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与薛定谔算子相关的Marcinkiewicz积分在混合Morrey空间上的加权估计

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摘 要

利用A_p的估计以及函数分解方法，借助L^p空间上的加权估计，证明了与薛定谔算子相关的Marcinkiewicz积分在混合Morrey空间上的加权有界性，并给出与薛定谔算子相关的Marcinkiewicz积分的BMO交换子的加权有界性。

关键词

薛定谔算子，Marcinkiewicz积分，交换子，混合Morrey空间，加权有界性

Weighted Estimates of Marcinkiewicz Integrals Associated with Schrödinger Operator on Mixed Morrey Space

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Abstract

By using the weight estimation of A_p and the function decomposition method, and owing to the weighted estimation on the L^p spaces, we proved the weighted boundedness of Marcinkiewicz integrals associated with Schrödinger operator on mixed Morrey spaces, and the weighted boundedness of BMO commutators of Marcinkiewicz integrals associated with Schrödinger operator.

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Keywords

Schrödinger Operator, Marcinkiewicz Integral, Commutator, Mixed Morrey Space, Weighted Boundedness

1. 引言及主要结果

我们考虑薛定谔算子

$$L=-\Delta+V(x),\ \mathbb{R}^d,\ d\geq 3$$

其中$V(x)$是非负位势，属于反向 Hölder 类 $RH_s, s\geq\frac{d}{2}$，因此，存在一个常数 C，使得对任意球 $B\subset\mathbb{R}^n$

$$\left(\frac{1}{|B|}\int_B V(y)^s\,\mathrm{d}y\right)^{\frac{1}{s}}\leq\frac{C}{|B|}\int_B V(y)\,\mathrm{d}y. \tag{1}$$

对任意 $V\in RH_s, s\geq\frac{d}{2}$，临界半径函数 $\rho(x)=\rho(x,V)$，给定

$$\rho(x):=\sup\left\{r>0:\frac{1}{r^{d-2}}\int_{B(x,r)}V(y)\,\mathrm{d}y\leq 1\right\}. \tag{2}$$

其中 $B(x,r)$ 是以 x 为中心，以 r 为半径的球。对任意 $x\in\mathbb{R}^d$ 这个辅助函数满足 $0<\rho(x)<\infty$。

与薛定谔算子 L 相关的 Marcinkiewicz 积分 $\mu_{j,\Omega}^L$ 定义为

$$\mu_{j,\Omega}^L(f)(x,t)=\left(\int_0^\infty\left|\int_{|x-y|\leq h}\left|\Omega(x-y)\right|K_j^L(x,y)f(y,t)\,\mathrm{d}y\right|^2\frac{\mathrm{d}h}{h^3}\right)^{\frac{1}{2}} \tag{3}$$

其中 $K_j^L(x,y)=\widetilde{K_j^L}(x,y)|x-y|$ 并且 $\widetilde{K_j^L}(x,y)$ 是 $R_j=\frac{\partial}{\partial x_j}L^{-\frac{1}{2}}, j=1,\cdots,n$ 的核。当 $V=0$ 时，

$K_j^\Delta(x,y)=\widetilde{K_j^\Delta}(x,y)|x-y|=\dfrac{\frac{x_j-y_j}{|x-y|}}{|x-y|^{n-1}}$ 并且 $\widetilde{K_j^\Delta}(x,y)$ 是 $R_j=\frac{\partial}{\partial x_j}\Delta^{-\frac{1}{2}}, j=1,\cdots,n$ 的核。

下面我们定义交换子$\left[b,\mu_{j,\Omega}^L\right]$

$$\left[b,\mu_{j,\Omega}^L\right](f)(x,t)=\left(\int_0^\infty\left|\int_{|x-y|\leq h}\left|\Omega(x-y)\right|K_j^L(x,y)\left[b(x)-b(y)\right]f(x,t)\,\mathrm{d}y\right|^2\frac{\mathrm{d}h}{h^3}\right)^{\frac{1}{2}} \tag{4}$$

当 $f(y,t)\equiv f(y)$时，上面定义的与薛定谔算子 L 相关的 Marcinkiewicz 积分 $\mu_{j,\Omega}^L$ 是一般的与薛定谔算子 L 相关的 Marcinkiewicz 积分 $\mu_{j,\Omega}^L$。2020 年，Ferit GÜRBÜZ 在文献[1]中得到了 $\mu_{j,\Omega}^L$ 与 $\mu_{j,\Omega,b}^L$ 在加权 Lebesgue 空间上的有界性，进一步关于与薛定谔算子 L 相关的 Marcinkiewicz 积分 $\mu_{j,\Omega}^L$ 的结果可参见文献[2] [3] [4] [5]。

Morrey 空间作为 Lebesgue 空间的一个重要推广，在调和分析及其偏微分方程等领域有非常重要的应用[6] [7] [8] [9]。2020 年，Wang 在文献[10]中定义了一类加权 Morrey 空间 $L^{p,k}_{\rho,\theta}(\mu,\nu)$。2017 年，Ragusa-Scapellato 在文献[11]中定义了一类时空混合范 Morry 空间 $L^{q,\mu}\left(0,T,L^{p,\lambda}\left(\mathbb{R}^n\right)\right)$，它的优点之一是允许我们将时间和空间分开对待，这一特点在研究进化算子(例如 Kolmogorov 算子)和抛物型偏微分方程有重要应用。文献[11] [12] [13]得到了 Riesz 位势，Marcinkiewicz 积分和带 Gaussian 核算子在时空混合范 Morry 空间 $L^{q,\mu}\left(0,T,L^{p,\lambda}\left(\mathbb{R}^n\right)\right)$ 上的有界性。本文将主要研究与薛定谔算子 L 相关的 Marcinkiewicz 积分 $\mu^L_{j,\Omega}$ 在时空混合范 Morrey 空间上的加权有界性，并给出 Marcinkiewicz 积分交换子的加权有界性。为此，我们首先引入下面定义。

定义 1 [10] 设 $1\le p<\infty$，$0<\lambda<1$ 并且对于 $\mathbb{R}^d$ 中两个权函数 u 和 v，给定 $0<\theta<\infty$，加权 Morrey 空间 $L^{p,k}_{\rho,\theta}(\mu,\nu)$ 定义为 $\mathbb{R}^d$ 上所有 p-局部可积函数 f 的集合，使

$$\left(\frac{1}{\upsilon(B)^{\lambda}}\int_B\left|f(x)\right|^p\mu(x)\mathrm{d}x\right)^{\frac{1}{p}}\le C\cdot\left(1+\frac{r}{\rho(x_0)}\right)^{\theta} \tag{5}$$

对 $\mathbb{R}^d$ 上任意的球 $B=B(x_0,r)$，则

$$\|f\|_{L^{p,\lambda}_{\rho,\theta}(\mu,\nu)}:=\sup_B\left(1+\frac{r}{\rho(x_0)}\right)^{-\theta}\cdot\left(\frac{1}{\nu(B)^{\lambda}}\int_B\left|f(x)\right|^p\mu(x)\mathrm{d}x\right)^{\frac{1}{p}}<\infty \tag{6}$$

定义 2 设 $T>0$，$1\le p,q<\infty$，$0<\lambda,\mu<1$。u，v 为 $\mathbb{R}^d$ 上的非负可测函数，时空混合 Morrey 空间定义为

$$L^{q,\mu}\left(0,T,L^{p,\lambda}_{\rho,\theta}(u,v)\right):=\left\{f(x,t):\|f\|_{L^{q,\mu}\left(0,T,L^{p,\lambda}_{\rho,\theta}(u,v)\right)}<\infty\right\},$$

其中，

$$\|f\|_{L^{q,\mu}\left(0,T,L^{p,\lambda}_{\rho,\theta}(u,v)\right)}:=\left(\sup_{t_0\in[0,T],\rho>0}\frac{1}{\rho^{\mu}}\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\sup_B\left(1+\frac{r}{\rho(x_0)}\right)^{-\theta q}\left(\frac{1}{v\left(B_\rho(x)\right)^{\lambda}}\int_B\left|f(x,t)\right|^p u(x)\mathrm{d}x\right)^{\frac{q}{p}}\mathrm{d}t\right)^{\frac{1}{q}}. \tag{7}$$

这里，$B_\rho(x)=\left\{y\in\mathbb{R}^d:|y-x|<\rho\right\}$。当 $\mu=\nu$ 时，简记为 $L^{q,\mu}\left(0,T,L^{p,\lambda}_{\rho,\theta}(u)\right)$。

容易看出，时空混合范加权 Morrey 空间 $L^{q,\mu}\left(0,T,L^{p,\lambda}_{\rho,\theta}(u,v)\right)$ 是文献中所定义的加权 Morrey 空间 $L^{p,\lambda}_{\rho,\theta}(u,v)$ 的一种自然推广。叙述本文主要结果之前，先回顾 $A^{\rho,\infty}_p$ 和 $A^{\rho,\infty}_{p,q}$ 权的定义[14]。

