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一类总人口变化的随机传染病模型的阈值动力学行为

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摘 要

为了研究死亡率、传染率和系统的随机扰动对疾病传播的影响，本文主要考虑了一个在变化的人口规模下，疾病的死亡率、传染率和系统都受到干扰的随机传染病模型。通过构造恰当的李雅普诺夫函数证明了解的存在唯一性。建立了决定疾病灭绝的阈值R_0^s，利用鞅的大数定理和$It\hat{o}$公式得到了传染病灭绝的充分和几乎必要条件。更具体地讲，如果$R_0^s < 1$，意味着疾病以指数方式灭绝。最终通过构造的阈值R_0^s可以发现，易感者的系统扰动会增强疾病的传播；而染病者的传染率、死亡率和系统扰动的随机波动会抑制疾病的传播。

关键词

SIRS模型，疾病灭绝，解的存在唯一性

The Threshold Dynamic Behavior of a Stochastic Infectious Disease Model with Varying Total Population Size

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Abstract

To study the effects of mortality, infection rates of random disturbances and random perturbation of the system on the spread of disease, in this paper, we consider a stochastic infectious disease model in which the mortality, transmission and system of the disease are disturbed with changing population size. The existence and uniqueness of knowledge is proved by constructing appropriate Lyapunov functions. Established thresholds R_0^s for determining disease extinction, applying Martingale's theorem of large numbers and *Itô* formulas, sufficient and almost necessary conditions have been obtained for the extinction of infectious diseases. More specifically, if $R_0^s < 1$, the disease will die out. Finally, by constructing threshold R_0^s, we find that the stochastic perturbations of the death rate and random perturbation of the system for susceptible population can enhance the spread of disease, while the stochastic perturbations of the death rate and random perturbation of the system for infectious population, as well as the transmission rate of the disease can suppress the spread of the disease.

Keywords

SIRS Model, Extinction of Disease, Existence and Uniqueness of Solutions

1. 引言

传染病严重影响着人类的身体健康和社会的经济发展, 研究是预防和控制传染病的重要方式. 数学模型可以很好地刻画传染病的传播规律及发展趋势、影响传染病传播的关键因素. 通过定量分析, 能够预测疾病的传播趋势及验证控制措施有效性. 为了使数学模型更好地刻画疾病的传播过程, 在建模过程中需要考虑影响疾病传播的现实因素.

在实际生活中，传染病的感染率、死亡率、复发率和免疫损失率往往受到季节、气温、环境湿度等环境因素的影响 [1–3]。已有大量的文献证明了随机环境扰动对传染病模型动力学行为的影响。例如，在Gray 等人 [4]的工作中，他们在传染率β中引入了随机环境影响，建立了一个随机SIS模型

的灭绝和持续的充分条件。Du等人的工作 [5]，引入了一个具有Beddington-DeAngelis功能响应的随机SIR模型，该模型中，出生率-死亡率和发病率同时对随机噪声产生干扰，他们通过找到阈值，建立了疾病持久性和灭绝的充分和几乎必要条件。陈等人 [6]通过在确定性模型中引入环境噪声，研究了具有一般非线性发病率的随机SIRS 传染病模型，建立了遍历平稳分布存在唯一性的充分条件，在一定条件下，证明了随机扰动对疾病灭绝的影响。韩等人 [7]考虑了传染率受到白噪声和彩色噪声的影响，提出了一类状态切换的随机SIRS传染病模型，得出了白噪声和彩色噪声能够抑制疾病爆发的结论并证明了在一定条件下无病平衡点是随机渐进稳定的。N.Tuerxun 等人 [8]假设传染率受到环境干扰，研究了一类具有非单调发生率和退化扩散的随机SIRS模型，利用马尔科夫半群理论证明了模型平稳分布的存在唯一性。蔡等人 [9,10] 假设传染率、恢复率和死亡率同时受到环境噪声的影响，提出了一类随机的SIS 传染病模型，并证明了系统解的存在唯一性和疾病的持久与灭绝。另外有一些文献也考虑了多个参数受环境扰动的传染病模型 [3,5,9,11–13]。蔡等人 [14]将环境噪声引入确定性模型中，即假设传染率因为环境噪声的波动而受到干扰，提出了一类在干预策略下的具有非线性发生率的随机SIRS模型，得出了环境噪声的强度能够抑制疾病的爆发与流行的结论。传染病模型受环境影响的研究还有很多 [3,12,15–22]。

赵等人 [23]假设传染率β受到随机环境因素的影响，提出了一类总人口变化且具有标准发生率的随机SIRS传染病模型如下：

$$\begin{cases} dS(t) = (bN(t) - \mu S(t) - \frac{\beta S(t)I(t)}{N(t)} + \delta R)dt - \sigma \frac{S(t)I(t)}{N(t)} dB(t), \\ dI(t) = [\frac{\beta S(t)I(t)}{N(t)} - (\mu + \alpha + \gamma)I(t)]dt + \sigma \frac{S(t)I(t)}{N(t)} dB(t), \\ dR(t) = [\gamma I(t) - (\mu + \delta)R(t)]dt. \end{cases} \tag{1}$$

