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带积分边界条件的非线性Caputo型分数阶微分方程边值问题解的存在唯一性

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摘 要

为了更好地利用分数阶微分方程的优良性质解决工程问题，探究分数阶微分方程边值问题解的性质就成为一项重要的任务。运用**Schaefer**不动点定理和**Banach**压缩映射原理，研究一类带积分边值条件的非线性**Caputo**型分数阶微分方程边值问题，得到了边值问题解的存在唯一性结果。

关键词

分数阶微分方程，边值问题，解，存在性，唯一性

Existence and Uniqueness of Solutions for Boundary Value Problems of Nonlinear Caputo Fractional Differential Equation with Integral Boundary Conditions

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Abstract

In order to make better use of the excellent properties of fractional differential equations to solve engineering problems, it is an important task to explore the properties of solutions of boundary value problems of fractional differential equations. By using Schaefer's fixed point theorem and

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Banach contraction mapping principle, the boundary value problem of a class of nonlinear Caputo type fractional differential equation with integral boundary value conditions is studied. The existence and uniqueness of solutions of the boundary value problem are obtained.

Keywords

Fractional Differential Equation, Boundary Value Problem, Solution, Existence, Uniqueness

1. 引言

分数微积分是数学理论中的一个重要概念，由于其良好的性质，在工程领域中得到了广泛的应用，所以研究分数阶微分方程边值问题解的性质是十分必要的。许多学者主要采用单调迭代法[1] [2]、不动点定理[3] [4] [5]、拓扑度理论等非线性工具，研究分数阶微分方程边值问题解的性质。研究人员利用上述工具研究了各种分数阶微分方程边值问题，其中带有积分边界条件的分数阶微分方程边值问题[6]是一个重要方向。

在文献[7]中，Lv 通过使用 Schauder 不动点定理，研究了下述非线性分数阶微分方程边值问题正解的存在性

$$\begin{cases} D_{0+}^{\alpha}u(t)+f(t,u(t))=0, & t\in(0,1), \\ u(0)=0, \quad D_{0+}^{\alpha}u(1)=\sum_{i=1}^{m-2}\xi_i D_{0+}^{\beta}u(\eta_i), \end{cases}$$

其中$1<\alpha\le 2$，$0\le\beta\le 1$，$0\le\alpha-\beta-1$，$0<\xi_i$，$\eta_i<1$，$i=1,2,\cdots,m-2$，$\sum_{i=1}^{m-2}\xi_i\eta_i^{\alpha-\beta-1}<1$，$D_{0+}^{\alpha}$，$D_{0+}^{\beta}$ 为标准的 Riemann-Liouville 分数阶导数，$f\in C([0,1]\times[0,+\infty),[0,+\infty))$。

在上述工作的启发下，本文研究了以下带有积分边值条件的非线性 Caputo 型分数阶微分方程边值问题

$$\begin{cases} \left({}^{C}D_{0+}^{q}u\right)(t)=f(t,u(t)), & t\in(0,1), \\ u(0)=u''(0)=0, \quad \left({}^{C}D_{0+}^{\sigma}u\right)(1)=\sum_{i=1}^{m}\lambda_i\left(I_{0+}^{\beta_i}u\right)(\eta_i), \end{cases} \tag{1}$$

其中$2<q<3$，$0<\sigma\le 1$，$0<\lambda_i,\eta_i<1$，$\Gamma(2-\sigma)\sum_{i=1}^{m}\lambda_i\eta_i^{\beta_i+1}<1$，$f\in C([0,1]\times\mathbb{R},\mathbb{R})$，${}^{C}D_{0+}^{q}$，${}^{C}D_{0+}^{\sigma}$ 为标准的 Caputo 分数阶导数，$I_{0+}^{\beta_i}$ 为标准的 Riemann-Liouville 分数阶积分。

2. 主要结果

引理 2.1 若 $y\in C[0,1]$，则边值问题

$$\begin{cases} \left({}^{C}D_{0+}^{q}u\right)(t)=y(t), & t\in(0,1), \\ u(0)=u''(0)=0, \quad \left({}^{C}D_{0+}^{\sigma}u\right)(1)=\sum_{i=1}^{m}\lambda_i\left(I_{0+}^{\beta_i}u\right)(\eta_i), \end{cases} \tag{2}$$