设 $1<p<\infty$，$0<\theta<\infty$，称非负可测函数 $w\in A^{\rho,\infty}_p$，如果对任意的球 $B\subseteq\mathbb{R}^d$，存在与 B 无关的常数 $C>0$，使得

$$\left(\frac{1}{|B|}\int_B w(x)\mathrm{d}x\right)^{\frac{1}{p}}\left(\frac{1}{|B|}\int_B w(x)^{-\frac{p'}{p}}\mathrm{d}x\right)^{\frac{1}{p'}}\le C\cdot\left(1+\frac{r}{\rho(x_0)}\right)^{\theta}. \tag{8}$$

设 $1<p<q<\infty$，$0<\theta<\infty$，称非负可测函数 $w\in A^{\rho,\infty}_{p,q}$，如果对任意的球 $B\subseteq\mathbb{R}^d$，存在与 B 无关的常数 $C>0$，使得

$$\left(\frac{1}{|B|}\int_B w(x)^q\mathrm{d}x\right)^{\frac{1}{q}}\left(\frac{1}{|B|}\int_B w(x)^{-p'}\mathrm{d}x\right)^{\frac{1}{p'}}\le C\cdot\left(1+\frac{r}{\rho(x_0)}\right)^{\theta}, \tag{9}$$

其中，$p'=\dfrac{p}{p-1}$ 为 p 的对偶指标。

设 $1<r<\infty$，$0<\theta<\infty$，如果对任意的球 $B\subseteq\mathbb{R}^d$，存在与 B 无关的常数 $C>0$，使得下面的反向 Hölder 不等式成立

$$\left(\frac{1}{|B|}\int_B w(x)^r\,\mathrm{d}x\right)^{\frac{1}{r}}\le C\left(\frac{1}{|B|}\int_B w(x)\,\mathrm{d}x\right)\left(1+\frac{r}{\rho(x_0)}\right)^{\theta},\tag{10}$$

记 $w\in RH_r^{\rho,\theta}$。

在文献[15]中引入了一类新的 $\mathrm{BMO}_{\rho,\infty}(\mathbb{R}^n)$ 空间的定义，即 $\mathrm{BMO}_{\rho,\infty}(\mathbb{R}^n):=\bigcup_{\theta>0}\mathrm{BMO}_{\rho,\infty}(\mathbb{R}^n)$ 其中 $0<\theta<\infty$，被 $\mathrm{BMO}_{\rho,\infty}(\mathbb{R}^n)$ 空间定义的局部可积函数 b 满足

$$\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)}|b(x)-b_B(x_0,r)|\le C\cdot\left(1+\frac{r}{\rho(x_0)}\right)^{\theta}\tag{11}$$

对所有的球 $B(x_0,r)$，其中 $x_0\in\mathbb{R}^d$，$r>0$，并且 $b_B(x_0,r)$ 被定义为 b 在 $B(x_0,r)$，即 $b_{B(x_0,r)}:=\dfrac{1}{|B(x_0,r)|}\int_{B(x_0,r)}b(y)\,\mathrm{d}y$。$b\in\mathrm{BMO}_{\rho,\infty}(\mathbb{R}^n)$ 的范数用 $\|b\|_{\mathrm{BMO}_{\rho,\theta}}$，即

$$\|b\|_{\mathrm{BMO}_{\rho,\theta}}:=\sup_{B(x_0,r)}\left(1+\frac{r}{\rho(x_0)}\right)^{-\theta}\cdot\left(\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)}\left|b(x)-b_{B(x_0,r)}\right|\mathrm{d}x\right),\tag{12}$$

其中上确界取所有的球 $B(x_0,r)$。

本文的主要结果如下。

定理 1 设 $\Omega\in L^s(S^{d-1})$，$1<s\le\infty$。则对任意 $1<p,q<\infty$，$0<\lambda<1$，$0<\mu<1$，且 $w\in A_p^{\rho,\infty}$，存在与 f 无关的常数 $C>0$，使得

$$\left\|\mu_{j,\Omega}^L(f)\right\|_{L^{q,\mu}\left(0,T,L_{\rho,\infty}^{p,\lambda}(w)\right)}\le C\|f\|_{L^{q,\mu}\left(0,T,L_{\rho,\infty}^{p,\lambda}(w)\right)}.$$

定理 2 设 $\Omega\in L^s(S^{d-1})$，$1<s\le\infty$，且 $b\in\mathrm{BMO}(\mathbb{R}^d\times[0,T])$。则对任意 $1<p,q<\infty$，$0<\lambda<1$，$0<\mu<1$，且 $w\in A_p^{\rho,\infty}$，存在与 f 无关的常数 $C>0$，使得

$$\left\|\left[b,\mu_{j,\Omega}^L\right](f)\right\|_{L^{q,\mu}\left(0,T,L_{\rho,\infty}^{p,\lambda}(w)\right)}\le C\|f\|_{L^{q,\mu}\left(0,T,L_{\rho,\infty}^{p,\lambda}(w)\right)}.$$

定理 3 设 $\Omega\in L^s(S^{d-1})$，$1<s\le\infty$，且 $b\in\mathrm{BMO}(\mathbb{R}^d\times[0,T])$。则对任意 $1<p,q<\infty$，$0<\lambda<1$，$0<\mu<1$，且 $w\in A_p^{\rho,\infty}$，存在与 f 无关的常数 $C>0$，使得

$$\left(\sup_{t_0\in(0,T),\rho>0}\frac{1}{\rho^{\mu}}\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\left\|\mu_{j,\Omega}^L(f)\right\|_{BMO_{\rho,\infty}}^q\mathrm{d}t\right)^{\frac{1}{q}}\le C\|f\|_{L^{q,\mu}\left(0,T,L_{\rho,\infty}^{p,\lambda}(w)\right)}.$$

2. 定理的证明

本节介绍定理证明中需要用到的结论和引理。

引理 1 [1] 设零阶齐次函数 $\Omega\in L^s(S^{d-1})(1<s\le\infty)$，对任意 $\mu>0$，$\Omega(\mu x)=\Omega(x)$，$x\in R^n\setminus 0$ 并且 $V\in RH_n$。则对任意 $q'<p<\infty$ 且 $w\in A_{\frac{p}{q'}}$，下列不等式成立

$$\left\|\mu_{j,\Omega}^{L}(f)\right\|_{L^{p}(w)}\le C\|f\|_{L^{p}(w)}.$$

引理 2 [1] 设零阶齐次函数 $\Omega\in L^{s}\left(S^{d-1}\right)(1<s\le\infty)$，对任意 $\mu>0$，$\Omega(\mu x)=\Omega(x)$，$x\in R^{n}\setminus 0$，$V\in RH_{n}$ 并且 $b\in \mathrm{BMO}\left(R^{n}\right)$。则对任意的 $q'<p<\infty$ 且 $w\in A_{\frac{p}{q'}}$，下列不等式成立

$$\left\|\left[b,\mu_{j,\Omega}^{L}\right](f)\right\|_{L^{p}(w)}\le C\|f\|_{L^{p}(w)}.$$

引理 3 [16] 设 $V\in RH_{d}$，则，

(a) 对任意 N，存在一个常数 C，使得

$$\left|K_{j}^{L}(x,z)\right|\le C\left(1+\frac{|x-z|}{\rho(x)}\right)^{-N}\cdot\frac{1}{|x-z|^{d-1}};$$

(b) 对任意 N 和 $0<\delta<\min 1,1-\frac{d}{q_{0}}$，存在一个常数 C，使得

$$\left|K_{j}^{L}(x,z)-K_{j}^{L}(y,z)\right|\le C\left(1+\frac{|x-z|}{\rho(x)}\right)^{-N}\frac{|x-y|^{\delta}}{|x-z|^{d-1+\delta}},$$

其中 $|x-y|<\frac{2}{3}|x-z|$。

引理 4 [16] 设 $V\in RH_{s}$ 且 $s\ge\frac{d}{2}$，则存在两个正常数 $C_{0}\ge 1$ 和 $N_{0}>0$，使得对任意的 $x,y\in\mathbb{R}^{d}$，都有

$$\frac{1}{C_{0}}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N_{0}}\le\frac{\rho(y)}{\rho(x)}\le C_{0}\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{N_{0}}{N_{0}+1}}.$$

作为上式的一个直接结果，我们可以得到，对任意的整数 $k\ge 1$，有下面的估计

$$1+\frac{2^{k}r}{\rho(y)}\ge\frac{1}{C_{0}}\left(1+\frac{r}{\rho(x)}\right)^{-\frac{N_{0}}{N_{0}+1}}\left(1+\frac{2^{k}r}{\rho(x)}\right),$$

