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有关(α, m)-对数凸函数的Hermite-Hadamard型积分不等式

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摘 要

本文主要建立一些新的涉及(α, m)对数凸函数的**Hermite-Hadamard**型积分不等式。

关键词

Hermite-Hadamard型不等式，(α, m)-对数凸函数，不等式

Integral Inequalities of Hermite-Hadamard Style for (α, m)-Logarithmically Convex Functions

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Abstract

The aim of this paper is to establish several new Hermite-Hadamard style inequalities for (α, m)-logarithmically convex function.

Keywords

Hermite-Hadamard's Inequality, (α,m)-Logarithmically Convex Function, Inequality

1. 引言

本文中设$\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_{++} = (0, \infty)$.

首先, 我们回忆一些重要的概念:

定义 1.1 [1] [2] 若函数$f : I \subset \mathbb{R} \to \mathbb{R}$ 满足

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y), \qquad \forall x, y \in I, t \in [0, 1], \tag{1.1}$$

则称函数f是I上的凸函数. 此外, 如果$-f$ 是凸函数, 就称f 是凹函数.

定义 1.2 [3]若实函数$f : I \subset \mathbb{R}_+ \to \mathbb{R}_{++}$ 满足

$$f(tx + (1-t)y) \le [f(x)]^{t^m} [f(y)]^{m(1-t)^m}, \qquad \forall x, y \in I, m \in (0, 1], t \in [0, 1], \tag{1.2}$$

就称函数f是m-对数凸函数.

近期, 文献 [4]将上面的定义进一步推广到(α, m)-对数凸函数, 如下:

定义 1.3 [4] 若实函数$f : I \subset \mathbb{R}_+ \to \mathbb{R}_{++}$

$$f(tx + m(1-t)y) \le [f(x)]^{t^\alpha} [f(y)]^{m(1-t^\alpha)}, \tag{1.3}$$

holds for all $x, y \in I, (\alpha, m) \in (0, 1] \times (0, 1], t \in [0, 1]$. 就称$f$是$(\alpha, m)$-对数凸函数.

设实函数$f: I \subset \mathbb{R} \to \mathbb{R}$ 是凸函数, 则著名的Hermite-Hadamard积分不等式为

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}, \quad a, b \in I\,,\ a < b. \tag{1.4}$$

此后, 众多学者推广了凸函数的概念, 得到有关几何凸函数、调和凸函数、m-凸函数以及(α, m)-凸函数的Hermite-Hadamard型积分不等式, 详细可阅读相关文献 [5–10].

Bai-Qi-Xi 在文献 [4]中获得了(α, m)-对数凸函函数的Hermite-Hadamard 型不等式, 如下:

定理 A [4] 设$I \subset \mathbb{R}_+$是开区间, 且函数$f: I \to \mathbb{R}_+$是I 上的可微函数, 且$f' \in L([a, b])$ ($0 \le a < b < \mathbb{R}_+$). 设$q \in [1, +\infty)$, $(\alpha, m) \in (0, 1] \times (0, 1]$, 若$|f'(x)|^q$ 是$[0, \frac{b}{m}]$ 上的(α, m)-对数凸函数, 则有

$$\left|\frac{f(a)+f(b)}{2} - \int_a^b f(x)dx\right| \le \frac{a+b}{2}\left|f'\left(\frac{b}{m}\right)\right|\left(\frac{1}{2}\right)^{1-1/q}[E_1(\alpha, mq)]^{1/q},$$

$$E(\alpha, m, q) = \begin{cases} \frac{1}{2}, & \mu = 1, \\ F(\mu, \alpha q), & 0 < \mu < 1, \quad \mu = \dfrac{|f'(a)|}{|f'(\frac{b}{m})|^m}, \\ \mu^{(1-\alpha)q}F(\mu, \frac{q}{\alpha}), & \mu > 1, \end{cases}$$

$$F(u, v) = \frac{1}{v^2(\ln u)^2}\left[v(u^v - 1)\ln u - 2(u^{v/2} - 1)^2\right], \quad u, v > 0, u \ne 1)$$

本文主要建立一些新的涉及$(\alpha,\ m)$-对数凸函数的Hermite-Hadamard型积分不等式.

