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一类k-Hessian 方程正径向解的存在性

岳存燕

西北师范大学数学与统计学院，甘肃 兰州
Email: yuecunyan@163.com

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摘 要

基于单调迭代方法，通过构造一个单调迭代序列，本文主要获得了一类 k-**Hessian** 方程正径向解的存在性.

关键词

k-**Hessian**方程，正径向解，单调迭代方法

The Existence of Positive Radial Solutions for a Class of k-Hessian Equation

Cunyan Yue

College of Mathematics and Statistics, Northwest Normal University, Lanzhou Gansu
Email: yuecunyan@163.com

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Abstract

Based on the monotone iterative method, we obtain the existence of positive radial so-

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lutions for a class of k-Hessian equation by constructing a momtone iterative sequence.

Keywords

k-Hessian Equation, Positive Radial Solution, Monotone Iterative Method

1. 介绍

k-Hessian 问题源于几何学, 流体力学和其他应用学科 [1]. 一般地, 我们定义如下的 k-Hessian 算子:

$$S_k(\lambda(D^2u)) = \sum_{1\le j_1<\ldots<j_k\le N} \lambda_{j_1}\lambda_{j_2}\ldots\lambda_{j_k}, k = 1, 2, \cdots, N,$$

其中 $u \in C^2(\mathbb{R}^N)$, D^2u 是二阶连续微分函数 u 的 Hessian 矩阵, $\lambda_1, \lambda_2, \ldots, \lambda_N$ 是 D^2u 的特征根, $\lambda(D^2u) = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ 是 D^2u 的特征值向量, $S_k(\lambda(D^2u))$ 是第 k 阶初等对称多项式, 是 Hessian 矩阵 D^2u 的所有 $k \times k$ 阶主子式的和. 特别地, 当 $k = 1$ 时, k-Hessian 算子退化为 Laplace 算子 $S_1(\lambda(D^2u)) = \sum_{i=1}^{N} \lambda_i = \Delta u$, 详见 [2–4]; 当 $k = N$ 时, k-Hessian 算子退化为 Monge-Ampère 算子 $S_N(\lambda(D^2u)) = \prod_{i=1}^{N} = \det(\lambda(D^2u))$, 详见 [5–7].

近年来, k-Hessian 问题引起了许多学者的广泛关注, 并取得了丰富的的成果 [2–13]. Caffarelli [8] 等人最先研究了 k-Hessian 方程 $S_k(\lambda(D^2u)) = f$ 光滑解的存在性和先验估计; 此后, Wei [9] 运用单调分离方法获得了 k-Hessian 方程 $S_k(\lambda(D^2u)) = f(-u)$ 径向解的唯一性结果; 2015年, Zhang 和 Zhou [10] 运用单调迭代方法和 Arzelà-Ascoli 定理获得了 k-Hessian 方程 $\sigma_k(\lambda(D^2u)) = p(|x|)f(u)$ 正径向解的存在性结果, 并建立了正径向解唯一存在的条件; 2020 年, Zhang [11] 进一步运用不动点定理获得了奇异超线性 k-Hessian 方程 $S_k(\lambda(D^2u)) = \lambda H(x)f(-u)$ 非平凡径向解的存在性, 这里考虑的是权函数 $H(x)$ 在边界 $\partial\Omega$ 附近奇异的情况, f 可能在 0 点处奇异或者在 ∞ 处 k-超线性增长. 值得注意的是, 上述文献所讨论的算子中均不含梯度项 $\eta|\nabla u|I$, 那么当 k-Hessian 算子中包含梯度项时是否依然可以得到解的存在性, 这是一个值得考虑的问题.