对任意 $y\in B(x,r)$，其中 $x\in\mathbb{R}^{n}$ 且 $r>0$，C_{0} 是(3)式中所定义的。

引理 5 [10] 设 $w\in A_{p}^{\rho,\theta}$，其中 $0<\theta<\infty$，$1\le p<\infty$。则存在两个数 $\delta,\eta>0$ 和常数 $C>0$，对球 B 的任意可测集 E，使得

$$\frac{\omega(E)}{\omega(B)}\le C\left(\frac{|E|}{|B|}\right)^{\delta}\left(1+\frac{r}{\rho(x_{0})}\right)^{\eta}.$$

引理 6 [10] 设 $b\in \mathrm{BMO}_{\rho,\infty}\left(\mathbb{R}^{d}\right)$ 且 $w\in A_{p}^{\rho,\infty}$，其中 $1\le p<\infty$，则存在常数 $C>0$ 和 $\mu>0$，使得对 $\mathbb{R}^{d}$ 上任意球 $B=B(x_{0},r)$，我们有

$$\left(\int_{B}\left|b(x)-b_{B}\right|^{p}w(x)\mathrm{d}x\right)^{\frac{1}{p}}\le C\cdot w(B)^{\frac{1}{p}}\left(1+\frac{r}{\rho(x_{0})}\right)^{\mu}.$$

引理 7 [10] 设 $b\in \mathrm{BMO}_{\rho,\theta}\left(\mathbb{R}^{d}\right)$，其中 $0<\theta<\infty$，则对任意整数 $k>0$，存在一个常数 $C>0$，使得对 $\mathbb{R}^{d}$ 上任意球 $B=B(x_{0},r)$，我们有

$$\left|b_{2^{k+1}B}-b_B\right| \le C\cdot(k+1)\left(1+\frac{2^{k+1}r}{\rho(x_0)}\right)^{\theta}.$$

定理 1 的证明 设 $B=B(x_0,r)$ 是 $\mathbb{R}^d$ 中的一个以 x_0 为中心，以 r 为半径的球，记 $f=f_1+f_2$，其中 $f_1=f\chi_{2B}$，χ_B 表示 B 的特征函数。则有

$$\begin{aligned}
&\frac{1}{w(B)^{\frac{\lambda}{p}}}\left(\int_B\left|\mu_{j,\Omega}^L f(x,t)\right|^p w(x)\mathrm{d}x\right)^{\frac{1}{p}}\\
&\le \frac{1}{w(B)^{\frac{\lambda}{p}}}\left(\int_B\left|\mu_{j,\Omega}^L f_1(x,t)\right|^p w(x)\mathrm{d}x\right)^{\frac{1}{p}}+\frac{1}{w(B)^{\frac{\lambda}{p}}}\left(\int_B\left|\mu_{j,\Omega}^L f_2(x,t)\right|^p w(x)\mathrm{d}x\right)^{\frac{1}{p}}\\
&:= I_1(t)+I_2(t).
\end{aligned}$$

由引理 1，可得

$$\begin{aligned}
I_1(t)&=C\frac{1}{w(B)^{\frac{\lambda}{p}}}\left(\int_B\left|\mu_{j,\Omega}^L f_1(x,t)\right|^p w(x)\mathrm{d}x\right)^{\frac{1}{p}}\\
&\le C\frac{w(2B)^{\frac{\lambda}{p}}}{w(B)^{\frac{\lambda}{p}}}\left(\frac{1}{w(2B)^{\lambda}}\int_{2B}\left|f(x,t)\right|^p w(x)\mathrm{d}x\right)^{\frac{1}{p}}\\
&\le C\|f\|_{L^{p,\lambda}_{\rho,\theta}(w)}\cdot\frac{w(2B)^{\frac{\lambda}{p}}}{w(B)^{\frac{\lambda}{p}}}\cdot\left(1+\frac{2r}{\rho(x_0)}\right)^{\theta}.
\end{aligned}$$

对任意权 $v\in A_p^{\rho,\theta'}$ 及 $\mathbb{R}^n$ 中任意球 B，则存在与 v 和 B 无关的常数 $C>0$，使得

$$v\left(2B(x_0,r)\right)\le C\cdot\left(1+\frac{2r}{\rho(x_0)}\right)^{p\theta'}v\left(B(x_0,r)\right),$$

事实上，对 $1<p<\infty$，应用 Hölder 不等式，可得

$$\begin{aligned}
&\frac{1}{2B}\int_{2B}\left|\hbar(x)\right|\mathrm{d}x=\frac{1}{2B}\int_{2B}\left|\hbar(x)\right|v(x)^{\frac{1}{p}}v(x)^{-\frac{1}{p}}\mathrm{d}x\\
&\le\frac{1}{2B}\left(\int_{2B}\left|\hbar(x)\right|^p v(x)\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{2B}v(x)^{-\frac{p'}{p}}\mathrm{d}x\right)^{\frac{1}{p'}}\\
&\le\frac{C}{v(2B)^{\frac{1}{p}}}\left(\int_{2B}\left|\hbar(x)\right|^p v(x)\mathrm{d}x\right)^{\frac{1}{p}}\left(1+\frac{2r}{\rho(x_0)}\right)^{\theta'}.
\end{aligned}$$

如果我们令 $\hbar(x)=\chi_B(x)$，则以上的表达式就为

$$\frac{|B|}{|2B|}\le C\cdot\frac{v(B)^{\frac{1}{t}}}{v(2B)^{\frac{1}{t}}}\left(1+\frac{2r}{\rho(x_0)}\right)^{p\theta'}.$$

即我们就得到

$$I_1(t)\le C\|f\|_{L^{p,\lambda}_{\rho,\theta}(w)}\cdot\left(1+\frac{2r}{\rho(x_0)}\right)^{p\theta'}\cdot\left(1+\frac{2r}{\rho(x_0)}\right)^{\theta}$$
$$\le C\|f\|_{L^{p,\lambda}_{\rho,\theta}(w)}\cdot\left(1+\frac{2r}{\rho(x_0)}\right)^{\vartheta'}.$$

其中 $\vartheta':=p\theta'+\theta$ 下面我们估计 $I_2(t)$，我们注意到，如果 $x\in B$， $y\in 2^{j+1}B\setminus 2^jB$， $j\ge 1$ 则应用引理 3(a)，可得

$$\mu^L_{j,\Omega}(f_2)(x,t)\le\left(\int_0^\infty\left|\int_{(2B)^c}|\Omega(x-y)|K^L_j(x,y)f(y,t)\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^3}\right)^{\frac12}$$
$$\le\sum_{j=1}^\infty\left(\int_{2^{j+1}B\setminus 2^jB}|\Omega(x-y)||K^L_j(x,y)||f(y,t)|\mathrm{d}y\right)\cdot\left(\int_{2^{j-1}r}^\infty\frac{\mathrm{d}h}{h^3}\right)^{\frac12}$$
$$\le\sum_{j=1}^\infty\frac{1}{|2^{j+1}B|^{\frac1n}}\int_{2^{j+1}B\setminus 2^jB}|\Omega(x-y)|\frac{1}{\left(1+\frac{|x-y|}{\rho(x)}\right)^N}\cdot\frac{1}{|x-y|^{n-1}}|f(y,t)|\mathrm{d}y.$$

我们注意到，如果 $x\in B$， $y\in(2B)^c$，则 $|y-x|\sim|y-x_0|$，应用 Hölder 不等式和引理 4。可得

$$\sum_{j=1}^\infty\frac{1}{|2^{j+1}B|^{\frac1n}}\int_{2^{j+1}B\setminus 2^jB}|\Omega(x-y)|\frac{1}{\left(1+\frac{|x-y|}{\rho(x)}\right)^N}\cdot\frac{1}{|x-y|^{n-1}}|f(y,t)|\mathrm{d}y$$
$$\le C\sum_{j=1}^\infty\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B\setminus 2^jB}|\Omega(x-y)|\left(1+\frac{2^jr}{\rho(x)}\right)^{-N}|f(y,t)|\mathrm{d}y$$
$$\le C\sum_{j=1}^\infty\frac{1}{|2^{j+1}B|}\left(1+\frac{r}{\rho(x_0)}\right)^{N\cdot\frac{N_0}{N_0+1}}\cdot\left(1+\frac{2^jr}{\rho(x_0)}\right)^{-N}\int_{2^{j+1}B\setminus 2^jB}|\Omega(x-y)||f(y,t)|\mathrm{d}y$$
$$\le C\sum_{j=1}^\infty\frac{1}{|2^{j+1}B|}\left(1+\frac{r}{\rho(x_0)}\right)^{N\cdot\frac{N_0}{N_0+1}}\left(1+\frac{2^jr}{\rho(x_0)}\right)^{-N}\left(\int_{2^{j+1}B\setminus 2^jB}|\Omega(x-y)|^s\mathrm{d}y\right)^{\frac1s}\cdot\left(\int_{2^{j+1}B\setminus 2^jB}|f(y,t)|^{s'}\mathrm{d}y\right)^{\frac{1}{s'}}.$$