2. 主要结果

本文主要结果如下:

定理 2.1 设$f: I \subset \mathbb{R}_+ \to \mathbb{R}_{++}$ 是(α, m)-对数凸函数, 且

$$J_1 = \frac{af(a)}{(b-a)^2}\int_a^b (b-x)f(x)dx,$$

$$J_2 = \frac{bf(b)}{(b-a)^2}\int_a^b (x-a)f(x)dx.$$

若$(\alpha, m) \in (0, 1]$, 则有

$$J_1 + J_2 \le \begin{cases} [f(\frac{a}{m})]^m\left[af(a)h(\delta^\alpha) + bf(b)g(\delta^\alpha)\right], & 当\ \delta \le 1; \\ [f(\frac{a}{m})]^m\left[af(a)h\left(\delta^{\frac{1}{\alpha}}\right) + bf(b)g\left(\delta^{\frac{1}{\alpha}}\right)\right], & 当\ \delta \ge 1. \end{cases} \tag{2.1}$$

其中$a, b \in I$ 且$a < b$, $\delta = \frac{f(b)}{[f(\frac{a}{m})]^m}$,

$$h(\beta) = \int_0^1 (1-t)\beta^t dt = \begin{cases} \dfrac{1}{2}, & \text{当 } \beta = 1; \\ \dfrac{\beta - 1 - \ln\beta}{(\ln\beta)^2}, & \text{当 } \beta \neq 1. \end{cases} \tag{2.2}$$

$$g(\beta) = \int_0^1 t\beta^t dt = \begin{cases} \dfrac{1}{2}, & \text{当 } \beta = 1; \\ \dfrac{\beta\ln\beta - \ln\beta + 1}{(\ln\beta)^2}, & \text{当 } \beta \neq 1. \end{cases} \tag{2.3}$$

.

证明: 设$x = (1-t)a + tb = m(1-t)\frac{a}{m} + tb$ $(t \in [0,1], m \in (0,1])$, 不难得到

$$J_1 = \frac{af(a)}{(b-a)^2}\int_a^b (b-x)f(x)dx = af(a)\int_0^1 (1-t)f(m(1-t)\frac{a}{m} + tb)dt, \tag{2.4}$$

$$J_2 = \frac{bf(b)}{(b-a)^2}\int_a^b (x-a)f(x)dx = bf(b)\int_0^1 tf(m(1-t)\frac{a}{m} + tb)dt. \tag{2.5}$$

因为$f(x)$是(α, m)-对数凸函数, 由(2.2)可得

$$\begin{aligned} J_1 &= af(a)\int_0^1 (1-t)f(m(1-t)\frac{a}{m} + tb)dt \\ &\le af(a)\int_0^1 (1-t)[f(\frac{a}{m})]^{m(1-t^\alpha)}[f(b)]^{t^\alpha}dt \\ &= af(a)[f(\frac{a}{m})]^m\int_0^1 (1-t)\left(\frac{[f(b)]}{[f(\frac{a}{m})]^m}\right)^{t^\alpha}dt. \end{aligned} \tag{2.6}$$

以及

$$\begin{aligned} J_2 &= bf(b)\int_0^1 tf(m(1-t)\frac{a}{m} + tb)dt \\ &\le bf(b)\int_0^1 t[f(\frac{a}{m})]^{m(1-t^\alpha)}[f(b)]^{t^\alpha}dt \\ &= bf(b)[f(\frac{a}{m})]^m\int_0^1 t\left(\frac{[f(b)]}{[f(\frac{a}{m})]^m}\right)^{t^\alpha}dt. \end{aligned} \tag{2.7}$$

若$0 < u \le 1 \le v$, $0 < \alpha, s \le 1$, 则

$$u^{\alpha^s} \le u^{\alpha s}, \quad v^{\alpha^s} \le v^{\frac{\alpha}{s}}. \tag{2.8}$$