- $S(t)$: 传染病中易感者的数量,
- $I(t)$: 传染病中染病者的数量,
- $R(t)$: 传染病中恢复者的数量,
- $N(t)$: 总人口的数量,
- b: 出生率,
- μ: 自然死亡率,
- β: 感染个体的人均接触率，即单位时间内暴露的接触者的平均数量,
- δ: 恢复个体丧失免疫力的速率,
- α: 感染个体的因病死亡率,
- γ: 感染个体的恢复率,
- σ: 白噪声强度,

这里$B(t)$ 是一维标准的布朗运动。赵等人 [23]得出了传染病模型(1)的灭绝和持久的充分条件如下：

(1) 如果(1) $\sigma^2 > \max\{(\beta-\alpha)^2/2(b+\gamma), \beta-\alpha\}$, (2) $\beta-\alpha \le 0$, $\tilde{R}_0 < 0$, (3) $\sigma^2 \le \beta-\alpha$, 其中$\tilde{R}_0 = \frac{\beta-\frac{\sigma^2}{2}}{b+\alpha+\gamma}$ 三个条件中有一个条件成立，则疾病灭绝。

(2) 如果$\tilde{R}_0 = \frac{\beta}{b+\alpha+\gamma} - \frac{\sigma^2}{2(b+\alpha+\gamma)} > 1$，则疾病持久。

在文献 [23]的基础上，进一步考虑模型(1)受到系统受到扰动，分别影响系统(1)中的$\frac{dS}{dt}$，$\frac{dI}{dt}$，$\frac{dR}{dt}$。

更具体讲，考虑的模型如下：

$$\begin{cases} dS(t) = [bN(t) - \mu S(t) - \frac{\beta S(t)I(t)}{N(t)} + \delta R(t)]dt - \sigma_1 \frac{S(t)I(t)}{N(t)} dB_1(t) + \sigma_2 S(t) dB_2(t), \\ dI(t) = [\frac{\beta S(t)I(t)}{N(t)} - (\mu+\alpha+\gamma)I(t)]dt + \sigma_1 \frac{S(t)I(t)}{N(t)} dB_1(t) + \sigma_3 I(t) dB_3(t), \\ dR(t) = [\gamma I(t) - (\mu+\delta)R(t)]dt + \sigma_4 R(t) dB_4(t). \end{cases} \tag{2}$$

$B_1(t)$，$B_2(t)$，$B_3(t)$ ，$B_4(t)$是四个独立的布朗运动，σ_1，σ_2，σ_3，σ_4为环境扰动的强度。假设模型(2)中的所有参数都是非负的。

从(2)式可以得到

$$dN(t) = [(b-\mu)N(t) - \alpha I(t)]dt + \sigma_2 S(t)dB_2(t) + \sigma_3 I(t)dB_3(t) + \sigma_4 R(t)dB_4(t).$$

令$x = \frac{S}{N}, y = \frac{I}{N}, z = \frac{R}{N}$，则系统(2)变为

$$\begin{cases} dx = [b(1-x) + (\alpha-\beta)xy + \delta z + \sigma_3^2 xy^2 - \sigma_2^2 x^2(1-x) + \sigma_4^2 xz^2]dt \\ \qquad -\sigma_1 xydB_1(t) + \sigma_2 x(1-x)dB_2(t) - \sigma_3 xydB_3(t) - \sigma_4 xzdB_4(t), \\ dy = [\beta xy - (b+\alpha+\gamma)y + \alpha y^2 - \sigma_3^2 y^2(1-y) + \sigma_2^2 x^2 y + \sigma_4^2 yz^2]dt \\ \qquad +\sigma_1 xydB_1(t) - \sigma_2 xydB_2(t) + \sigma_3 y(1-y)dB_3(t) - \sigma_4 yzdB_4(t), \\ dz = [\gamma y - (b+\delta)z + \alpha yz + \sigma_3^2 y^2 z + \sigma_2^2 x^2 z - \sigma_4^2 z^2(1-z)]dt - \sigma_2 xzdB_2(t) \\ \qquad -\sigma_3 yzdB_3(t) + \sigma_4 z(1-z)dB_4(t). \end{cases} \tag{3}$$