下面利用球坐标变换，我们估计

$$\left(\int_{2^{j+1}B\setminus 2^jB}|\Omega(x-y)|^s\mathrm{d}y\right)^{\frac1s}=\left(\int_{2^{j-1}r\le|z|<2^{j+2}r}|\Omega(z)|^s\mathrm{d}y\right)^{\frac1s}$$
$$=\left(\int_{2^{j-1}r}^{2^{j+2}r}\int_{S^{n-1}}|\Omega(z)|^s\rho^{n-1}\mathrm{d}\sigma(z)\mathrm{d}\rho\right)^{\frac1s}$$
$$\le C\left(\int_{S^{n-1}}|\Omega(z)|^s\mathrm{d}\sigma(z)\right)^{\frac1s}\cdot\left(\int_{2^{j-1}r}^{2^{j+2}r}\rho^{n-1}\mathrm{d}\rho\right)^{\frac1s}$$
$$\le C\|\Omega\|_{L^s(S^{n-1})}|2^{j+1}B|^{\frac1s}.$$

下面应用 Hölder 不等式，且 $w\in A_p^{\rho,\theta'}$，则

$$\left(\int_{2^{j+1}B\backslash 2^{j}B}\left|f(y,t)\right|^{s'}\mathrm{d}y\right)^{\frac{1}{s'}}\le\left(\int_{2^{j+1}B}\left|f(y,t)\right|^{s'}w(y)^{\frac{1}{p_1}-\frac{1}{p_1}}\mathrm{d}y\right)^{\frac{1}{s'}}$$

$$\le\left(\int_{2^{j+1}B}\left|f(y,t)\right|^{p_1 s'}w(y)\mathrm{d}y\right)^{\frac{1}{p_1 s'}}\cdot\left(\int_{2^{j+1}B}w(y)^{-\frac{p_1'}{p_1}}\mathrm{d}y\right)^{\frac{1}{p_1' s'}}$$

$$\le C\left(\int_{2^{j+1}B}\left|f(y,t)\right|^{p}w(y)\mathrm{d}y\right)^{\frac{1}{p}}\cdot\left(\int_{2^{j+1}B}w(y)^{-\frac{1}{p_1-1}}\mathrm{d}y\right)^{\frac{p_1-1}{p}}$$

$$\le C\|f\|_{L^{p,\lambda}_{\rho,\theta}(w)}w\left(2^{j+1}B\right)^{\frac{\lambda-1}{p}}\left(1+\frac{2^{j+1}r}{\rho(x_0)}\right)^{\theta+\theta'}\left|2^{j+1}B\right|^{\frac{1}{p_1 s'}+\frac{1}{p_1' s'}}$$

$$\le C\|f\|_{L^{p,\lambda}_{\rho,\theta}(w)}w\left(2^{j+1}B\right)^{\frac{\lambda-1}{p}}\left(1+\frac{2^{j+1}r}{\rho(x_0)}\right)^{\theta+\theta'}\left|2^{j+1}B\right|^{\frac{1}{s'}}.$$

因此，由上面的估计我们可以得到

$$\left|\mu^{L}_{j,\Omega}f_2(x,t)\right|\le C\|f\|_{L^{p,\lambda}_{\rho,\theta}(w)}\sum_{j=1}^{\infty}w\left(2^{j+1}B\right)^{\frac{\lambda-1}{p}}\left(1+\frac{2^{j+1}r}{\rho(x_0)}\right)^{-N+\theta+\theta'}\cdot\left(1+\frac{r}{\rho(x_0)}\right)^{N\left(\frac{N_0}{N_0+1}\right)}.$$

因此，可以得到

$$I_2(t)\le C\|f\|_{L^{p,\lambda}_{\rho,\theta}(w)}\left(1+\frac{r}{\rho(x_0)}\right)^{N\left(\frac{N_0}{N_0+1}\right)}\cdot\sum_{j=1}^{\infty}\frac{w(B)^{\frac{1-\lambda}{p}}}{w\left(2^{j+1}B\right)^{\frac{1-\lambda}{p}}}\left(1+\frac{2^{j+1}r}{\rho(x_0)}\right)^{-N+\theta+\theta'}$$

$$\le C\|f\|_{L^{p,\lambda}_{\rho,\theta}(w)}\left(1+\frac{r}{\rho(x_0)}\right)^{N\left(\frac{N_0}{N_0+1}\right)}\cdot\sum_{j=1}^{\infty}\left(\frac{B}{2^{j+1}B}\right)^{\delta\left(\frac{1-\lambda}{p}\right)}\left(1+\frac{2^{j+1}r}{\rho(x_0)}\right)^{-N+\theta+\theta'+\eta\left(\frac{1-\lambda}{p}\right)}.$$

因此，取 N 足够大，使得 $N>\theta+\theta'+\eta\left(\frac{1-\lambda}{p}\right)$ 我们可得到

$$I_2(t)\le C\|f\|_{L^{p,\lambda}_{\rho,\theta}(w)}\left(1+\frac{r}{\rho(x_0)}\right)^{N\left(\frac{N_0}{N_0+1}\right)}\cdot\sum_{j=1}^{\infty}\left(\frac{B}{2^{j+1}B}\right)^{\delta\left(\frac{1-\lambda}{p}\right)}$$

$$\le C\|f\|_{L^{p,\lambda}_{\rho,\theta}(w)}\left(1+\frac{r}{\rho(x_0)}\right)^{N\left(\frac{N_0}{N_0+1}\right)}.$$

结合 $I_1(t)$ 和 $I_2(t)$ 的估计，我们令 $\vartheta=\max\left\{\vartheta',N\left(\frac{N_0}{N_0+1}\right)\right\}$，可以得到下面式子

$$\sup_{x\in R^n,\rho>0}\left(1+\frac{r}{\rho(x_0)}\right)^{-\vartheta}\left(\frac{1}{w\left(B_\rho(x)\right)^{\lambda}}\int_{B_\rho(x)}\left|\mu^{L}_{j,\Omega}f(x,t)\right|^{p}w(x)\mathrm{d}x\right)^{\frac{1}{p}}$$

$$\le C\sup_{x\in R^n,\rho>0}\left(1+\frac{r}{\rho(x_0)}\right)^{-\theta}\left(\frac{1}{w\left(B_\rho(x)\right)^{\lambda}}\int_{B_\rho(x)}\left|f(x,t)\right|^{p}w(x)\mathrm{d}x\right)^{\frac{1}{p}}.$$

在上式两边 q 次方，再在 $(0,T)\cap(t_0-\rho,t_0+\rho)$ 上积分，得到

$$\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\sup_{x\in R^n,\rho>0}\left(1+\frac{r}{\rho(x_0)}\right)^{-\vartheta q}\left(\frac{1}{w(B_\rho(x))^\lambda}\int_{B_\rho(x)}\left|\mu_{j,\Omega}^L f(x,t)\right|^p w(x)\mathrm{d}x\right)^{\frac{q}{p}}\mathrm{d}t$$

$$\le C\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\sup_{x\in R^n,\rho>0}\left(1+\frac{r}{\rho(x_0)}\right)^{-\theta q}\left(\frac{1}{w(B_\rho(x))^\lambda}\int_{B_\rho(x)}\left|f(x,t)\right|^p w(x)\mathrm{d}x\right)^{\frac{q}{p}}\mathrm{d}t.$$

不等号两边乘 $\frac{1}{\rho^\mu}$，再取上确界，两边再 $\frac{1}{q}$ 次方，就可得到

$$\left(\sup_{t_0\in(0,T),\rho>0}\frac{1}{\rho^\mu}\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\sup_{x\in R^n,\rho>0}\left(1+\frac{r}{\rho(x_0)}\right)^{-\vartheta q}\left(\frac{1}{w(B_\rho(x))^\lambda}\int_{B_\rho(x)}\left|\mu_{j,\Omega}^L f(x,t)\right|^p w(x)\mathrm{d}y\right)^{\frac{q}{p}}\mathrm{d}t\right)^{\frac{1}{q}}$$

$$\le\left(\sup_{t_0\in(0,T),\rho>0}\frac{1}{\rho^\mu}\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\sup_{x\in R^n,\rho>0}\left(1+\frac{r}{\rho(x_0)}\right)^{-\theta q}\left(\frac{1}{w(B_\rho(x))^\lambda}\int_{B_\rho(x)}\left|f(x,t)\right|^p w(x)\mathrm{d}x\right)^{\frac{q}{p}}\mathrm{d}t\right)^{\frac{1}{q}}.$$