受以上文献的启发, 本文将尝试运用单调迭代方法考虑 k-Hessian 方程

$$S_k(\lambda(D^2u + \eta|\nabla u|I)) = h(|x|)f(u), \; x \in \mathbb{R}^N \tag{1.1}$$

正径向解的存在性, 其中 $\eta \in [0,+\infty)$, $h \in C([0,+\infty),[0,+\infty))$ 单调递增. 相较文献 [10] 和 [11] 而言, 由于算子中 $\eta|\nabla u|I$ 的出现使问题更加复杂.

2. 预备知识

为了获得方程 (1.1) 径向解的存在性, 本节给出一些必要的引理. 令 $r = |x| = \sqrt{\sum_{i=1}^{N} x_i^2}$, $B_R := \{x \in \mathbb{R}^N : |x| < R\}$, 则有如下性质成立.

引理 2.1 [13] 设 $v(r) \in C^2[0,R)$ 是一个径向对称函数且 $v'(0) = 0$, 则函数 $u(|x|) = v(r)$ 是 $C^2(B_R)$ 的, 且

$$\lambda(D^2u+\eta|\nabla u|I) = \begin{cases} (v''(r)+\eta v'(r), (\frac{1}{r}+\eta)v'(r), \ldots, (\frac{1}{r}+\eta)v'(r)), & r \in (0,R), \\ (v''(0), v''(0), \ \ldots, v''(0)), & r = 0; \end{cases}$$

$$S_k(\lambda(D^2u+\eta|\nabla u|I)) = \begin{cases} C_{N-1}^{k-1}(v''(r)+\eta v'(r))((\frac{1}{r}+\eta)v'(r))^{k-1} + C_{N-1}^{k}((\frac{1}{r}+\eta)v'(r))^k, & r \in (0,R), \\ C_N^k (v''(0))^k, & r = 0, \end{cases}$$

其中 $C_N^k = \frac{N!}{k!(N-k)!}$.

引理 2.2 若 $v(r) \in C[0,R] \cap C^1(0,R)$ 是 Cauchy 问题

$$\begin{cases} v'(r) = \left(\frac{k}{C_{N-1}^{k-1}} e^{-\psi_k(r)} \int_0^r e^{\psi_k(s)} \left(\frac{s}{1+\eta s}\right)^{k-1} h(s) f(v(s)) ds\right)^{\frac{1}{k}}, \ r \in (0,R), \\ v(0) = \gamma, \quad \gamma > 0 \end{cases} \tag{2.1}$$

的解, 则 $v(r) \in C^2[0,R)$ 是常微分方程

$$C_{N-1}^{k-1}(v''(r))(v'(r))^{k-1} r + r(C_{N-1}^{k-1}\eta + C_{N-1}^{k}(\frac{1}{r}+\eta))(v'(r))^k = \frac{r^k}{(1+\eta r)^{k-1}} h(r) f(v(r)), \ r > 0 \tag{2.2}$$

的解且 $v'(0) = 0$.

证明: 显然, $v(r) \in C^2[0,R)$, 由 (2.1) 可知

$$(v'(r))^k = \frac{k}{C_{N-1}^{k-1}} e^{-\psi_k(r)} \int_0^r e^{\psi_k(s)} \left(\frac{s}{1+\eta s}\right)^{k-1} h(s) f(v(s)) ds, \ r \geq 0. \tag{2.3}$$

则

$$\left[\frac{C_{N-1}^{k-1}}{k} e^{\psi_k(r)} (v'(r))^k\right]' = e^{\psi_k(r)} \left(\frac{r}{1+\eta r}\right)^{k-1} h(r) f(v(r)), \quad r \geq 0. \tag{2.4}$$

对 (2.4) 关于 r 求导, 得

$$C_{N-1}^{k-1}(v^{''}(r))(v^{'}(r))^{k-1}r+r(C_{N-1}^{k-1}\eta+C_{N-1}^{k}(\frac{1}{r}+\eta))(v^{'}(r))^{k}=\frac{r^k}{(1+\eta r)^{k-1}}h(r)f(v(r)),\quad r>0$$

且 $v^{'}(0)=0$.