对于模型(3)，因为有$x+y+z=1$ 成立，则模型(3) 也可以写成

$$\begin{cases} dx = [b(1-x) + (\alpha-\beta)xy + \delta(1-x-y) + \sigma_3^2 xy^2 - \sigma_2^2 x^2(1-x) \\ \qquad +\sigma_4^2 x(1-x-y)^2]dt - \sigma_1 xydB_1(t) + \sigma_2 x(1-x)dB_2(t) \\ \qquad -\sigma_3 xydB_3(t) - \sigma_4 x(1-x-y)dB_4(t), \\ dy = [\beta xy - (b+\alpha+\gamma)y + \alpha y^2 - \sigma_3^2 y^2(1-y) + \sigma_2^2 x^2 y \\ \qquad +\sigma_4^2 y(1-x-y)^2]dt + \sigma_1 xydB_1(t) \\ \qquad -\sigma_2 xydB_2(t) + \sigma_3 y(1-y)dB_3(t) - \sigma_4 y(1-x-y)dB_4(t), \end{cases} \tag{4}$$

在分析传染病模型的过程中，一些学者尝试着建立了疾病灭绝和持久的阈值条件，例如Du等人 [5]提出了一类具有Beddington-DeAngelis 功能反应函数的随机SIR 模型，得到了疾病灭绝和持久的阈值条件。Dieu 等人 [24] 提出了一类具有退化扩散的SIR随机微分方程，得到了疾病灭绝和持久的阈值条件。Du 等人 [25] 研究了一类具有Beddington-DeAngelis 发生率的随机SIR模型，提

出了传染病模型在退化和非退化情况下的持久和遍历的充分必要条件。这篇文章中，我们的目的是在文献 [23]的基础上考虑传染率、死亡率和系统受到环境影响，得到了推广模型(4)，并试着建立了模型(4) 的疾病灭绝的阈值条件。

文章剩余部分的内容如下。在第二部分中将给出本文会用到的基本定义和公式。在第三部分中将证明模型(4)的全局正解的存在唯一性并且给出了模型(4)的正不变集。第四部分通过建立合适的李雅普诺夫函数，证明了随机模型(4)关于疾病灭绝的阈值条件。第五部分进行了结论总结和未来展望。

2. 预备知识

本节将给出本文所用到的部分微分方程的相关知识，内容详见文献 [26–33].

定义2.1 [**27, 28, 34**] 若某个函数 $\tau : \Omega \to T \cup \{\infty\}$ 满足条件 $\{\tau \le t\} \in \mathscr{F}_t, \forall\, t \in T$, 则称 τ 为一个停时或 $\mathscr{F}_t$ 停时.

定义2.2 [**28**] 让 $P : \mathcal{F} \to [0, 1]$ 如果

(1) $P(\Omega) = 1$.

(2) 对 $A_i \in \mathcal{F}$ 且 $A_i \cap A_j = \emptyset (i \ne j)$, i,j=1, 2, ...有 $P\big(\cup_{i=1}^{\infty} A_i\big) = \Sigma_{i=1}^{\infty} P(A_j)$ 成立，那么称为概率测度, 称 $(\Omega, \mathcal{F}, P)$ 为概率空间. 如果 $(\Omega, \mathcal{F}, P)$ 为一概率空间. 令 $\bar{\mathcal{F}} = \Big\{A \in \mathcal{F} : \exists\, B,\ C \in \mathcal{F}$ 使得 $B \subset A \subset C, P(B) = P(C)\Big\}$, 那么 $\bar{\mathcal{F}}$ 是一个 σ- 代数, 称 $\bar{\mathcal{F}}$ 是 $\mathcal{F}$ 的完备化. 若 $\bar{\mathcal{F}} = \mathcal{F}$, 称 $\big(\Omega, \mathcal{F}, P\big)$ 为完备的概率空间.

定理 1.3.2.3 [**27, 34**](鞅的强大数定律) 设 $M(t) = \{M_t\}_{t\ge 0}$ 是 $t = 0$ 时为零的实值连续局部鞅, 则有

$$\lim_{t\to+\infty} \langle M, M\rangle_t = \infty \text{ a.s.} \Rightarrow \lim_{t\to+\infty} \frac{M_t}{\langle M, M\rangle_t} = 0 \qquad \text{a.s.}$$

以及

$$\limsup_{t\to+\infty} \frac{\langle M, M\rangle_t}{t} < \infty \ \text{ a.s.} \Rightarrow \lim_{t\to+\infty} \frac{M_t}{t} = 0 \qquad \text{a.s.}$$

3. 全局正解的存在唯一性

在这部分中，将证明不变域$\Lambda = \{x > 0, y > 0, x + y < 1\}$是系统(4)的正不变集且模型(4)存在唯一的全局正解。首先给出证明需要的引理3.1。

引理3.1. 记$\Gamma = \{(x, y, z) : x > 0, y > 0, z > 0, x + y + z = 1\}$。*对于任何给定的初始值*$(x_0, y_0, z_0) \in \Gamma$，*方程(3)存在唯一的正解*$(x(t), y(t), z(t), t \ge 0)$*且解将以概率1保持在*$\Gamma$ *中，也就是说，对任意的*$t \ge 0$，$(x(t), y(t), z(t)) \in \Gamma$。

证明. 模型(3)的系数是局部Lipschitz连续的，对于任何初始值$(x_0, y_0, z_0) \in \mathbb{R}^3_+$且$t \in (0, \tau_e)$，存在