定理 2 的证明 设 $B=B(x_0,r)$ 是 $\mathbb{R}^d$ 中的一个以 x_0 为中心，以 r 为半径的球，记 $f=f_1+f_2$，其中 $f_1=f\chi_{2B}$，χ_B 表示 B 的特征函数。则有

$$\frac{1}{w(B)^{\frac{\lambda}{p}}}\left(\int_B\left|\left[b,\mu_{j,\Omega}^L\right]f(x,t)\right|^p w(x)\mathrm{d}x\right)^{\frac{1}{p}}$$

$$\le\frac{1}{w(B)^{\frac{\lambda}{p}}}\left(\int_B\left|\left[b,\mu_{j,\Omega}^L\right]f_1(x,t)\right|^p w(x)\mathrm{d}x\right)^{\frac{1}{p}}+\frac{1}{w(B)^{\frac{\lambda}{p}}}\left(\int_B\left|\left[b,\mu_{j,\Omega}^L\right]f_2(x,t)\right|^p w(x)\mathrm{d}x\right)^{\frac{1}{p}}$$

$$:=J_1(t)+J_2(t).$$

由引理 2 可得

$$J_1(t)=\frac{1}{w(B)^{\frac{\lambda}{p}}}\left(\int_B\left|\left[b,\mu_{j,\Omega}^L\right]f_1(x,t)\right|^p w(x)\mathrm{d}x\right)^{\frac{1}{p}}$$

$$\le C\frac{1}{w(B)^{\frac{\lambda}{p}}}\left(\int_{2B}\left|f(x,t)\right|^p w(x)\mathrm{d}x\right)^{\frac{1}{p}}$$

$$\le C\frac{w(2B)^{\frac{\lambda}{p}}}{w(B)^{\frac{\lambda}{p}}}\left(\frac{1}{w(2B)^\lambda}\int_{2B}\left|f(x,t)\right|^p w(x)^p\,\mathrm{d}x\right)^{\frac{1}{p}}$$

$$\le C\|f\|_{L_{\rho,\theta}^{p,\lambda}(w)}\cdot\frac{w(2B)^{\frac{\lambda}{p}}}{w(B)^{\frac{\lambda}{p}}}\cdot\left(1+\frac{2r}{\rho(x_0)}\right)^{\theta}.$$

下面我们估计 $J_2(t)$，对任意的 $x\in B$，我们就要

$$\left|\left[b,\mu_{j,\Omega}^{L}\right]f_2(x,t)\right|\le\left|b(x)-b_B\right|\left(\int_0^\infty\left|\int_{(2B)^c\cap\{y:|x-y|\le h\}}\left|\Omega(x-y)\right|K_j^L(x,y)f(y,t)\mathrm{d}y\right|^2\frac{\mathrm{d}h}{h^3}\right)^{\frac{1}{2}}$$
$$+\left(\int_0^\infty\left|\int_{(2B)^c\cap\{y:|x-y|\le h\}}\left|\Omega(x-y)\right|\left|b(y)-b_B\right|K_j^L(x,y)f(y,t)\mathrm{d}y\right|^2\frac{\mathrm{d}h}{h^3}\right)^{\frac{1}{2}}$$
$$:=J_{21}+J_{22}.$$

由定理 1 的证明可知

$$\left|\mu_{j,\Omega}^{L}f_2(x,t)\right|\le C\|f\|_{L_{\rho,\theta}^{p,\lambda}(w)}\sum_{j=1}^{\infty}w\left(2^{j+1}B\right)^{\frac{\lambda-1}{p}}\left(1+\frac{2^{j+1}r}{\rho(x_0)}\right)^{-N+\theta+\theta'}\cdot\left(1+\frac{r}{\rho(x_0)}\right)^{N\left(\frac{N_0}{N_0+1}\right)}$$

则

$$J_{21}\le C\left|b(x)-b_B\right|\|f\|_{L_{\rho,\theta}^{p,\lambda}(w)}\sum_{j=1}^{\infty}w\left(2^{j+1}B\right)^{\frac{\lambda-1}{p}}\left(1+\frac{2^{j+1}r}{\rho(x_0)}\right)^{-N+\theta+\theta'}\cdot\left(1+\frac{r}{\rho(x_0)}\right)^{N\left(\frac{N_0}{N_0+1}\right)}$$

因此，应用 Hölder 不等式和引理 5，可得

$$\frac{1}{w(B)^{\frac{\lambda}{p}}}\left(\int_B J_{21}^p w(x)\mathrm{d}x\right)^{\frac{1}{p}}$$
$$\le C\|f\|_{L_{\rho,\theta}^{p,\lambda}(w)}\sum_{j=1}^{\infty}\frac{w(B)^{\frac{1-\lambda}{p}}}{w\left(2^{j+1}B\right)^{\frac{1-\lambda}{p}}}\left(1+\frac{2^{j+1}r}{\rho(x_0)}\right)^{-N+\theta+\theta'}\left(1+\frac{r}{\rho(x_0)}\right)^{N\left(\frac{N_0}{N_0+1}\right)}\left(\frac{1}{w(B)}\int_B\left|b(x)-b_B\right|^p w(x)\mathrm{d}x\right)^{\frac{1}{p}}$$
$$\le C\|f\|_{L_{\rho,\theta}^{p,\lambda}(w)}\left(1+\frac{r}{\rho(x_0)}\right)^{N\left(\frac{N_0}{N_0+1}\right)+\mu}\cdot\sum_{j=1}^{\infty}\frac{w(B)^{\frac{1-\lambda}{p}}}{w\left(2^{j+1}B\right)^{\frac{1-\lambda}{p}}}\left(1+\frac{2^{j+1}r}{\rho(x_0)}\right)^{-N+\theta+\theta'}$$
$$\le C\|f\|_{L_{\rho,\theta}^{p,\lambda}(w)}\left(1+\frac{r}{\rho(x_0)}\right)^{N\left(\frac{N_0}{N_0+1}\right)+\mu}\cdot\sum_{j=1}^{\infty}\left(\frac{|B|}{\left|2^{j+1}B\right|}\right)^{\delta\left(\frac{1-\lambda}{p}\right)}\left(1+\frac{2^{j+1}r}{\rho(x_0)}\right)^{-N+\theta+\theta'+\eta\left(\frac{1-\lambda}{p}\right)}.$$

下面我们估计 J_{22}，我们注意到，如果 $x\in B$， $y\in(2B)^c$，则 $|y-x|\sim|y-x_0|$。因此应用引理 3，可得

$$J_{22}\le C\sum_{j=1}^{\infty}\frac{1}{\left|2^{j+1}B\right|^{\frac{1}{n}}}\int_{2^{j+1}B\backslash 2^jB}\left|\Omega(x-y)\right|\left|b(y)-b_B\right|\left|K_j^L(x,y)f(y,t)\right|\mathrm{d}y$$
$$\le C\sum_{j=1}^{\infty}\frac{1}{\left|2^{j+1}B\right|^{\frac{1}{n}}}\int_{2^{j+1}B\backslash 2^jB}\left|\Omega(x-y)\right|\left|b(y)-b_B\right|\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}\frac{1}{|x-y|^{n-1}}\left|f(y,t)\right|\mathrm{d}y$$
$$\le C\sum_{j=1}^{\infty}\frac{1}{\left|2^{j+1}B\right|}\int_{2^{j+1}B\backslash 2^jB}\left|\Omega(x-y)\right|\left|b(y)-b_B\right|\left(1+\frac{|x_0-y|}{\rho(x)}\right)^{-N}\left|f(y,t)\right|\mathrm{d}y$$
$$\le C\sum_{j=1}^{\infty}\frac{1}{\left|2^{j+1}B\right|}\int_{2^{j+1}B\backslash 2^jB}\left|\Omega(x-y)\right|\left|b(y)-b_B\right|\left(1+\frac{|2^jr|}{\rho(x)}\right)^{-N}\left|f(y,t)\right|\mathrm{d}y$$
$$\le C\sum_{j=1}^{\infty}\frac{1}{\left|2^{j+1}B\right|}\int_{2^{j+1}B\backslash 2^jB}\left|\Omega(x-y)\right|\left|b(y)-b_B\right|\left(1+\frac{|r|}{\rho(x)}\right)^{-N\left(\frac{N_0}{N_0+1}\right)}\left(1+\frac{|2^{j+1}r|}{\rho(x)}\right)^{-N}\left|f(y,t)\right|\mathrm{d}y$$
$$\le C\sum_{j=1}^{\infty}\left(1+\frac{|r|}{\rho(x)}\right)^{-N\left(\frac{N_0}{N_0+1}\right)}\left(1+\frac{|2^{j+1}r|}{\rho(x)}\right)^{-N}\frac{1}{\left|2^{j+1}B\right|}\int_{2^{j+1}B\backslash 2^jB}\left|\Omega(x-y)\right|\left|b(y)-b_B\right|\left|f(y,t)\right|\mathrm{d}y.$$