有唯一解

$$u(t)=\int_0^1 G(t,s)y(s)\mathrm{d}s,$$

其中

$$G(t,s)=G_{\circ}(t,s)+G_{*}(t,s).$$

$G_{\circ}(t,s)$，$G_{*}(t,s)$ 分别为

$$G_{\circ}(t,s)=\begin{cases}\dfrac{(t-s)^{q-1}}{\Gamma(q)}-\dfrac{\dfrac{1-\sum_{i=1}^{m}\lambda_i\eta_i^{\beta_i+1}}{A}t(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)}, & 0\le s\le t\le 1,\\ -\dfrac{\dfrac{1-\sum_{i=1}^{m}\lambda_i\eta_i^{\beta_i+1}}{A}t(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)}, & 0\le t\le s\le 1,\end{cases}$$

$$G_{*}(t,s)=\begin{cases}-\dfrac{t}{A}\left[\sum_{0\le s\le\eta_i}\left(\dfrac{\lambda_i\eta_i^{\beta_i+1}(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)}-\dfrac{\lambda_i(\eta_i-s)^{q+\beta_i+1}}{\Gamma(q+\beta_i)}\right)\right], & t\in[0,1],\\ -\dfrac{t}{A}\sum_{\eta_i\le s\le 1}\dfrac{\lambda_i\eta_i^{\beta_i+1}(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)}, & t\in[0,1].\end{cases}$$

证明：由方程(2)可得

$$u(t)=\left(I_{0+}^{q}y\right)(t)-c_0-c_1t-c_2t^2,\quad t\in[0,1]. \tag{3}$$

所以

$$u'(t)=\left(I_{0+}^{q-1}y\right)(t)-c_1-2c_2t\ ,\quad t\in[0,1],$$

$$u''(t)=\left(I_{0+}^{q-2}y\right)(t)-2c_2\ ,\quad t\in[0,1].$$

考虑到方程(3)和边值条件 $u(0)=u''(0)=0$，可得

$$c_0=c_2=0,$$

$$u(t)=\left(I_{0+}^{q}y\right)(t)-c_1(t),\quad t\in[0,1].$$

进一步，可得

$$\left({}^{C}D_{0+}^{\sigma}u\right)(1)=\left(I_{0+}^{q-\sigma}y\right)(1)-c_1\frac{\Gamma(2)}{\Gamma(2-\sigma)},\quad t\in[0,1].$$

由边值条件 $\left({}^{C}D_{0+}^{\sigma}u\right)(1)=\sum_{i=1}^{m}\lambda_i\left(I_{0+}^{\beta_i}y\right)(\eta_i)$，可得

$$\begin{aligned}c_1&=\frac{1}{A}\left[\left(I_{0+}^{q-\sigma}y\right)(1)-\sum_{i=1}^{m}\lambda_i\left(I_{0+}^{q+\beta_i}y\right)(\eta_i)\right]\\&=\frac{1-\sum_{i=1}^{m}\lambda_i\eta_i^{\beta_i+1}}{A}\int_0^1\frac{(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)}y(s)\mathrm{d}s+\frac{1}{A}\sum_{i=1}^{m}\lambda_i\eta_i^{\beta_i+1}\int_0^1\frac{(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)}y(s)\mathrm{d}s\\&\quad-\frac{1}{A}\sum_{i=1}^{m}\lambda_i\int_0^{\eta_i}\frac{(\eta_i-s)^{q+\beta_i-1}}{\Gamma(q+\beta_i)}y(s)\mathrm{d}s.\end{aligned}$$