唯一的局部解$(x(t), y(t), z(t))$ a.s，其中τ_e是爆破时间。接着只需要证明$\tau_e = \infty$ a.s. 令$n_0 \geq 1$是充分大的使得(x_0, y_0, z_0)都在区间$[\frac{1}{n_0}, 1]$中。对所有整数$n \geq n_0$，定义停时

$$\tau_n = \inf\{t \in [0, \tau_e) : \min\{x(t), y(t), z(t)\} \leq \frac{1}{n}\}; \tag{5}$$

集合$\inf \emptyset = \infty$($\emptyset$ 表示空集). 根据τ_n的定义，可以看出随着$n \to \infty$，τ_n是递增的并且记$\tau_\infty = \lim\limits_{n\to\infty} \tau_n$，则$\tau_\infty \leq \tau_e$ a.s。若$\tau_\infty = \infty$，则$\tau_e = \infty$ a.s。接下来，若有$\tau_\infty = \infty$成立，则可以得出正解。若$\tau_\infty \neq \infty$，也就是说，$\exists\, T \geq 0$和$\varepsilon \in (0, 1)$使得

$$P\{\tau_\infty \leq T\} > \varepsilon.$$

因此存在整数$n_1 \geq n_0$使得

$$P\{\tau_n \leq T\} \geq \varepsilon, \quad \forall n \geq n_1. \tag{6}$$

定义C^2-函数$D : \mathbb{R}^3_+ \to \mathbb{R}$，其中

$$D(x, y, z) = \ln xyz,$$

很明显，对任意$x, y, z \in (0, 1)$，函数$D(x, y, z)$为负。
对函数$D(x, y, z)$应用$It\hat{o}$公式，可以得到

$$\begin{aligned}
\mathscr{L}D(x,y,z) \;\; = \;\; & \frac{1}{x}[b(1-x) + (\alpha-\beta)xy + \delta z + \sigma_3^2 xy^2 - \sigma_2^2 x^2(1-x) + \sigma_4^2 xz^2] \\
& -\frac{1}{2x^2}[-\sigma_1 xydB_1(t) + \sigma_2 x(1-x)dB_2(t) - \sigma_3 xydB_3(t) - \sigma_4 xzdB_4(t)], \\
& +\frac{1}{y}[\beta xy - (b+\alpha+\gamma)y + \alpha y^2 - \sigma_3^2 y^2(1-y) + \sigma_2^2 x^2 y + \sigma_4^2 yz^2] \\
& -\frac{1}{2y^2}[\sigma_1 xydB_1(t) - \sigma_2 xydB_2(t) + \sigma_3 y(1-y)dB_3(t) - \sigma_4 yzdB_4(t)] \\
& +\frac{1}{z}[\gamma y - (b+\delta)z + \alpha yz + \sigma_3^2 y^2 z + \sigma_2^2 x^2 z - \sigma_4^2 z^2(1-z)] \\
& -\frac{1}{2z^2}[-\sigma_2 xzdB_2(t) - \sigma_3 yzdB_3(t) + \sigma_4 z(1-z)dB_4(t)] \\
= \;\; & [\frac{b}{x} - b + (\alpha-\beta)y + \frac{\delta z}{x} + \sigma_3^2 y^2 - \sigma_2^2 x(1-x) + \sigma_4^2 z^2] \\
& +[\beta x - (b+\alpha+\gamma) + \alpha y - \sigma_3^2 y(1-y) + \sigma_2^2 x^2 + \sigma_4^2 z^2] \\
& +[\frac{\gamma y}{z} - (b+\delta) + \alpha y + \sigma_3^2 y^2 + \sigma_2^2 x^2 - \sigma_4^2 z(1-z)] \\
& -\frac{1}{2}[\sigma_1^2 y^2 + \sigma_2^2 (1-x)^2 + \sigma_3^2 y^2 + \sigma_4^2 z^2] \\
& -\frac{1}{2}[\sigma_1^2 x^2 + \sigma_2^2 x^2 + \sigma_3^2 (1-y)^2 + \sigma_4^2 z^2] \\
& -\frac{1}{2}[\sigma_2^2 x^2 + \sigma_3^2 y^2 + \sigma_4^2 (1-z)^2]
\end{aligned}$$