(1) 若$\frac{[f(b)]}{[f(\frac{a}{m})]^m} \leq 1$,由(2.6-2.8)可得

$$J_1 \leq af(a)[f(\frac{a}{m})]^m \int_0^1 (1-t)\left(\frac{[f(b)]}{[f(\frac{a}{m})]^m}\right)^{t\alpha} dt = af(a)[f(\frac{a}{m})]^m h\left(\left[\frac{f(b)}{[f(\frac{a}{m})]^m}\right]^{\alpha}\right) \tag{2.9}$$

$$J_2 \leq bf(b)[f(\frac{a}{m})]^m \int_0^1 t\left(\frac{[f(b)]}{[f(\frac{a}{m})]^m}\right)^{t\alpha} dt = bf(b)[f(\frac{a}{m})]^m g\left(\left[\frac{f(b)}{[f(\frac{a}{m})]^m}\right]^{\alpha}\right). \tag{2.10}$$

(2) 若$\frac{[f(b)]}{[f(a)]^m} \geq 1$, 由(2.6-2.8) 可得

$$J_1 \leq af(a)[f(\frac{a}{m})]^m \int_0^1 (1-t)\left(\frac{[f(b)]}{[f(\frac{a}{m})]^m}\right)^{\frac{t}{\alpha}} dt = af(a)[f(\frac{a}{m})]^m h\left(\left[\frac{f(b)}{[f(\frac{a}{m})]^m}\right]^{\frac{1}{\alpha}}\right) \tag{2.11}$$

$$J_2 \leq bf(b)[f(\frac{a}{m})]^m \int_0^1 t\left(\frac{[f(b)]}{[f(\frac{a}{m})]^m}\right)^{\frac{t}{\alpha}} dt = bf(b)[f(\frac{a}{m})]^m g\left(\left[\frac{f(b)}{[f(\frac{a}{m})]^m}\right]^{\frac{1}{\alpha}}\right). \tag{2.12}$$

由(2.9) 和(2.12) 可得(2.1).□

定理 2.2 设实函数$f: I \subset \mathbb{R}_+ \to \mathbb{R}_{++}$. 若$[f(x)]^q$ $(q \geq 1)$ 是I上的$(\alpha; m)$-对数凸函数, 则有

$$J_1 + J_2 \leq \begin{cases} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} [f(\frac{a}{m})]^m \left[af(a)\left[h\left(\delta^{q\alpha}\right)\right]^{\frac{1}{q}} + bf(b)\left[g\left(\delta^{q\alpha}\right)\right]^{\frac{1}{q}}\right], & \text{当 } \delta \leq 1; \\ \\ \left(\frac{1}{p+1}\right)^{\frac{1}{p}} [f(\frac{a}{m})]^m \left[af(a)\left[h\left(\delta^{\frac{q}{\alpha}}\right)\right]^{\frac{1}{q}} + bf(b)\left[g\left(\delta^{\frac{q}{\alpha}}\right)\right]^{\frac{1}{q}}\right], & \text{当 } \delta \geq 1. \end{cases} \tag{2.13}$$

其中$a, b \in I$ 且$a < b$, $(\alpha, m) \in (0, 1]$, $p \geq 1$ 且$\frac{1}{p} + \frac{1}{q} = 1$.

证明: 因为$[f(x)]^q$是$(\alpha; m)$对数凸函数, 并结合(2.2), (2.3), 以及Hölder 不等式，可得

$$\begin{aligned} J_1 &= af(a)\int_0^1 (1-t)f(m(1-t)\frac{a}{m} + tb)dt \\ &\leq af(a)\left(\int_0^1 (1-t)^p dt\right)^{\frac{1}{p}} \left(\int_0^1 [f(m(1-t)\frac{a}{m} + tb)]^q dt\right)^{\frac{1}{q}} \\ &\leq af(a)\left(\int_0^1 (1-t)^p dt\right)^{\frac{1}{p}} \left(\int_0^1 [f(\frac{a}{m})]^{qm(1-t^\alpha)}[f(b)]^{qt^\alpha} dt\right)^{\frac{1}{q}} \\ &= \left(\frac{1}{p+1}\right)^{\frac{1}{p}} af(a)[f(\frac{a}{m})]^m \left(\int_0^1 \left(\frac{[f(b)]}{[f(\frac{a}{m})]^m}\right)^{qt^\alpha} dt\right)^{\frac{1}{q}}. \end{aligned} \tag{2.14}$$