因此边值问题(2)有唯一解

$$
\begin{aligned}
u(t)=&\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}y(s)\mathrm{d}s-\frac{1-\sum_{i=1}^{m}\lambda_i\eta_i^{\beta_i+1}}{A}t\int_0^1\frac{(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)}y(s)\mathrm{d}s\\
&-\frac{t}{A}\sum_{i=1}^{m}\lambda_i\eta_i^{\beta_i+1}\int_0^1\frac{(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)}y(s)\mathrm{d}s+\frac{t}{A}\sum_{i=1}^{m}\lambda_i\int_0^{\eta_i}\frac{(\eta_i-s)^{q+\beta_i-1}}{\Gamma(q+\beta_i)}y(s)\mathrm{d}s\\
=&\int_0^t\left[\frac{(t-s)^{q-1}}{\Gamma(q)}-\frac{\frac{1-\sum_{i=1}^{m}\lambda_i\eta_i^{\beta_i+1}}{A}t(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)}\right]y(s)\mathrm{d}s\\
&-\int_t^1\left[\frac{\frac{1-\sum_{i=1}^{m}\lambda_i\eta_i^{\beta_i+1}}{A}t(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)}\right]y(s)\mathrm{d}s\\
&-\frac{t}{A}\lambda_1\eta_1^{\beta_1+1}\int_0^{\eta_1}\frac{(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)}y(s)\mathrm{d}s-\frac{t}{A}\lambda_1\eta_1^{\beta_1+1}\int_{\eta_1}^1\frac{(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)}y(s)\mathrm{d}s\\
&+\frac{t}{A}\lambda_1\int_0^{\eta_1}\frac{(\eta_1-s)^{q+\beta_1-1}}{\Gamma(q+\beta_1)}y(s)\mathrm{d}s\\
&-\cdots\\
&-\frac{t}{A}\lambda_m\eta_m^{\beta_m+1}\int_0^{\eta_m}\frac{(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)}y(s)\mathrm{d}s-\frac{t}{A}\lambda_m\eta_m^{\beta_m+1}\int_{\eta_m}^1\frac{(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)}y(s)\mathrm{d}s\\
&+\frac{t}{A}\lambda_m\int_0^{\eta_m}\frac{(\eta_m-s)^{q+\beta_m-1}}{\Gamma(q+\beta_m)}y(s)\mathrm{d}s\\
=&\int_0^1G_{\circ}(t,s)y(s)\mathrm{d}s+\int_0^1G_{*}(t,s)y(s)\mathrm{d}s\\
=&\int_0^1G(t,s)y(s)\mathrm{d}s,\qquad t\in[0,1].
\end{aligned}
$$

证明结束。

定义算子 $T:C[0,1]\to C[0,1]$ 为

$$(Tu)(t)=\int_0^1G(t,s)f(s,u(s))\mathrm{d}s,\qquad t\in[0,1].$$

边值问题(1)有解当且仅当算子 T 有不动点。

定理 2.1 [8] 令 X 是 Banach 空间。若 $T:X\to X$ 全连续算子，并且

$$V=\{u\in X\,|\,u=\mu Tu,\ 0<\mu<1\}$$

是有界的，则算子 T 在空间 X 中有不动点。

定理 2.2 若下面两个条件成立：

1) 对于任意的 $(t,u)\in([0,1]\times\mathbb{R})$，存在函数 $a(t),b(t)\in C([0,1]\times\mathbb{R})$ 使得

$$\left|f\left(t,u\right)\right|\le a\left(t\right)+b\left|u\left(t\right)\right|, \tag{2}$$

$$\|b\|\left[\frac{1}{\Gamma\left(q+1\right)}+\frac{1}{A\Gamma\left(q-\sigma+1\right)}+\frac{1}{A}\sum_{i=1}^{m}\frac{\lambda_i\eta_i^{q+\beta_i}}{\Gamma\left(q+\beta_i+1\right)}\right]<1.$$

则边值问题(1)至少有一个解。

证明：显然，算子T是连续的。接下来证明算子T是紧算子。令$\Omega=\left\{u\in C\left[0,1\right]:\|u\|<r\right\}$，对于任意的$u\in\Omega$，存在正数$K$使得$\left|f\left(t,u\left(t\right)\right)\right|\le K$，可得