下面对任意的整数 $j \ge 1$，我们有

$$\begin{aligned}
&\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B\backslash 2^j B}|\Omega(x-y)||b(y)-b_B||f(y,t)|\mathrm{d}y \\
&\le \frac{1}{|2^{j+1}B|}\int_{2^{j+1}B\backslash 2^j B}|\Omega(x-y)||b(y)-b_{2^{j+1}B}||f(y,t)|\mathrm{d}y \\
&\quad + \frac{1}{|2^{j+1}B|}|b_{2^{j+1}B}-b_B|\int_{2^{j+1}B\backslash 2^j B}|\Omega(x-y)||f(y,t)|\mathrm{d}y \\
&=: I_{21}+I_{22}.
\end{aligned}$$

下面应用 Hölder 不等式和引理 6 可得

$$\begin{aligned}
I_{21} &\le \frac{1}{|2^{j+1}B|}\left(\int_{2^{j+1}B}|b(y)-b_{2^{j+1}B}|^{s'}|f(y,t)|^{s'}\mathrm{d}y\right)^{\frac{1}{s'}}\left(\int_{2^{j+1}B}|\Omega(x-y)|^{s}\right)^{\frac{1}{s}} \\
&\le C\frac{1}{|2^{j+1}B|^{1-\frac{1}{s}}}\left(\int_{2^{j+1}B}|f(y,t)|^{p_1 s'}w(y)\mathrm{d}y\right)^{\frac{1}{p_1 s'}}\left(\int_{2^{j+1}B}|b(y)-b_{2^{j+1}B}|^{p_1' s'}w(y)^{-\frac{p_1'}{p_1}}\mathrm{d}y\right)^{\frac{1}{p_1' s'}} \\
&\le C\|f\|_{L^{p,\lambda}_{\rho,\theta}(w)}\frac{w(2^{j+1}B)^{\frac{\lambda}{p}}}{|2^{j+1}B|}\left(1+\frac{2^{j+1}r}{\rho(x_0)}\right)^{\theta} w^{-\frac{p'}{p}}(2^{j+1}B)^{\frac{1}{p'}}\left(1+\frac{2^{j+1}r}{\rho(x_0)}\right)^{\mu} \\
&\le C\|f\|_{L^{p,\lambda}_{\rho,\theta}(w)}\frac{w(2^{j+1}B)^{\frac{\lambda}{p}}}{|2^{j+1}B|}\left(1+\frac{2^{j+1}r}{\rho(x_0)}\right)^{\theta+\mu}|2^{j+1}B|^{\frac{1}{p'}}\frac{|2^{j+1}B|^{\frac{1}{p}}}{w(2^{j+1}B)^{\frac{1}{p}}}\left(1+\frac{2^{j+1}r}{\rho(x_0)}\right)^{\theta'} \\
&\le C\|f\|_{L^{p,\lambda}_{\rho,\theta}(w)}w(2^{j+1}B)^{\frac{\lambda-1}{p}}\left(1+\frac{2^{j+1}r}{\rho(x_0)}\right)^{\theta+\mu+\theta'}.
\end{aligned}$$

由定理 1 的证明可得

$$\begin{aligned}
&\int_{2^{j+1}B\backslash 2^j B}|\Omega(x-y)||f(y,t)|\mathrm{d}y \\
&\le C\|\Omega\|_{L^s(S^{n-1})}|2^{j+1}B|^{\frac{1}{s}}\|f\|_{L^{p,\lambda}_{\rho,\theta}}(w)w(2^{j+1}B)^{\frac{\lambda-1}{p}}\left(1+\frac{2^{j+1}r}{\rho(x_0)}\right)^{\theta+\theta'}|2^{j+1}B|^{\frac{1}{s'}}.
\end{aligned}$$

因此，应用引理 7，可得

$$\begin{aligned}
I_{22} &= \frac{1}{|2^{j+1}B|}|b_{2^{j+1}B}-b_B|\int_{2^{j+1}B\backslash 2^j B}|\Omega(x-y)||f(y,t)|\mathrm{d}y \\
&\le C\|f\|_{L^{p,\lambda}_{\rho,\theta}(w)}(j+1)w(2^{j+1}B)^{\frac{\lambda-1}{p}}\left(1+\frac{2^{j+1}r}{\rho(x_0)}\right)^{\theta''+\theta+\theta'}.
\end{aligned}$$

因此，结合 I_{21} 和 I_{22} 的估计可得

$$\begin{aligned}
&\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B\backslash 2^j B}|\Omega(x-y)||b(y)-b_B||f(y,t)|\mathrm{d}y \\
&\le C\|f\|_{L^{p,\lambda}_{\rho,\theta}(w)}(j+1)w(2^{j+1}B)^{\frac{\lambda-1}{p}}\left(1+\frac{2^{j+1}r}{\rho(x_0)}\right)^{\theta''+\theta+\theta'+\mu}
\end{aligned}$$

因此，应用引理 5 可得

$$\frac{1}{w(B)^{\frac{\lambda}{p}}}\left(\int_B J_{22}^p \mathrm{d}x\right)^{\frac{1}{p}}$$
$$\le C\|f\|_{L_{\rho,\theta}^{p,\lambda}(w)}\left(1+\frac{r}{\rho(x_0)}\right)^{N\left(\frac{N_0}{N_0+1}\right)}\sum_{j=1}^{\infty}(j+1)\frac{w(B)^{\frac{1-\lambda}{p}}}{w(2^{j+1}B)^{\frac{1-\lambda}{p}}}\left(1+\frac{2^{j+1}r}{\rho(x_0)}\right)^{\theta''+\theta+\theta'+\mu-N}$$
$$\le C\|f\|_{L_{\rho,\theta}^{p,\lambda}(w)}\left(1+\frac{r}{\rho(x_0)}\right)^{N\left(\frac{N_0}{N_0+1}\right)}\sum_{j=1}^{\infty}(j+1)\left(\frac{|B|}{|2^{j+1}B|}\right)^{\delta\left(\frac{1-\lambda}{p}\right)}\left(1+\frac{2^{j+1}r}{\rho(x_0)}\right)^{\theta''+\theta+\theta'+\mu-N+\eta\left(\frac{1-\lambda}{p}\right)}.$$

结合以上的估计，我们可得

$$J_2(t)=\frac{1}{w(B)^{\frac{\lambda}{p}}}\left(\int_B J_{21}^p \mathrm{d}x\right)^{\frac{1}{p}}+\frac{1}{w(B)^{\frac{\lambda}{p}}}\left(\int_B J_{22}^p \mathrm{d}x\right)^{\frac{1}{p}}$$
$$\le C\|f\|_{L_{\rho,\theta}^{p,\lambda}(w)}\left(1+\frac{r}{\rho(x_0)}\right)^{\mu+N\left(\frac{N_0}{N_0+1}\right)}\sum_{j=1}^{\infty}(j+1)\left(\frac{|B|}{|2^{j+1}B|}\right)^{\delta\left(\frac{1-\lambda}{p}\right)}\left(1+\frac{2^{j+1}r}{\rho(x_0)}\right)^{\theta''+\theta+\theta'+\mu-N+\eta\left(\frac{1-\lambda}{p}\right)}.$$

取 N 足够大，使得 $N>\theta''+\theta+\theta'+\mu+\eta\left(\frac{1-\lambda}{p}\right)$，则

$$J_2(t)\le C\|f\|_{L_{\rho,\theta}^{p,\lambda}(w)}\left(1+\frac{r}{\rho(x_0)}\right)^{\mu+N\left(\frac{N_0}{N_0+1}\right)}\sum_{j=1}^{\infty}(j+1)\left(\frac{|B|}{|2^{j+1}B|}\right)^{\delta\left(\frac{1-\lambda}{p}\right)}$$
$$\le C\|f\|_{L_{\rho,\theta}^{p,\lambda}(w)}\left(1+\frac{r}{\rho(x_0)}\right)^{\mu+N\left(\frac{N_0}{N_0+1}\right)}.$$

结合 $J_1(t)$ 和 $J_2(t)$ 的估计，我们令 $\vartheta=\max\left\{\vartheta',\mu+N\left(\frac{N_0}{N_0+1}\right)\right\}$，则可以得到

$$\sup_{x\in R^n,\rho>0}\left(1+\frac{r}{\rho(x_0)}\right)^{-\vartheta}\left(\frac{1}{w(B_\rho(x))^{\lambda}}\int_{B_\rho(x)}\left|\left[b,\mu_{j,\Omega}^L\right]f(x,t)\right|^p w(x)\mathrm{d}x\right)^{\frac{1}{p}}$$
$$\le C\sup_{x\in R^n,\rho>0}\left(1+\frac{r}{\rho(x_0)}\right)^{-\theta}\left(\frac{1}{w(B_\rho(x))^{\lambda}}\int_{B_\rho(x)}|f(x,t)|^p w(x)\mathrm{d}x\right)^{\frac{1}{p}}.$$

