphi(b)\cdot\phi^3(d)\big)-\phi\big(\varphi^2(b)\cdot\phi\varphi(c)+\varphi^2(c)\cdot\phi\varphi(b)\big)\cdot\big(\phi^2\varphi(a)\cdot\phi^3(d)\big)\\
&=\rho\big(\alpha^2\beta^2(z)\big)\phi\varphi^{-1}\rho\big(\beta^2(x)*\alpha\beta(y)\big)\phi^2\varphi(d)+\rho\big(\alpha^2\beta^2(y)\big)\phi\varphi^{-1}\rho\big(\beta^2(x)*\alpha\beta(z)\big)\\
&\quad\phi^2\varphi(d)+\rho\big(\alpha^2\beta^2(x)\big)\phi\varphi^{-1}\rho\big(\beta^2(y)*\alpha\beta(z)\big)\phi^2\varphi(d)-\rho\Big(\alpha\big(\beta^2(x)*\alpha\beta(y)\big)\Big)\\
&\quad\rho\big(\alpha^2\beta(z)\big)\phi^3(d)-\rho\Big(\alpha\big(\beta^2(x)*\alpha\beta(z)\big)\Big)\rho\big(\alpha^2\beta(y)\big)\phi^3(d)-\rho\Big(\alpha\big(\beta^2(y)\\
&\quad*\alpha\beta(z)\big)\Big)\rho\big(\alpha^2\beta(x)\big)\phi^3(d)\\
&=0,
\end{aligned}$$

所以，$(V,\cdot,\phi,\varphi)$ 是一个 BiHom-pre-Jordan 代数。

5. 结论

本文给出了 BiHom-Jordan 代数表示和$\mathcal{O}$-算子的定义，同时研究了 BiHom-Jordan 代数与 BiHom-pre-Jordan 代数之间的关系，但是没有给出 BiHom-pre-Jordan 代数的表示，未来还需要进一步研究。

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