令

$$T(r)=\int_0^r\left(\frac{k}{C_{N-1}^{k-1}}e^{-\psi_k(t)}\int_0^t e^{\psi_k(s)}\left(\frac{s}{1+\eta s}\right)^{k-1}h(s)ds\right)^{\frac{1}{k}}dt,\quad r\geq 0,\quad T(\infty):=\lim_{r\to\infty}T(r).$$

$$F(r)=\int_\gamma^r\frac{1}{f(t)}dt,\quad r\geq\gamma>0,\quad F(\infty):=\lim_{r\to\infty}F(r),$$

其中

$$\psi_k(r)=\frac{k}{C_{N-1}^{k-1}}\left(C_{N-1}^{k-1}\eta r+C_{N-1}^{k}\ln r+C_{N-1}^{k}\eta r\right).$$

在本文中, 我们总假设以下条件成立:

(H1) $F(\infty)=\infty$, $T(\infty)<\infty$;

(H2) $f\in C([0,+\infty),[0,+\infty))$ 是单调递增的且当 $t>0$ 时, $f(t)>0$.

由引理 (2.2) 知, 方程 (1.1) 等价于常微分方程 (2.2). 因此, 为获得方程 (1.1) 的正径向解的存在性, 只需证明常微分方程 (2.2) 有正解即可.

3. 正径向解的存在性

定理 3.1 若 (H1) 和 (H2) 成立, 则 k-Hessian 方程 (1) 有无穷多个正径向解 $u\in C^2[0,+\infty)$.

证明: 对方程 (2.4) 积分, 得

$$v(r)=v(0)+\int_0^r\left(\frac{k}{C_{N-1}^{k-1}}e^{-\psi_k(t)}\int_0^t e^{\psi_k(s)}\left(\frac{s}{1+\eta s}\right)^{k-1}h(s)f(v(s))ds\right)^{\frac{1}{k}}dt,\ r\geq 0. \tag{3.1}$$

结合引理 2.2, 取初值 $v_0(r)=v(0)=\gamma>0$, 在 $[0,+\infty)$ 上定义序列 $\{v_m(r)\}_{m\geq 0}$, 进行如下迭代:

$$\begin{cases} v_0(r)=\gamma, \\ v_m(r)=\gamma+\int_0^r\left(\frac{k}{C_{N-1}^{k-1}}e^{-\psi_k(t)}\int_0^t e^{\psi_k(s)}\left(\frac{s}{1+\eta s}\right)^{k-1}h(s)f(v_{m-1}(s))ds\right)^{\frac{1}{k}}dt,\ r\geq 0. \end{cases} \tag{3.2}$$

首先证明 $\{v_m(r)\}_{m\geq 0}$ 在 $[0,+\infty)$ 上是非减的. 显然, 当 $m=0$ 时, $v_m(r)=v_0(r)<v_1(r)$, 假设 $v_{m-1}(r)\leq v_m(r)$ 成立, 要证 $\{v_m(r)\}_{m\geq 0}$ 非减, 只需证 $v_m(r)\leq v_{m+1}(r)$ 即可. 易知对 $\forall m\geq 0$, $r\in[0,+\infty)$ 有

$$v_m(r) = \gamma + \int_0^r \left(\frac{k}{C_{N-1}^{k-1}} e^{-\psi_k(t)} \int_0^t e^{\psi_k(s)} \left(\frac{s}{1+\eta s} \right)^{k-1} h(s) f(v_m(s)) ds \right)^{\frac{1}{k}} dt$$

$$\le \gamma + \int_0^r \left(\frac{k}{C_{N-1}^{k-1}} e^{-\psi_k(t)} \int_0^t e^{\psi_k(s)} \left(\frac{s}{1+\eta s} \right)^{k-1} h(s) f(v_m(s)) ds \right)^{\frac{1}{k}} dt \tag{3.3}$$