在上式两边 q 次方，再在 $(0,T)\cap(t_0-\rho,t_0+\rho)$ 上积分，得到

$$\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\sup_{x\in R^n,\rho>0}\left(1+\frac{r}{\rho(x_0)}\right)^{-\vartheta q}\left(\frac{1}{w(B_\rho(x))^{\lambda}}\int_{B_\rho(x)}\left|\left[b,\mu_{j,\Omega}^L\right]f(x,t)\right|^p w(x)\mathrm{d}x\right)^{\frac{q}{p}}\mathrm{d}t$$
$$\le C\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\sup_{x\in R^n,\rho>0}\left(1+\frac{r}{\rho(x_0)}\right)^{-\theta q}\left(\frac{1}{w(B_\rho(x))^{\lambda}}\int_{B_\rho(x)}|f(x,t)|^p w(x)\mathrm{d}x\right)^{\frac{q}{p}}\mathrm{d}t.$$

不等号两边乘 $\frac{1}{\rho^{\mu}}$，再取上确界，两边再 $\frac{1}{q}$ 次方，就可得到

$$\left(\sup_{t_0\in(0,T),\rho>0}\frac{1}{\rho^{\mu}}\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\sup_{x\in R^n,\rho>0}\left(1+\frac{r}{\rho(x_0)}\right)^{-\vartheta q}\left(\frac{1}{w\left(B_\rho(x)\right)^{\lambda}}\int_{B_\rho(x)}\left|\left[b,\mu_{j,\Omega}^L\right]f(x,t)\right|^p w(x)\mathrm{d}x\right)^{\frac{q}{p}}\mathrm{d}t\right)^{\frac{1}{q}}$$

$$\le\left(\sup_{t_0\in(0,T),\rho>0}\frac{1}{\rho^{\mu}}\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\sup_{x\in R^n,\rho>0}\left(1+\frac{r}{\rho(x_0)}\right)^{-\theta q}\left(\frac{1}{w\left(B_\rho(x)\right)^{\lambda}}\int_{B_\rho(x)}\left|f(x,t)\right|^p w(x)\mathrm{d}x\right)^{\frac{q}{p}}\mathrm{d}t\right)^{\frac{1}{q}}.$$

定理 3 的证明 设 $B=B(x_0,r)$ 是 $\mathbb{R}^d$ 中的一个以 x_0 为中心，以 r 为半径的球，记 $f=f_1+f_2$，其中 $f_1=f\chi_{4B}$，χ_B 表示 B 的特征函数。则有

$$\begin{aligned}&\frac{1}{|B|}\int_B\left|\mu_{j,\Omega}^L f(x,t)-\left(\mu_{j,\Omega}^L f\right)_B\right|\mathrm{d}x\\&\le\frac{1}{|B|}\int_B\left|\mu_{j,\Omega}^L f_1(x,t)-\left(\mu_{j,\Omega}^L f_1\right)_B\right|\mathrm{d}x+\frac{1}{|B|}\int_B\left|\mu_{j,\Omega}^L f_2(x,t)-\left(\mu_{j,\Omega}^L f_2\right)_B\right|\mathrm{d}x\\&=:K_1(t)+K_2(t).\end{aligned}$$

下面我们估计 $K_1(t)$，令 $\lambda=\frac{p}{q}$ 由 Hölder 不等式和引理 1 可得

$$\begin{aligned}K_1(t)&\le\frac{2}{|B|}\int_B\left|\mu_{j,\Omega}^L f_1(x,t)\right|\mathrm{d}x\\&\le\frac{2}{|B|}\left(\int_B\left|\mu_{j,\Omega}^L f_1(x,t)\right|^q w(x)^q\,\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_B w(x)^{-q'}\,\mathrm{d}x\right)^{\frac{1}{q'}}\\&\le\frac{C}{|B|}\left(\int_{4B}\left|f(x,t)\right|^p w(x)^p\,\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_B w(x)^{-q'}\,\mathrm{d}x\right)^{\frac{1}{q'}}\\&\le C\|f\|_{L_{\rho,\theta}^{p,\lambda}\left(w^p,w^q\right)}\cdot\frac{w^q(4B)^{\frac{\lambda}{p}}}{|B|}\left(\int_B w(x)^{-q'}\,\mathrm{d}x\right)^{\frac{1}{q'}}\left(1+\frac{4r}{\rho(x_0)}\right)^{\theta}\\&\le C\|f\|_{L_{\rho,\theta}^{p,\lambda}\left(w^p,w^q\right)}\cdot\frac{w^q(4B)^{\frac{1}{q}}}{w^q(B)^{\frac{1}{q}}}\left(1+\frac{r}{\rho(x_0)}\right)^{\theta'}\left(1+\frac{4r}{\rho(x_0)}\right)^{\theta}\\&\le C\|f\|_{L_{\rho,\theta}^{p,\lambda}\left(w^p,w^q\right)}\left(1+\frac{4r}{\rho(x_0)}\right)^{2\theta'}\left(1+\frac{r}{\rho(x_0)}\right)^{\theta'}\left(1+\frac{4r}{\rho(x_0)}\right)^{\theta}.\end{aligned}$$

因此，我们令 $\vartheta':=3\theta'+\theta$，则

$$K_1(t)\le C\|f\|_{L_{\rho,\theta}^{p,\lambda}\left(w^p,w^q\right)}\left(1+\frac{r}{\rho(x_0)}\right)^{\vartheta'}$$

下面我们估计 $K_2(t)$，对任意 $x\in B(x_0,r)$，应用引理 3(b)，则

$$\begin{aligned}
&\left|\mu_{j,\Omega}^{L} f_2(x,t)-\left(\mu_{j,\Omega}^{L} f_2\right)_B\right| \\
&=\left|\frac{1}{|B|}\int_B\left[\mu_{j,\Omega}^{L} f_2(x,t)-\mu_{j,\Omega}^{L} f_2(y,t)\right]\mathrm{d}y\right| \\
&=\left|\frac{1}{|B|}\int_B\left(\int_0^{\infty}\left|\int_{(4B)^c}|\Omega(x-z)|K_j^L(x,z)f(z,t)\mathrm{d}z\right|\frac{\mathrm{d}h}{h^3}\right)^{\frac{1}{2}}\mathrm{d}y\right| \\
&\quad-\left|\frac{1}{|B|}\int_B\left(\int_0^{\infty}\left|\int_{(4B)^c}|\Omega(y-z)|K_j^L(y,z)f(z,t)\mathrm{d}z\right|\frac{\mathrm{d}h}{h^3}\right)^{\frac{1}{2}}\mathrm{d}y\right| \\
&\le C\frac{1}{|B|}\sum_{j=2}^{\infty}\frac{1}{\left|2^{j+1}B\right|^{\frac{1}{n}}}\int_B\left(\int_{2^{j+1}B\setminus 2^jB}\left|\Omega(x-z)-\Omega(y-z)\right|\left|K_j^L(x,z)-K_j^L(y,z)\right|\cdot\left|f(z,t)\right|\mathrm{d}z\right)\mathrm{d}y \\
&\le C\frac{1}{|B|}\sum_{j=2}^{\infty}\frac{1}{\left|2^{j+1}B\right|^{\frac{1}{n}}}\int_B\left(\int_{2^{j+1}B\setminus 2^jB}\left|\Omega(x-z)-\Omega(y-z)\right|\left(1+\frac{|x-z|}{\rho(x)}\right)^{-N}\frac{|x-y|^{\delta}}{|x-z|^{n-1+\delta}}\cdot\left|f(z,t)\right|\mathrm{d}z\right)\mathrm{d}y.
\end{aligned}$$