$$
\begin{aligned}
&= [\frac{b}{x}-2b-\delta+(\alpha-\beta)y+\frac{\delta z}{x}+3\sigma_3^2y^2-\sigma_2^2x+3\sigma_4^2z^2 \\
&\quad +\beta x-(b+\alpha+\gamma)+\alpha y-\sigma_3^2y+3\sigma_2^2x^2+\frac{\gamma y}{z}+\alpha y-\sigma_4^2z] \\
&\quad -\frac{1}{2}[(\sigma_1^2+2\sigma_2^2)x^2+(\sigma_1^2+2\sigma_3^2)y^2 \\
&\quad +2\sigma_4^2z^2+\sigma_2^2(1-x)^2+\sigma_3^2(1-y)^2+\sigma_4^2(1-z)^2] \\
&\geq -2b-\delta-\beta y-(\sigma_3^2+\sigma_2^2)x-(b+\alpha+\gamma)-\sigma_3^2y-\sigma_4^2z \\
&\quad -\frac{1}{2}[(\sigma_1^2+2\sigma_2^2)x^2+(\sigma_1^2+2\sigma_3^2)y^2+2\sigma_4^2z^2+\sigma_2^2(1-x)^2 \\
&\quad +\sigma_3^2(1-y)^2+\sigma_4^2)(1-z)^2]. \\
&\triangleq H(x,y,z).
\end{aligned}
$$

由于有条件$x+y+z=1$且函数$H(x,y,z)$是连续的，则$\exists C<0$，使得对任意的$(x,y,z)\in\Gamma$，有$H(x,y,z)\geq C$；因此

$$
\begin{aligned}
dD(x,y,z) &\geq Cdt+\sigma_1(x-y)dB_1(t)+\sigma_2(3x-1)dB_2(t) \\
&\quad +\sigma_3(3y-1)dB_3(t)+\sigma_4(3z-1)dB_4(t). \qquad (7)
\end{aligned}
$$

再对方程(7)两边从0到$\tau_n\wedge T$积分可得

$$
\begin{aligned}
\int_0^{\tau_n\wedge T}dD(x,y,z) &\geq \int_0^{\tau_n\wedge T}Cdt+\int_0^{\tau_n\wedge T}\sigma_1(x-y)dB_1(t)+\int_0^{\tau_n\wedge T}\sigma_2(3x-1)dB_2(t) \\
&\quad +\int_0^{\tau_n\wedge T}\sigma_3(3y-1)dB_3(t)+\int_0^{\tau_n\wedge T}\sigma_4(3z-1)dB_4(t).
\end{aligned}
$$

对上述不等式取期望，可以得到

$$
\mathbb{E}D(x(\tau_n\wedge T),y(\tau_n\wedge T),z(\tau_n\wedge T))\geq CT+D(x_0,y_0,z_0)>-\infty.
$$

对于$n\geq n_1$，令集合$\Omega_n=\{\omega\in\Omega:\tau_n=\tau_n(\omega)\leq T\}$。由(6)式，有$P(\Omega_n)\geq\varepsilon$成立。此外，对任意$\omega\in\Omega_n$，在$x(\tau_n,\omega),y(\tau_n,\omega),z(\tau_n,\omega)$中至少有一个等于$\frac{1}{n}$。因此，

$$
D(x(\tau_n,\omega),y(\tau_n,\omega),z(\tau_n,\omega))\leq\ln\frac{1}{n}.
$$

可以得到

$$
\begin{aligned}
\mathbb{E}[D(x(\tau_n\wedge T),y(\tau_n\wedge T),z(\tau_n\wedge T))] &\leq \mathbb{E}[\ln y(T\wedge\tau_n)] \\
&\leq \mathbb{E}[1_{\Omega_n}\ln y(\tau_n)] \\
&\leq \varepsilon\ln\frac{1}{n}.
\end{aligned}
$$

其中1_{Ω_k}是函数Ω_k的特征函数，因为当$n\to\infty$时，$-\infty<CT<-\infty$是矛盾的，因此有$\tau_\infty=\infty$，也就是$\tau_n=\infty$，则$\tau_e=\infty$ a.s.。引理3.1的证明完成。 □

在引理3.1的基础上，可以看出对任意$t\geq 0$，有$x(t)>0, y(t)>0, z(t)>0$且$x+y+z=1$，定理3.1的结果明显是成立的。

定理3.1. 记$\Lambda=\{(x,y):x>0,y>0,x+y<1\}$，则$\Lambda$是模型*(4)*的正不变集。此外，对于任何初始值$(x_0,y_0)\in\Lambda$，模型*(4)*存在一个唯一的正解$(x(t),y(t))$且解以概率*1*存在于$\Lambda$中，即对任意的$t\geq 0$，$(x(t),y(t))\in\Lambda a.s.$。

4. 疾病的灭绝

在本节中，将讨论模型(4)关于疾病灭绝的充分和几乎必要的条件。首先，为了方便后面的证明，提出引理4.1和标记4.1中给出一些符号。

引理4.1. 对方程

$$dx=[b(1-x)+\delta(1-x)-\sigma_2^2x^2(1-x)+\sigma_4^2x(1-x)]dt+\sigma_2x(1-x)dB_2-\sigma_4x(1-x)dB_4.$$