$$= v_{m+1}(r),$$

故序列 $\{v_m(r)\}_{m\ge 0}$ 在 $[0,+\infty)$ 上是非减的. 由 (H2) 和 $\{v_m(r)\}_{m\ge 0}$ 的单调性得

$$\left[\frac{C_{N-1}^{k-1}}{k} e^{\psi_k(r)} \left((v_m(r))' \right)^k \right]' = e^{\psi_k(r)} \left(\frac{r}{1+\eta r} \right)^{k-1} h(r) f(v_{m-1}(r))$$

$$\le e^{\psi_k(r)} \left(\frac{r}{1+\eta r} \right)^{k-1} h(r) f(v_m(r)), \tag{3.4}$$

故

$$(v_m(r))' \le \left(\frac{k}{C_{N-1}^{k-1}} e^{-\psi_k(r)} \int_0^r e^{\psi_k(s)} \left(\frac{s}{1+\eta s} \right)^{k-1} h(s) ds \right)^{\frac{1}{k}} f(v_m(r)). \tag{3.5}$$

由于 $v_m(r) \ge 0$, 则

$$\int_0^r \frac{(v_m(t))'}{f(v_m(t))} dt \le \int_0^r \left(\frac{k}{C_{N-1}^{k-1}} e^{-\psi_k(t)} \int_0^t e^{\psi_k(s)} \left(\frac{s}{1+\eta s} \right)^{k-1} h(s) ds \right)^{\frac{1}{k}} dt = T(r),$$

进而得

$$\int_\gamma^{v_m(r)} \frac{1}{f(\tau)} d\tau \le T(r).$$

因此

$$F(v_m(r)) \le T(r), \quad \forall r \ge 0. \tag{3.6}$$

显然, F 是双射且 $F'(r) = \frac{1}{f(v(r))} > 0$ $(r > 0)$, 所以 F^{-1} (F 的逆映射) 在 $[0, F(\infty))$ 上严格递增. 又 $F(\infty) = \infty$ 且 $T(\infty) < \infty$, 则 $F^{-1}(\infty) = \infty$ 且

$$v^{(m)}(r) \le F^{-1}T(r) \le F^{-1}T(\infty) < \infty, \quad \forall r \ge 0. \tag{3.7}$$

此外, 由于 $h \in C([0,+\infty),[0,+\infty))$ 单调递增, 那么对取定的常数 $c_0 > 0$, $\forall\, \varepsilon > 0$, r_1, $r_2 \in [0, c_0]$,

$$
\begin{aligned}
|v_m(r_2)-v_m(r_1) &= \left|\int_{r_1}^{r_2}\left(\frac{k}{C_{N-1}^{k-1}}e^{-\psi_k(t)}\int_0^t e^{\psi_k(s)}\left(\frac{s}{1+\eta s}\right)^{k-1}h(s)f(v_{m-1}(s))ds\right)^{\frac{1}{k}}dt\right| \\
&\le \left(\frac{k}{C_{N-1}^{k-1}}\right)^{\frac{1}{k}}\left|\int_{r_1}^{r_2}\left(t^k(\frac{1}{1+\eta t})^{k-1}h(t)f(v_{m-1}(t))\right)^{\frac{1}{k}}dt\right| \\
&\le \left(\frac{k}{C_{N-1}^{k-1}}\right)^{\frac{1}{k}}\left(c_0^k(\frac{1}{1+\eta c_0})^{k-1}h(c_0)f(v_{m-1}(c_0))\right)^{\frac{1}{k}}|r_2-r_1|.
\end{aligned}
$$