我们注意到$|x-y|\le\frac{|x-z|}{2}$，若$x,y\in B$，$z\in(4B)^c$，则$|x-z|\sim|x_0-z|$，因此，可得

$$\begin{aligned}
&\left|\mu_{j,\Omega}^{L} f_2(x,t)-\left(\mu_{j,\Omega}^{L} f_2\right)_B\right| \\
&\le C\frac{1}{|B|}\sum_{j=2}^{\infty}\frac{1}{\left|2^{j+1}B\right|^{\frac{1}{n}}}\int_B\left(\int_{2^{j+1}B\setminus 2^jB}\left|\Omega(x-z)-\Omega(y-z)\right|\left(1+\frac{|x_0-z|}{\rho(x)}\right)^{-N}\frac{r^{\delta}}{|x_0-z|^{n-1+\delta}}\cdot\left|f(z,t)\right|\mathrm{d}z\right)\mathrm{d}y \\
&\le C\sum_{j=2}^{\infty}\frac{1}{2^{j\delta}}\cdot\frac{1}{\left|2^{j+1}B\right|}\int_{2^{j+1}B\setminus 2^jB}\left|\Omega(x-z)-\Omega(y-z)\right|\left(1+\frac{2^jr}{\rho(x)}\right)^{-N}\left|f(z,t)\right|\mathrm{d}z \\
&\le C\sum_{j=2}^{\infty}\frac{1}{2^{j\delta}}\cdot\left(1+\frac{r}{\rho(x)}\right)^{-N\left(\frac{N_0}{N_0+1}\right)}\left(1+\frac{2^{j+1}r}{\rho(x)}\right)^{-N}\frac{1}{\left|2^{j+1}B\right|}\int_{2^{j+1}B\setminus 2^jB}\left|\Omega(x-z)-\Omega(y-z)\right|\left|f(z,t)\right|\mathrm{d}z \\
&\le C\sum_{j=2}^{\infty}\frac{1}{2^{j\delta}}\cdot\left(1+\frac{r}{\rho(x)}\right)^{-N\left(\frac{N_0}{N_0+1}\right)}\left(1+\frac{2^{j+1}r}{\rho(x)}\right)^{-N}\omega_{\infty}\left(2\frac{|x-y|}{|x-z|}\right)\cdot\frac{1}{\left|2^{j+1}B\right|}\int_{2^{j+1}B\setminus 2^jB}\left|f(z,t)\right|\mathrm{d}z.
\end{aligned}$$

下面令$k=\frac{p}{q}$且应用Hölder不等式和引理5可得

$$\begin{aligned}
&\frac{1}{\left|2^{j+1}B\right|}\int_{2^{j+1}B\setminus 2^jB}\left|f(z,t)\right|\mathrm{d}z \\
&\le\frac{1}{\left|2^{j+1}B\right|}\left(\int_{2^{j+1}B}\left|f(z,t)\right|^p w(z)^p\mathrm{d}z\right)^{\frac{1}{p}}\left(\int_{2^{j+1}B}w(z)^{-p'}\mathrm{d}z\right)^{\frac{1}{p'}} \\
&\le C\|f\|_{L_{\rho,\theta}^{p,\lambda}\left(w^p,w^q\right)}\left(1+\frac{2^{j+1}r}{\rho(x_0)}\right)^{\theta}\cdot\frac{w^q\left(2^{j+1}B\right)^{\frac{\lambda}{p}}}{w^q\left(2^{j+1}B\right)^{\frac{1}{q}}}\cdot\left(1+\frac{2^{j+1}r}{\rho(x_0)}\right)^{\theta'} \\
&\le C\|f\|_{L_{\rho,\theta}^{p,\lambda}\left(w^p,w^q\right)}\left(1+\frac{2^{j+1}r}{\rho(x_0)}\right)^{\theta+\theta'}.
\end{aligned}$$

因此，取 N 足够大，使得 $N>\theta+\theta'$

$$
\begin{aligned}
&\left|\mu_{j,\Omega}^{L}f_2(x,t)-\left(\mu_{j,\Omega}^{L}f_2\right)_B\right|\\
&\le C\|f\|_{L_{\rho,\theta}^{p,\lambda}(w^p,w^q)}\sum_{j=2}^{\infty}\frac{1}{2^{j\delta}}\cdot\left(1+\frac{r}{\rho(x)}\right)^{-N\left(\frac{N_0}{N_0+1}\right)}\left(1+\frac{2^{j+1}r}{\rho(x)}\right)^{-N+\theta+\theta'}\cdot\omega_\infty\left(2\frac{|x-y|}{|x-z|}\right)\\
&\le C\|f\|_{L_{\rho,\theta}^{p,\lambda}(w^p,w^q)}\left(1+\frac{r}{\rho(x)}\right)^{-N\left(\frac{N_0}{N_0+1}\right)}\sum_{j=2}^{\infty}\frac{1}{2^{j\delta}}\cdot\omega_\infty\left(2\frac{|x-y|}{|x-z|}\right)\\
&\le C\|f\|_{L_{\rho,\theta}^{p,\lambda}(w^p,w^q)}\left(1+\frac{r}{\rho(x)}\right)^{-N\left(\frac{N_0}{N_0+1}\right)}\cdot\sum_{j=2}^{\infty}\omega_\infty\left(\frac{1}{2^{k-1}}\right).
\end{aligned}
$$

由于

$$
\sum_{j=2}^{\infty}\omega_\infty\left(\frac{1}{2^{k-1}}\right)=\frac{1}{\log 2}\sum_{j=2}^{\infty}\omega_\infty\left(\frac{1}{2^{k-1}}\right)\int_{\frac{1}{2^{k-1}}}^{\frac{1}{2^{k-2}}}\frac{\mathrm{d}\delta}{\delta}\le\frac{1}{\log 2}\sum_{j=2}^{\infty}\int_{\frac{1}{2^{k-1}}}^{\frac{1}{2^{k-2}}}\omega_\infty(\delta)\frac{\mathrm{d}\delta}{\delta}\le\frac{1}{\log 2}\int_0^1\omega_\infty(\delta)\frac{\mathrm{d}\delta}{\delta}<\infty.
$$

因此

$$
\left|\mu_{j,\Omega}^{L}f_2(x,t)-\left(\mu_{j,\Omega}^{L}f_2\right)_B\right|\le C\|f\|_{L_{\rho,\theta}^{p,\lambda}(w^p,w^q)}\left(1+\frac{r}{\rho(x)}\right)^{-N\left(\frac{N_0}{N_0+1}\right)}.
$$

我们令 $\vartheta=\max\left\{\vartheta',N\left(\frac{N_0}{N_0+1}\right)\right\}$，则可以得到下面不等式

$$
\begin{aligned}
&\sup_{x\in R^n,\rho>0}\left(1+\frac{r}{\rho(x_0)}\right)^{-\vartheta}\cdot\frac{1}{|B|}\int_{B_\rho(x)}\left|\mu_{j,\Omega}^{L}f(x,t)-\left(\mu_{j,\Omega}^{L}f\right)_B\right|\mathrm{d}x\\
&\le C\sup_{x\in R^n,\rho>0}\left(1+\frac{r}{\rho(x_0)}\right)^{-\theta}\left(\frac{1}{w^q\left(B_\rho(x)\right)^{\lambda}}\int_{B_\rho(x)}|f(x,t)|^p w(x)^p\,\mathrm{d}x\right)^{\frac{1}{p}}.
\end{aligned}
$$

在上式两边 q 次方，再在 $(0,T)\cap(t_0-\rho,t_0+\rho)$ 上积分，得到

$$
\begin{aligned}
&\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\sup_{x\in R^n,\rho>0}\left(1+\frac{r}{\rho(x_0)}\right)^{-\vartheta q}\cdot\left(\frac{1}{|B|}\int_{B_\rho(x)}\left|\mu_{j,\Omega}^{L}f(x,t)-\left(\mu_{j,\Omega}^{L}f\right)_B\right|\mathrm{d}x\right)^{q}\mathrm{d}t\\
&\le C\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\sup_{x\in R^n,\rho>0}\left(1+\frac{r}{\rho(x_0)}\right)^{-\theta q}\left(\frac{1}{w^q\left(B_\rho(x)\right)^{\lambda}}\int_{B_\rho(x)}|f(x,t)|^p w(x)^p\,\mathrm{d}x\right)^{\frac{q}{p}}\mathrm{d}t.
\end{aligned}
$$

不等号两边乘 $\frac{1}{\rho^{\mu}}$，再取上确界，两边再 $\frac{1}{q}$ 次方，就可得到

$$
\begin{aligned}
&\left(\sup_{t_0\in(0,T),\rho>0}\frac{1}{\rho^{\mu}}\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\sup_{x\in R^n,\rho>0}\left(1+\frac{r}{\rho(x_0)}\right)^{-\vartheta q}\cdot\left(\frac{1}{|B|}\int_{B_\rho(x)}\left|\mu_{j,\Omega}^{L}f(x,t)-\left(\mu_{j,\Omega}^{L}f\right)_B\right|\mathrm{d}x\right)^{q}\mathrm{d}t\right)^{\frac{1}{q}}\\
&\le\left(\sup_{t_0\in(0,T),\rho>0}\frac{1}{\rho^{\mu}}\int_{(0,T)\cap(t_0-\rho,t_0+\rho)}\sup_{x\in R^n,\rho>0}\left(1+\frac{r}{\rho(x_0)}\right)^{-\theta q}\left(\frac{1}{w^q\left(B_\rho(x)\right)^{\lambda}}\int_{B_\rho(x)}|f(x,t)|^p w(x)^p\,\mathrm{d}x\right)^{\frac{q}{p}}\mathrm{d}t\right)^{\frac{1}{q}}.
\end{aligned}
$$

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