和

$$
\begin{aligned}
J_2 &= bf(b)\int_0^1 tf(m(1-t)\frac{a}{m}+tb)dt \\
&\le bf(b)\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 [f(m(1-t)\frac{a}{m}+tb)]^q dt\right)^{\frac{1}{q}} \\
&\le bf(b)\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 [f(\frac{a}{m})]^{qm(1-t^{\alpha})}[f(b)]^{qt^{\alpha}} dt\right)^{\frac{1}{q}} \\
&= \left(\frac{1}{p+1}\right)^{\frac{1}{p}} bf(b)[f(\frac{a}{m})]^m\left(\int_0^1 \left(\frac{[f(b)]}{[f(\frac{a}{m})]^m}\right)^{qt^{\alpha}} dt\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.15}
$$

(1) 若 $\frac{[f(b)]}{[f(\frac{a}{m})]^m}\le 1$, 利用(2.8), (2.14) 和(2.15) 可得

$$
\begin{aligned}
J_1 &\le \left(\frac{1}{p+1}\right)^{\frac{1}{p}} af(a)[f(\frac{a}{m})]^m\left(\int_0^1 \left(\frac{[f(b)]}{[f(\frac{a}{m})]^m}\right)^{qt\alpha} dt\right)^{\frac{1}{q}} \\
&= \left(\frac{1}{p+1}\right)^{\frac{1}{p}} af(a)[f(\frac{a}{m})]^m\left[h\left(\left[\frac{f(b)}{[f(a)]^m}\right]^{q\alpha}\right)\right]^{\frac{1}{q}}
\end{aligned} \tag{2.16}
$$

$$
\begin{aligned}
J_2 &\le \left(\frac{1}{p+1}\right)^{\frac{1}{p}} bf(b)[f(\frac{a}{m})]^m\left(\int_0^1 \left(\frac{[f(b)]}{[f(\frac{a}{m})]^m}\right)^{qt\alpha} dt\right)^{\frac{1}{q}} \\
&= \left(\frac{1}{p+1}\right)^{\frac{1}{p}} bf(b)[f(\frac{a}{m})]^m\left[g\left(\left[\frac{f(b)}{[f(\frac{a}{m})]^m}\right]^{q\alpha}\right)\right]^{\frac{1}{q}}.
\end{aligned} \tag{2.17}
$$

(2) 若 $\frac{[f(b)]}{[f(\frac{a}{m})]^m}\ge 1$, 再由(2.8), (2.14) 和(2.15) 可得

$$
\begin{aligned}
J_1 &\le \left(\frac{1}{p+1}\right)^{\frac{1}{p}} af(a)[f(\frac{a}{m})]^m\left(\int_0^1 \left(\frac{[f(b)]}{[f(\frac{a}{m})]^m}\right)^{\frac{qt}{\alpha}} dt\right) \\
&= \left(\frac{1}{p+1}\right)^{\frac{1}{p}} af(a)[f(\frac{a}{m})]^m\left[h\left(\left[\frac{f(b)}{[f(a)]^m}\right]^{\frac{q}{\alpha}}\right)\right]^{\frac{1}{q}}
\end{aligned} \tag{2.18}
$$

$$
\begin{aligned}
J_2 &\le \left(\frac{1}{p+1}\right)^{\frac{1}{p}} bf(b)[f(\frac{a}{m})]^m\left(\int_0^1 \left(\frac{[f(b)]}{[f(\frac{a}{m})]^m}\right)^{\frac{qt}{\alpha}} dt\right)^{\frac{1}{q}} \\
&= \left(\frac{1}{p+1}\right)^{\frac{1}{p}} bf(b)[f(\frac{a}{m})]^m\left[g\left(\left[\frac{f(b)}{[f(\frac{a}{m})]^m}\right]^{\frac{q}{\alpha}}\right)\right]^{\frac{1}{q}}.
\end{aligned} \tag{2.19}
$$