假设条件$b+\delta>\frac{1}{2}\sigma_2^2$ 成立，则对任意的$x\in(0,1)$，则方程*(8)*的解x一致收敛于*1*。

证明. 对于方程(8)，应用Itô公式，可以得到

$$\begin{aligned} d\ln(1-x) &= [-b-\delta+\frac{1}{2}\sigma_2^2x^2-\sigma_4^2x-\frac{1}{2}\sigma_4^2x^2]dt+\sigma_2xdB_2-\sigma_4xdB_4 \\ &\leq [-b-\delta+\frac{1}{2}\sigma_2^2-\sigma_4^2x-\frac{1}{2}\sigma_4^2x^2]dt+\sigma_2xdB_2-\sigma_4xdB_4 \\ &\triangleq \Upsilon_1(x)dt+\sigma_2xdB_2-\sigma_4xdB_4. \end{aligned} \tag{8}$$

其中$\Upsilon_1(x)=-b-\delta+\frac{1}{2}\sigma_2^2-\sigma_4^2x-\frac{1}{2}\sigma_4^2x^2$。因为假设条件$b+\delta>\frac{1}{2}\sigma_2^2$ 成立和函数$\Upsilon_1(x)$的连续性，对于任意的$x\in(1-\varrho_1,1)$，选择充分小的常数ϱ_1 使得$\Upsilon_1(x)<0$。对等式(8)两边从0到t积分并除以t，可以得出

$$\frac{\ln(1-x(t))}{t} \leq \frac{\ln(1-x(0))}{t}+\frac{\int_0^t\Upsilon_1(x)dt}{t}+\frac{\int_0^t\sigma_2xdB_2(t)}{t}-\frac{\int_0^t\sigma_4xdB_4(t)}{t}.$$

对上式两边取上确界并运用鞅的大数定理可得

$$\limsup_{t\to\infty}\frac{\ln(1-x(t))}{t} \leq \limsup_{t\to\infty}\frac{\int_0^t\Upsilon_1(x)dt}{t}<0,\ \forall\ x\in(1-\varrho_1,1).$$

因此，对任意的$x\in(1-\varrho,1)$，可以得出x一致趋于1。接着，证明方程(8)的任意解x，从$(0,1)$开始最终可进入到区域$(1-\varrho_1,1)$。定义函数$V^*=-(2x+1)^{k^*}$，其中k^*为常数，在后面的证明过程中会

被确定，由Itô公式可得

$$\begin{aligned}\mathscr{L}V^* &= -2k^*(2x+1)^{k^*-1}[b(1-x)+\delta(1-x)-\sigma_2^2x^2(1-x)+\sigma_4^2x(1-x)] \\ &\quad -\frac{4k^*(k^*-1)(2x+1)^{k^*-2}}{2}[\sigma_2^2x^2(1-x)^2+\sigma_4^2(1-x)^2] \\ &= -2k^*(2x+1)^{k^*-2}[(2x+1)(1-x)(b+\delta-(\sigma_1^2+\sigma_2^2)x^2+\sigma_4^2x) \\ &\quad +(k^*-1)x^2(1-x)^2(\sigma_2^2+\sigma_4^2)] \\ &\triangleq -2k^*(2x+1)^{k^*-2}\cdot\Pi\end{aligned}$$

其中$\Pi=(2x+1)(1-x)(b+\delta-(\sigma_1^2+\sigma_2^2)x^2+\sigma_4^2x)+(k^*-1)x^2(1-x)^2(\sigma_2^2+\sigma_4^2)$。
当$x\in(0,1-\varrho_1]$，不等式$(2x+1)(1-x)(b+\sigma_4^2x)>(b+\sigma_4^2)(1-1+\varrho_1)=\varrho_1 b$成立，则可以找到足够大的常数$k^*$ 使得

$$\begin{aligned}\Pi &> \varrho_1 b-(2x+1)(\sigma_1^2+\sigma_2^2)x^2+(k^*-1)(\sigma_2^2+\sigma_4^2)x^2(1-x)^2 \\ &> \frac{1}{4}\varrho_1 b,\ \ \forall\ \ x\in(0,1-\varrho_1].\end{aligned}$$

因此，

$$\mathscr{L}V^*=-2k^*(2x+1)^{k^*-2}\cdot\Pi<-2k^*\cdot\frac{1}{4}\varrho_1 b<-\frac{1}{2}k^*\varrho_1 b,\ \ \text{for all}\ \ x\in(0,1-\varrho_1].$$

对上式从0到$\tau_{\varrho_1}\wedge t$积分且由Dynkin公式，可以得到

$$\begin{aligned}\mathbb{E}V^*(x(\tau_{\varrho_1}\wedge t)) &= V^*(x(0))+\mathbb{E}\int_0^{\tau_{\varrho_1}\wedge t}\mathscr{L}V^*(\Psi(s))ds \\ &< V^*(x(0))-\frac{1}{2}k^*\varrho_1 b\cdot\mathbb{E}(\tau_{\varrho_1}\wedge t).\end{aligned}$$