$$\begin{aligned}
&\left|\left(Tu\right)\left(t\right)\right|\\
&=\left|\int_0^1 G\left(t,s\right)f\left(s,u\left(s\right)\right)\mathrm{d}s\right|\\
&\le\left|\int_0^t\frac{\left(t-s\right)^{q-1}}{\Gamma\left(q\right)}f\left(s,u\left(s\right)\right)\mathrm{d}s-\frac{t}{A}\int_0^1\frac{\left(1-s\right)^{q-\sigma-1}}{\Gamma\left(q-\sigma\right)}f\left(s,u\left(s\right)\right)\mathrm{d}s\right|\\
&\quad+\left|\frac{t}{A}\sum_{i=1}^{m}\lambda_i\int_0^{\eta_i}\frac{\left(\eta_i-s\right)^{q+\beta_i-1}}{\Gamma\left(q+\beta_i\right)}f\left(s,u\left(s\right)\right)\mathrm{d}s\right|\\
&\le\int_0^t\frac{\left(t-s\right)^{q-1}}{\Gamma\left(q\right)}\left|f\left(s,u\left(s\right)\right)\right|\mathrm{d}s+\frac{t}{A}\int_0^1\frac{\left(1-s\right)^{q-\sigma-1}}{\Gamma\left(q-\sigma\right)}\left|f\left(s,u\left(s\right)\right)\right|\mathrm{d}s\\
&\quad+\frac{t}{A}\sum_{i=1}^{m}\lambda_i\int_0^{\eta_i}\frac{\left(\eta_i-s\right)^{q+\beta_i-1}}{\Gamma\left(q+\beta_i\right)}\left|f\left(s,u\left(s\right)\right)\right|\mathrm{d}s\\
&\le\frac{K}{\Gamma(q)}\int_0^t\left(t-s\right)^{q-1}\mathrm{d}s+\frac{Kt}{A\Gamma\left(q-\sigma\right)}\int_0^1\left(1-s\right)^{q-\sigma-1}\mathrm{d}s\\
&\quad+\frac{Kt}{A}\sum_{i=1}^{m}\lambda_i\int_0^{\eta_i}\frac{\left(\eta_i-s\right)^{q+\beta_i-1}}{\Gamma\left(q+\beta_i\right)}\mathrm{d}s\\
&\le K\left[\frac{1}{\Gamma\left(q+1\right)}+\frac{1}{A\Gamma\left(q-\sigma+1\right)}+\frac{1}{A}\sum_{i=1}^{m}\frac{\lambda_i\eta_i^{q+\beta_i}}{\Gamma\left(q+\beta_i+1\right)}\right].
\end{aligned}$$

因此，$u\in T\left(\Omega\right)$是一致有界的。

对于任意的$\varepsilon>0$，$t_1,t_2\in\left[0,1\right]$ $\left(t_1<t_2\right)$存在

$$\delta=\frac{\varepsilon}{\dfrac{5K}{\Gamma\left(q+1\right)}+\dfrac{K}{A\Gamma\left(q-\sigma+1\right)}+\dfrac{K}{A}\displaystyle\sum_{i=1}^{m}\frac{\lambda_i\eta_i^{q+\beta_i}}{\Gamma\left(q+\beta_i+1\right)}}>0,$$

若$t_2-t_1<\delta$，则

$$\begin{aligned}
&\left|\left(Tu\right)\left(t_2\right)-\left(Tu\right)\left(t_1\right)\right|\\
&=\left|\int_0^1 G\left(t_2,s\right)f\left(s,u\left(s\right)\right)\mathrm{d}s-\int_0^1 G\left(t_1,s\right)f\left(s,u\left(s\right)\right)\mathrm{d}s\right|\\
&\le\int_0^{t_1}\frac{\left(t_2-s\right)^{q-1}-\left(t_1-s\right)^{q-1}}{\Gamma\left(q\right)}\left|f\left(s,u\left(s\right)\right)\right|\mathrm{d}s+\int_{t_1}^{t_2}\frac{\left(t_2-s\right)^{q-1}}{\Gamma\left(q\right)}\left|f\left(s,u\left(s\right)\right)\right|\mathrm{d}s\\
&\quad+\frac{t_2-t_1}{A}\int_0^1\frac{\left(1-s\right)^{q-\sigma-1}}{\Gamma\left(q-\sigma\right)}\left|f\left(s,u\left(s\right)\right)\right|ds+\frac{t_2-t_1}{A}\sum_{i=1}^{m}\lambda_i\int_0^{\eta_i}\frac{\left(\eta_i-s\right)^{q+\beta_i-1}}{\Gamma\left(q+\beta_i\right)}\left|f\left(s,u\left(s\right)\right)\right|\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{K\left[2(t_2-t_1)^q+(t_2^q-t_1^q)\right]}{\Gamma(q+1)}+\frac{K(t_2-t_1)}{A\Gamma(q-\sigma)}\int_0^1(1-s)^{q-\sigma-1}\mathrm{d}s\\
&\quad+\frac{K(t_2-t_1)}{A}\sum_{i=1}^m\lambda_i\int_0^{\eta_i}\frac{(\eta_i-s)^{q+\beta_i-1}}{\Gamma(q+\beta_i)}\mathrm{d}s\\
&\le\left[\frac{5K}{\Gamma(q+1)}+\frac{K}{A\Gamma(q-\sigma+1)}+\frac{K}{A}\sum_{i=1}^m\frac{\lambda_i\eta_i^{q+\beta_i}}{\Gamma(q+\beta_i+1)}\right](t_2-t_1)\\
&<\varepsilon.
\end{aligned}
$$