由(2.16) 和(2.19)可得(2.13). □

定理 2.3 设函数$f: I\subset \mathbb{R}_+ \to \mathbb{R}_{++}$ 是(α,m)对数凸函数. 若$a,b\in I$ 且$a<b$, 则有

$$\frac{1}{b-a}\int_a^b f(x)f(a+b-x)dx$$
$$\leq \begin{cases} \frac{1}{2}\left[[f(\frac{a}{m})]^{2m}p(u^{2\alpha})+[f(\frac{b}{m})]^{2m}p(v^{\frac{2}{\alpha}})\right], & 当\ u\leq 1,\ v\geq 1; \\ \frac{1}{2}\left[[f(\frac{a}{m})]^{2m}p(u^{2\alpha})+[f(\frac{b}{m})]^{2m}p(v^{2\alpha})\right], & 当\ u\leq 1,\ v\leq 1; \\ \frac{1}{2}\left[[f(\frac{a}{m})]^{2m}p(u^{\frac{2}{\alpha}})+[f(\frac{b}{m})]^{2m}p(v^{2\alpha})\right], & 当\ u\geq 1,\ v\leq 1; \\ \frac{1}{2}\left[[f(\frac{a}{m})]^{2m}p(u^{\frac{2}{\alpha}})+[f(\frac{b}{m})]^{2m}p(v^{\frac{2}{\alpha}})\right], & 当\ u\geq 1,\ v\geq 1. \end{cases}, \tag{2.20}$$

其中$u=\frac{f(b)}{f(\frac{a}{m})}$, $v=\frac{f(a)}{f(\frac{b}{m})}$,

$$p(\beta)=\int_0^1 \beta^t dt=\begin{cases} 1\ , & 当\ \beta=1; \\ \frac{\beta-1}{\ln\beta}, & 当\ \beta\neq 1. \end{cases}$$

证明: 因为f是(α,m)对数凸函数, 则对任意的$a,b\in I$, $t\in[0,1]$, $(\alpha,m)\in(0,1]\times(0,1]$, 有

$$f(m(1-t)a+tb)\leq [f(a)]^{m(1-t^\alpha)}[f(b)]^{t^\alpha}, \tag{2.21}$$

$$f((ta+m(1-t)b)\leq [f(a)]^{t^\alpha}[f(b)]^{m(1-t^\alpha)}, \tag{2.22}$$

设$x=(1-t)a+tb=m(1-t)\frac{a}{m}+tb$ $(t\in[0,1], m\in(0,1])$, 并应用基本不等式$\sqrt{uv}\leq\sqrt{\frac{u^2+v^2}{2}}$ $(u,v>0)$,可得

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(x)f(a+b-x)dx \\ &=\frac{1}{b-a}\int_0^1 f((1-t)a+tb)f((ta+(1-t)b)(b-a)dt \\ &\leq\frac{1}{2}\int_0^1\{[f((1-t)a+tb)]^2+[f((ta+(1-t)b)]^2\}dt. \end{aligned}\tag{2.23}$$

将(2.21) 和(2.22) 代入(2.23), 可得

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(x)f(a+b-x)dx \\ &\leq\frac{1}{2}\left[\int_0^1\{[f(\frac{a}{m})]^{m(1-t^\alpha)}[f(b)]^{t^\alpha}\}^2dt+\int_0^1\{[f(a)]^{t^\alpha}[f(\frac{b}{m})]^{m(1-t^\alpha)}\}^2dt\right] \\ &=\frac{1}{2}\left[[f(\frac{a}{m})]^{2m}\int_0^1\left[\frac{f(b)}{f(\frac{a}{m})}\right]^{2t^\alpha}dt+[f(\frac{b}{m})]^{2m}\int_0^1\left[\frac{f(a)}{f(\frac{b}{m})}\right]^{2t^\alpha}dt\right]. \end{aligned}\tag{2.24}$$

(1) 若 $\frac{f(b)}{f(\frac{a}{m})} \leq 1$, $\frac{f(a)}{f(\frac{b}{m})} \geq 1$, 由(2.8) 和(2.24) 可得