令 $D:=(\frac{k}{C_{N-1}^{k-1}})^{\frac{1}{k}}\left(c_0^k(\frac{1}{1+\eta c_0})^{k-1}h(c_0)f(v_{m-1}(c_0))\right)^{\frac{1}{k}}$, 取 $\delta=\frac{\varepsilon}{D}$, 当 $|r_2-r_1|<\delta$ 时, 有

$$
|v_m(r_2)-v_m(r_1)<\varepsilon,
$$

故序列 $\{v_m(r)\}_{m\ge 0}$ 在 $[0,c_0]$ 上有界且等度连续. 根据 Arzelà-Ascoli 定理, $\{v_m(r)\}_{m\ge 0}$ 在 $[0,c_0]$ 上一致收敛到 $v(r)$. 由 $r>0$ 知 $v(r)$ 是常微分方程 (2.2) 的一个正解. 由 $\gamma\in(0,\infty)$ 的任意性可知, 方程 (2.2) 有无穷多个正解.

下证 $v\in C^2[0,+\infty)$. 显然, $v\in C^2(0,+\infty)$, 只需证 $v^{'}(r)$ 和 $v^{''}(r)$ 在 $r=0$ 处连续. 实际上,

$$
\begin{aligned}
v^{'}(0) &= \lim_{r\to 0}\frac{v(r)-v(0)}{r} \\
&= \lim_{r\to 0}\frac{\int_0^r\left(\frac{k}{C_{N-1}^{k-1}}e^{-\psi_k(t)}\int_0^t e^{\psi_k(s)}\left(\frac{s}{1+\eta s}\right)^{k-1}h(s)f(v(s))ds\right)^{\frac{1}{k}}dt}{r} \\
&= \lim_{r\to 0}\left(\frac{k}{C_{N-1}^{k-1}}e^{-\psi_k(r)}\int_0^r e^{\psi_k(s)}\left(\frac{s}{1+\eta s}\right)^{k-1}h(s)f(v(s))ds\right)^{\frac{1}{k}} \\
&= \left(\frac{k}{C_{N-1}^{k-1}}\right)^{\frac{1}{k}}\lim_{r\to 0}\left(\frac{\int_0^r re^{\psi_k(s)}\left(\frac{s}{1+\eta s}\right)^{k-1}h(s)f(v(s))ds}{re^{\psi_k(r)}}\right)^{\frac{1}{k}} \\
&= \left(\frac{k}{C_{N-1}^{k-1}}\right)^{\frac{1}{k}}\lim_{r\to 0}\left(\frac{r\left(\frac{r}{1+\eta r}\right)^{k-1}h(r)f(v(r))}{1+r\psi_k^{'}}\right)^{\frac{1}{k}} \\
&= \left(\frac{k}{C_{N-1}^{k-1}}\right)^{\frac{1}{k}}\lim_{r\to 0}\left(\frac{r^kh(r)f(v(r))}{(1-k)(1+\eta r)^{k-1}+N(1+\eta r)^k}\right)^{\frac{1}{k}} \\
&= 0
\end{aligned}
$$

且

$$v^{'}(r)=\left(\frac{k}{C_{N-1}^{k-1}}e^{-\psi_k(r)}\int_0^r e^{\psi_k(s)}\left(\frac{s}{1+\eta s}\right)^{k-1}h(s)f(v(s))ds\right)^{\frac{1}{k}},$$

$$\lim_{r\to 0}v^{'}(r)=\lim_{r\to 0}\left(\frac{k}{C_{N-1}^{k-1}}e^{-\psi_k(r)}\int_0^r e^{\psi_k(s)}\left(\frac{s}{1+\eta s}\right)^{k-1}h(s)f(v(s))ds\right)^{\frac{1}{k}}=0,$$