因此，由 Arzela-Ascoli 定理，算子 T 是全连续的。

接下来，考虑集合 $V=\{u\in X\,|\,u=\mu Tu,\ 0<\mu<1\}$，下证集合 V 是有界的。

$$
\begin{aligned}
|u(t)|&=|\mu(Tu)(t)|\\
&\le\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,u(s))|\mathrm{d}s+\frac{t}{A}\int_0^1\frac{(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)}|f(s,u(s))|\mathrm{d}s\\
&\quad+\frac{t}{A}\sum_{i=1}^m\lambda_i\int_0^{\eta_i}\frac{(\eta_i-s)^{q+\beta_i-1}}{\Gamma(q+\beta_i)}|f(s,u(s))|\mathrm{d}s\\
&\le\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}(\|a\|+\|b\|\|u\|)ds+\frac{t}{A}\int_0^1\frac{(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)}(\|a\|+\|b\|\|u\|)\mathrm{d}s\\
&\quad+\frac{t}{A}\sum_{i=1}^m\lambda_i\int_0^{\eta_i}\frac{(\eta_i-s)^{q+\beta_i-1}}{\Gamma(q+\beta_i)}(\|a\|+\|b\|\|u\|)\mathrm{d}s\\
&\le(\|a\|+\|b\|\|u\|)\left[\frac{1}{\Gamma(q+1)}+\frac{1}{A\Gamma(q-\sigma+1)}+\frac{1}{A}\sum_{i=1}^m\frac{\lambda_i\eta_i^{q+\beta_i}}{\Gamma(q+\beta_i+1)}\right].
\end{aligned}
$$

因此，集合 V 是有界的。由定理 2.1 可知算子 T 至少有一个不动点，也就是边值问题(1)至少有一个解。

定理 2.3 假设 $f\in C([0,1]\times\mathbb{R},\mathbb{R})$，对于任意的 $t\in[0,1]$，$x,y\in\mathbb{R}$，存在正数 L 使得

$$
|f(t,x)-f(t,y)|\le L|x-y|,
$$

其中

$$
L<\frac{1}{\dfrac{1}{\Gamma(q+1)}+\dfrac{1}{A\Gamma(q-\sigma+1)}+\dfrac{1}{A}\sum_{i=1}^m\dfrac{\lambda_i\eta_i^{q+\beta_i}}{\Gamma(q+\beta_i+1)}}.
$$

则边值问题(1)存在唯一解。

证明： 对于任意 $t\in[0,1]$，$u(t),v(t)\in C[0,1]$，可得

$$
\begin{aligned}
&|(Tu)(t)-(Tv)(t)|\\
&=\left|\int_0^1G(t,s)f(s,u(s))\mathrm{d}s-\int_0^1G(t,s)f(s,v(s))\mathrm{d}s\right|\\
&\le\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,u(s))-f(s,v(s))|+\frac{t}{A}\int_0^1\frac{(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)}|f(s,u(s))-f(s,v(s))|\mathrm{d}s\\
&\quad+\frac{t}{A}\sum_{i=1}^m\lambda_i\int_0^{\eta_i}\frac{(\eta_i-s)^{q+\beta_i-1}}{\Gamma(q+\beta_i)}|f(s,u(s))-f(s,v(s))|\mathrm{d}s
\end{aligned}
$$

$$\le L|u-v|\left[\frac{1}{\Gamma(q+1)}+\frac{1}{A\Gamma(q-\sigma+1)}+\frac{1}{A}\sum_{i=1}^{m}\frac{\lambda_i\eta_i^{q+\beta_i}}{\Gamma(q+\beta_i+1)}\right].$$

因此，由 Banach 压缩映射原理，可知算子 T 有唯一不动点，也就相当于边值问题(1)存在唯一解。

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