使得$t\to\infty$且应用Fatou引理，能够得到

$$\mathbb{E}V^*(x(\tau_{\varrho_1}))<V^*(x(0))-\frac{1}{2}k^*\varrho_1 b\cdot\mathbb{E}(\tau_{\varrho_1}).$$

根据V^*在$(0,1)$的有界性，可以得到$\mathbb{E}(\tau_{\varrho_1})<\infty$。因此，对于$x\in(0,1)$，当$t\to\infty$时，有$x\to 1$。引理证明完成。 □

标记4.1. *我们记系统(4)的解为$\Psi(t)=(x(t),y(t))$。特别地，表示初值为$\psi=(x_0,y_0)\in\Lambda$。记函数*

$$\begin{aligned}f(x,y) &= \beta x-(b+\alpha+\gamma)+\alpha y-\sigma_3^2y(1-y)+\frac{1}{2}(\sigma_2^2-\sigma_1^2)x^2 \\ &\quad +\frac{1}{2}\sigma_4^2(1-x-y)^2-\frac{1}{2}\sigma_3^2(1-y)^2.\end{aligned}\tag{9}$$

将平衡点$(1,0)$代入上述方程，可以得到确定疾病灭绝的阈值条件，即得到

$$\begin{aligned}f(1,0) &= [\beta-(b+\alpha+\gamma)+\frac{\sigma_2^2-\sigma_1^2-\sigma_3^2}{2}] \\ &= (b+\alpha+\gamma)[R_0^s-1].\end{aligned}\tag{10}$$

其中$R_0^s=\frac{\beta}{b+\alpha+\gamma}+\frac{\sigma_2^2-\sigma_1^2-\sigma_3^2}{2(b+\alpha+\gamma)}$。

下面将证明，在某些条件下，受感染的人数呈指数下降至零，即疾病灭绝。

定理4.1. 对初值$\psi=(x_0,y_0)$，记$\Psi(t)=(x(t),y(t))$ 为模型(4)的解。如果$R_0^s<1$且$b+\delta>\frac{1}{2}\sigma_2^2$，对任意初值$(x_0,y_0)\in\Lambda$，当$t\to\infty$，则$(x(t),y(t))\to(1,0)$ $a.s.$。疾病灭绝。更多地，

$$\mathbb{P}\left\{\lim_{t\to\infty}\frac{\ln y(t)}{t}=(b+\alpha+\gamma)(R_0^s-1)<0\right\}=1,\ \ \forall\ \ (x_0,y_0)\in\Lambda,y>0. \tag{11}$$

证明. 因为$R_0^s<1$，则从(10)式中可以看出$f(1,0)<0$。对$(x,y)\in\Lambda$，模型(4)运用Itô公式可得

$$\begin{aligned}\mathscr{L}\ln y &= \frac{1}{y}[\beta xy-(b+\alpha+\gamma)y+\alpha y^2-\sigma_3^2y^2(1-y)+\sigma_2^2x^2y+\sigma_4^2y(1-x-y)^2]\\ &\quad -\frac{1}{2y^2}[\sigma_1^2x^2y^2+\sigma_2^2x^2y^2+\sigma_3^2y^2(1-y)^2+\sigma_4^2y^2(1-x-y)^2]\\ &= \beta x-(b+\alpha+\gamma)+\alpha y-\sigma_3^2y(1-y)+\frac{1}{2}(\sigma_2^2-\sigma_1^2)x^2\\ &\quad +\frac{1}{2}\sigma_4^2(1-x-y)^2-\frac{1}{2}\sigma_3^2(1-y)^2\\ &= f(x,y)\end{aligned}$$

$$\begin{aligned}d\ln y &= f(x,y)+\sigma_1xdB_1(t)-\sigma_2xdB_2+\sigma_3(1-y)dB_3\\ &\quad -\sigma_4(1-x-y)dB_4.\end{aligned}$$

因为函数$f(x,y)$是连续的并且$f(1,0)<0$，对任意的$(x,y)\in U_{\varrho_2}=\{(x,y),x\in(1-\varrho_2,1),y\in(0,\varrho_2)\}$。选正常数$\varrho_2\in(0,1)$充分小使得$f(x,y)<0$成立。
对上式两边从0到t积分并除以t，可以得到

$$\begin{aligned}\frac{\ln y}{t} &= \frac{\ln y_0}{t}+\frac{1}{t}\int_0^t f(x,s)ds+\frac{1}{t}\int_0^t\sigma_1xdB_1(t)-\frac{1}{t}\int_0^t\sigma_2xdB_2\\ &\quad +\frac{1}{t}\int_0^t\sigma_3(1-y)dB_3-\frac{1}{t}\int_0^t\sigma_4(1-x-y)dB_4.\end{aligned}$$

对上式两边取上确界并且根据鞅的大数定理可得

$$\limsup_{t\to\infty}\frac{\ln y}{t}=\limsup_{t\to\infty}f(x,y)<0,\ \ \forall\ \ (x,y)\in U_{\varrho_2}. \tag{12}$$