故

$$\lim_{r\to 0}v^{'}(r)=v^{'}(0)=0.$$

因此, $v^{'}(r)$ 在 $r=0$ 处连续. 类似地,

$$\begin{aligned}
v^{''}(0)&=\lim_{r\to 0}\frac{v^{'}(r)-v^{'}(0)}{r}\\
&=\lim_{r\to 0}\frac{\left(\frac{k}{C_{N-1}^{k-1}}e^{-\psi_k(r)}\int_0^r e^{\psi_k(s)}\left(\frac{s}{1+\eta s}\right)^{k-1}h(s)f(v(s))ds\right)^{\frac{1}{k}}}{r}\\
&=\left(\frac{k}{C_{N-1}^{k-1}}\right)^{\frac{1}{k}}\lim_{r\to 0}\left(\frac{\int_0^r e^{\psi_k(s)}\left(\frac{s}{1+\eta s}\right)^{k-1}h(s)f(v(s))ds}{r^k e^{\psi_k(r)}}\right)^{\frac{1}{k}}\\
&=\left(\frac{k}{C_{N-1}^{k-1}}\right)^{\frac{1}{k}}\lim_{r\to 0}\left(\frac{\left(\frac{r}{1+\eta r}\right)^{k-1}h(r)f(v(r))}{kr^{k-1}+r^k\psi_k^{'}(r)}\right)^{\frac{1}{k}}\\
&=\left(\frac{k}{C_{N-1}^{k-1}}\right)^{\frac{1}{k}}\lim_{r\to 0}\left(\frac{h(r)f(v(r))}{N(1+\eta r)^k}\right)^{\frac{1}{k}}\\
&=\left(\frac{k}{C_{N-1}^{k-1}}\right)^{\frac{1}{k}}\left(\frac{h(0)f(v(0))}{N}\right)^{\frac{1}{k}}.
\end{aligned}$$

通过计算, 得

$$\begin{aligned}
v^{''}(r)=\frac{1}{k}\left(\frac{k}{C_{N-1}^{k-1}}\right)^{\frac{1}{k}}\Bigg[&-\psi_k^{'}(r)\left(e^{-\psi_k(r)}\int_0^r e^{\psi_k(s)}\left(\frac{s}{1+\eta s}\right)^{k-1}h(s)f(v(s))ds\right)^{\frac{1}{k}}\\
&+\left(\frac{r}{1+\eta r}\right)^{k-1}h(r)f(v(r))\left(e^{-\psi_k(r)}\int_0^r e^{\psi_k(s)}\left(\frac{s}{1+\eta s}\right)^{k-1}h(s)f(v(s))ds\right)^{\frac{1}{k}-1}\Bigg].
\end{aligned}$$

令

$$V_1(r)=-\psi_k^{'}(r)\left(e^{-\psi_k(r)}\int_0^r e^{\psi_k(s)}\left(\frac{s}{1+\eta s}\right)^{k-1}h(s)f(v(s))ds\right)^{\frac{1}{k}},$$

$$V_2(r)=\left(\frac{r}{1+\eta r}\right)^{k-1}h(r)f(v(r))\left(e^{-\psi_k(r)}\int_0^r e^{\psi_k(s)}\left(\frac{s}{1+\eta s}\right)^{k-1}h(s)f(v(s))ds\right)^{\frac{1}{k}-1}.$$

则

$$\lim_{r\to 0} V_1(r) = -\lim_{r\to 0}\psi_k'(r)\left(e^{-\psi_k(r)}\int_0^r e^{\psi_k(s)}\left(\frac{s}{1+\eta s}\right)^{k-1} h(s)f(v(s))ds\right)^{\frac{1}{k}}$$

$$= -\lim_{r\to 0}\psi_k'(r)\lim_{r\to 0}\left(e^{-\psi_k(r)}\int_0^r e^{\psi_k(s)}\left(\frac{s}{1+\eta s}\right)^{k-1} h(s)f(v(s))ds\right)^{\frac{1}{k}}$$

$$= \lim_{r\to 0}(-r\psi_k'(r))\lim_{r\to 0}\left(\frac{\int_0^r e^{\psi_k(s)}\left(\frac{s}{1+\eta s}\right)^{k-1} h(s)f(v(s))ds}{r^k e^{\psi_k(r)}}\right)^{\frac{1}{k}}$$