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)f(a+b-x)dx \\
&\quad \leq \frac{1}{2}\left[[f(\frac{a}{m})]^{2m}\int_0^1\left[\frac{f(b)}{f(\frac{a}{m})}\right]^{2t\alpha}dt+[f(\frac{b}{m})]^{2m}\int_0^1\left[\frac{f(a)}{f(\frac{b}{m})}\right]^{\frac{2t}{\alpha}}dt\right] \\
&\quad = \frac{1}{2}\left[[f(\frac{a}{m})]^{2m}p\left(\left[\frac{f(b)}{f(\frac{a}{m})}\right]^{2\alpha}\right)+[f(\frac{b}{m})]^{2m}p\left(\left[\frac{f(a)}{f(\frac{b}{m})}\right]^{\frac{2}{\alpha}}\right)\right].
\end{aligned} \tag{2.25}$$

(2) 若 $\frac{f(b)}{f(\frac{a}{m})} \leq 1$, $\frac{f(a)}{f(\frac{b}{m})} \leq 1$, 则

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)f(a+b-x)dx \\
&\quad \leq \frac{1}{2}\left[[f(\frac{a}{m})]^{2m}\int_0^1\left[\frac{f(b)}{f(\frac{a}{m})}\right]^{2t\alpha}dt+[f(\frac{b}{m})]^{2m}\int_0^1\left[\frac{f(a)}{f(\frac{b}{m})}\right]^{2t\alpha}dt\right] \\
&\quad = \frac{1}{2}\left[[f(\frac{a}{m})]^{2m}p\left(\left[\frac{f(b)}{f(\frac{a}{m})}\right]^{2\alpha}\right)+[f(\frac{b}{m})]^{2m}p\left(\left[\frac{f(a)}{f(\frac{b}{m})}\right]^{2\alpha}\right)\right].
\end{aligned} \tag{2.26}$$

(3) 若 $\frac{f(b)}{f(\frac{a}{m})} \geq 1$, $\frac{f(a)}{f(\frac{b}{m})} \leq 1$, 则

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)f(a+b-x)dx \\
&\quad \leq \frac{1}{2}\left[[f(\frac{a}{m})]^{2m}\int_0^1\left[\frac{f(b)}{f(\frac{a}{m})}\right]^{\frac{2t}{\alpha}}dt+[f(\frac{b}{m})]^{2m}\int_0^1\left[\frac{f(a)}{f(\frac{b}{m})}\right]^{2t\alpha}dt\right] \\
&\quad = \frac{1}{2}\left[[f(\frac{a}{m})]^{2m}p\left(\left[\frac{f(b)}{f(\frac{a}{m})}\right]^{\frac{2}{\alpha}}\right)+[f(\frac{b}{m})]^{2m}p\left(\left[\frac{f(a)}{f(\frac{b}{m})}\right]^{2\alpha}\right)\right].
\end{aligned} \tag{2.27}$$

(4) 若 $\frac{f(b)}{f(\frac{a}{m})} \geq 1$, $\frac{f(a)}{f(\frac{b}{m})} \geq 1$, 则

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)f(a+b-x)dx \\
&\quad \leq \frac{1}{2}\left[[f(\frac{a}{m})]^{2m}\int_0^1\left[\frac{f(b)}{f(\frac{a}{m})}\right]^{\frac{2t}{\alpha}}dt+[f(\frac{b}{m})]^{2m}\int_0^1\left[\frac{f(a)}{f(\frac{b}{m})}\right]^{\frac{2t}{\alpha}}dt\right] \\
&\quad = \frac{1}{2}\left[[f(\frac{a}{m})]^{2m}p\left(\left[\frac{f(b)}{f(\frac{a}{m})}\right]^{\frac{2}{\alpha}}\right)+[f(\frac{b}{m})]^{2m}p\left(\left[\frac{f(a)}{f(\frac{b}{m})}\right]^{\frac{2}{\alpha}}\right)\right].
\end{aligned} \tag{2.28}$$

由(2.25) 和(2.28)即可得(2.20)成立.□

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