因此，可以看出对任意的$(x,y)\in U_{\varrho_2}$，y一致趋于0。

接下来，我们将证明对任意的$(x,y)\in U_{\varrho_2}$，$x\to 1$。对任意小的$\varrho_2>0$使得$y<\varrho_2$，可以得到

$$\begin{aligned}\mathscr{L}\ln(1-x) &= -b-\frac{\alpha-\beta}{1-x}xy-\frac{\delta}{1-x}(1-x-y)-\sigma_3^2\frac{xy^2}{1-x}-\sigma_4^2\frac{x(1-x-y)^2}{1-x}\\&\quad+\frac{\sigma_2^2}{2}x^2-\sigma_4^2\frac{x^2(1-x-y)^2}{2(1-x)^2}-(\sigma_1^2+\sigma_3^2)\frac{x^2y^2}{2(1-x)^2}\\&\leq -b+\beta\frac{x\rho_2}{1-x}-\frac{\delta}{1-x}(1-x-\rho_2)-\sigma_4^2\frac{x(1-x-\rho_2)^2}{1-x}\\&\quad+\frac{\sigma_2^2}{2}x^2-\sigma_4^2\frac{x^2(1-x-\rho_2)^2}{2(1-x)^2}\\&\triangleq \Upsilon(x).\end{aligned}$$

其中$\Upsilon(x)=-b+\beta\frac{x\rho_2}{1-x}-\frac{\delta}{1-x}(1-x-\rho_2)-\sigma_4^2\frac{x(1-x-\rho_2)^2}{1-x}+\frac{\sigma_2^2}{2}x^2-\sigma_4^2\frac{x^2(1-x-\rho_2)^2}{2(1-x)^2}$。根据函数$\Upsilon(x)$的连续性和引理8的结论，意味着对任意的$x\in(1-\varrho_2,1)$，有$\Upsilon(x)<0$成立。因此

$$\begin{aligned}d\ln(1-x) &\leq \Upsilon(x)+\sigma_1\frac{xy}{1-x}dB_1(t)-\sigma_2 x dB_2(t)+\sigma_3\frac{xy}{1-x}dB_3(t)\\&\quad+\sigma_4\frac{x(1-x-y)}{1-x}dB_4(t)\end{aligned}\tag{13}$$

对(13)式两边从0到t积分并且除以t，可以得到

$$\begin{aligned}\frac{\ln(1-x)}{t} &\leq \frac{\ln(1-x_0)}{t}+\frac{\Upsilon(x)}{t}+\frac{1}{t}\int_0^t\sigma_1\frac{xy}{1-x}dB_1(t)-\frac{1}{t}\int_0^t\sigma_2 x dB_2(t)\\&\quad+\frac{1}{t}\int_0^t\sigma_3\frac{xy}{1-x}dB_3(t)+\frac{1}{t}\int_0^t\sigma_4\frac{x(1-x-y)}{1-x}dB_4(t).\end{aligned}\tag{14}$$

取(14)式两边的上极限并且根据鞅的大数定理有

$$\limsup_{t\to\infty}\frac{\ln(1-x)}{t}\leq\limsup_{t\to\infty}\frac{\Upsilon(x)}{t}<0\ \ \forall\ \ x\in(1-\varrho_2,1).$$

因此，

$$\limsup_{t\to\infty}\frac{\ln(1-x)}{t}<0\ \ \forall\ \ (x,y)\in U_{\varrho_2}.\tag{15}$$

则根据不等式(15)和(12)，可以得出对任意的$(x,y)\in U_{\varrho_2}$，(x,y)一致收敛于(1,0)。因此对任意的$\bar{\varepsilon}>0$，可以发现当取$0<\varrho<\varrho_2$有

$$\mathbb{P}\left\{\lim_{t\to\infty}(x(t),y(t))=(1,0)\right\}\geq 1-\bar{\varepsilon},(x_0,y_0)\in U_\varrho.\tag{16}$$

□

5. 结论

本文在文献 [23]的基础上，考虑疾病的死亡率和恢复率会受到环境的扰动，建立了一个具

有人口规模变化的随机SIRS模型。证明了模型(4)存在唯一的全局正解。接着通过建立决定疾病灭绝的阈值$R_0^s = \frac{\beta}{b+\alpha+\gamma} + \frac{\sigma_2^2-\sigma_1^2-\sigma_3^2}{2(b+\alpha+\gamma)}$，得出了疾病灭绝的充分几乎必要条件，即当$R_0^s < 1$ 且条件$b+\delta > \frac{1}{2}\sigma_2^2$ 成立，疾病灭绝。

在后续研究工作中，还可以考虑本文未完成的工作和探究其他建模思想。在本提出的模型基础上，还可以考虑具有人口logistic增长的随机传染病模型。当然，在系统(4)中引入Ornstein-Uhlenbeck过程或加入电报噪声也是值得考虑。这些问题可以在接下来的工作中进行讨论。

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