$$= \lim_{r\to 0}(k-N(1+\eta r))\lim_{r\to 0}\left(\frac{h(r)f(v(r))}{N(1+\eta r)^k}\right)^{\frac{1}{k}}$$

$$= (k-N)\left(\frac{h(0)f(v(0))}{N}\right)^{\frac{1}{k}},$$

$$\lim_{r\to 0} V_2(r) = \lim_{r\to 0}\left(\frac{r}{1+\eta r}\right)^{k-1} h(r)f(v(r))\left(e^{-\psi_k(r)}\int_0^r e^{\psi_k(s)}\left(\frac{s}{1+\eta s}\right)^{k-1} h(s)f(v(s))ds\right)^{\frac{1}{k}-1}$$

$$= \lim_{r\to 0}\left(\frac{1}{1+\eta r}\right)^{k-1}\lim_{r\to 0} h(r)f(r,v(r))\lim_{r\to 0}\left(\frac{\int_0^r e^{\psi_k(s)}\left(\frac{s}{1+\eta s}\right)^{k-1} h(s)f(v(s))ds}{r^k e^{\psi_k(r)}}\right)^{\frac{1}{k}-1}$$

$$= N\left(\frac{h(0)f(v(0))}{N}\right)^{\frac{1}{k}},$$

故

$$\lim_{r\to 0} v''(r) = \frac{1}{k}\left(\frac{k}{C_{N-1}^{k-1}}\right)^{\frac{1}{k}}\left(\lim_{r\to 0}V_1(r)+\lim_{r\to 0}V_2(r)\right) = \left(\frac{k}{C_{N-1}^{k-1}}\right)^{\frac{1}{k}}\left(\frac{h(0)f(v(0))}{N}\right)^{\frac{1}{k}} = v''(0).$$

因此 $v \in C^2[0,+\infty)$.

综上, 方程 (2.2) 有无穷多个正解 $v \in C^2[0,+\infty)$, 即 k-Hessian 方程 (1.1) 有无穷多个正径向解 $u \in C^2[0,+\infty)$.

4. 应用举例

考虑 5-Hessian 方程

$$S_5(\lambda(D^2u+\eta|\nabla u|I)) = h(|x|)f(u),\ x\in\mathbb{R}^6. \tag{4.1}$$

其中 $\eta = \frac{1}{6}$, $H(r) = (\frac{1+r}{r})^4 e^{-r-\ln r}$, $f(u) = u+1$. 显然, $f(u)$ 满足 (H2) 且

$$\psi_5(r) = \frac{5}{C_5^4}\left(\frac{C_5^4 r}{6} + C_5^5\ln r + \frac{C_5^5 r}{6}\right) = r+\ln r.$$

此外,

$$
\begin{aligned}
T(\infty) &= \int_0^{\infty} \left(\frac{5}{C_5^4} e^{-t-\ln t} \int_0^t e^{s+\ln s} \left(\frac{s}{1+s} \right)^4 \left(\frac{1+s}{s} \right)^4 e^{-s-\ln s} ds \right)^{\frac{1}{5}} dt \\
&= \int_0^{\infty} \left(e^{-t-\ln t} \int_0^t 1 ds \right)^{\frac{1}{5}} dt \\
&= \int_0^{\infty} \left(\frac{t}{e^{t+\ln t}} \right)^{\frac{1}{5}} dt \\
&< \int_0^{\infty} \left(\frac{t+\ln t}{e^{t+\ln t}} \right)^{\frac{1}{5}} dt < \infty, \\
F(\infty) &= \int_{\gamma}^{\infty} \frac{1}{f(t)} dt = \int_{\gamma}^{\infty} \frac{1}{t+1} dt = \infty,
\end{aligned}
$$

即 (H1) 成立. 因此 5-Hessian方程 (4.1) 有无穷多个正径向解 $u \in C^2[0, +\infty)$.

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