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四阶非线性Schrödinger方程几乎质量守恒的低正则算法

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摘 要

本文研究了四阶非线性Schrödinger方程的具有几乎质量守恒的一种低正则算法．该算法不仅保持一阶收敛, 并且还具有几乎质量守恒性．我们通过严格的误差分析, 证明了当初值属于$H^{\gamma+3}(\mathbb{T}^d)$时, 对固定的$T>0$和$\gamma>\frac{d}{2}$, 存在常$C=C(\|u\|_{L^\infty(0,T;H^{\gamma+3})})>0$, 使得

$$\|u(t_n)-u^n\|_{H^{\gamma+3}}\le C\tau,$$

其中u^n是四阶非线性Schrödinger方程在$t_n=n\tau$时的数值解. 此外, 数值解的质量$M(u^n)$ 满足

$$|M(u^n)-M(u_0)|\le C\tau^3.$$

关键词

四阶非线性Schrödinger方程，低正则，一阶收敛，质量守恒

A Low-Regularity Integrator for the Fourth-Order Nonlinear Schrödinger Equation with Almost Mass Conservation

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Abstract

In this paper, we introduce a low-regularity integrator for the fourth-order nonlinear Schrödinger equation with almost mass conservation. The algorithm can not only achieve first-order convergence, but also obey almost mass conservation law. By rigorous error analysis, for rough initial data in $H^{\gamma+3}(\mathbb{T}^d)$ with $\gamma > \frac{d}{2}$, up to some fixed time of T, there exists $C = C(\|u\|_{L^\infty(0,T;H^{\gamma+3})}) > 0$, such that

$$\|u(t_n) - u^n\|_{H^{\gamma+3}} \leq C\tau,$$

where u^n denotes the numerical solution at $t_n = n\tau$. Moreover, the mass of the numerical solution $M(u^n)$ verifies

$$|M(u^n) - M(u_0)| \geq C\tau^3.$$

Keywords

Fourth-Order Nonlinear Schrödinger Equation, Low-Regularity Integrator, First-Order Convergent, Mass Conversation

1. 引言

本文考虑下述四阶非线性Schrödinger方程

$$\begin{cases} iu_t = (-\Delta)^2 u + \mu|u|^2 u, & (t, \boldsymbol{x}) \in (\mathbb{R}^+, \mathbb{T}^d), \\ u(0, x) = u_0(x), \end{cases} \tag{1}$$

其中$\mathbb{T}^d=(0,2\pi)^d$, $u(t,\boldsymbol{x})$是$(\mathbb{R}^+,\mathbb{T}^d)$ 上的复值函数, $u_0(\boldsymbol{x})\in H^\gamma(\mathbb{T}^d)(\gamma\geq 0)$为给定初值, $\mu=\pm 1$, 并且$d\geq 2$. 该方程在L^2中的解, 满足质量守恒定律：

$$M(u(t,\boldsymbol{x}))=\frac{1}{2\pi}\int_{\mathbb{T}^d}|u(t,\boldsymbol{x})|^2d\boldsymbol{x}=M(u_0).$$

在理论研究方面, Zhu等[1]证明了聚焦型四阶非线性Schrödinger方程质量临界时基态解的变分结构以及解的爆破性. Guo [2] 证明了聚焦型四阶非线性Schrödinger方程质量超临界时的全局适定性以及径向初值解的散射性. 其他型的Schrödinger方程也被广泛研究, 可参见[3–10] 等.

在数值研究方面, 对于具有光滑的初始条件的方程, 常见的数值方法有分割法[11, 12], 有限差分方法[13, 14], 有限元方法[15, 16], 谱方法[17, 18], 不连续Galerkin方法[19, 20], 指数积分器法[21, 22]等. 当方程的解不够光滑时, 则要开发低正则性的算法, 如Strange Splitting算法[23], 新型指数型算法[24], Fourier 积分算法[25]等. 对于四阶非线性Schrödinger方程, Ning [26]提出了一种数值格式, 使得在损失三阶导数的情况下达到一阶收敛, 即初始值属于$H^{\gamma+3}$ 时方程的解在H^γ中一阶收敛.

结合文献[26], 我们构造一种新的低正则算法, 可以得到四阶非线性Schrödinger方程(1) 的一阶收敛, 此外还满足几乎质量守恒. 在给出具体算法前, 我们首先定义一些要用到的函数. 令

$$\varphi(z)=\begin{cases}\dfrac{e^z-1}{z}, & z\neq 0,\\ 1, & z=0,\end{cases}$$

并且

$$\Psi(f)=e^{-i\tau(-\Delta)^2}f-i\mu\tau e^{-i\tau(-\Delta)^2}[f^2\cdot\varphi(2i\tau(-\Delta)^2)\bar{f}].$$

定义下列修正算子

$$I(f)=\Psi(f)-e^{-i\tau(-\Delta)^2}f,$$

和

$$J(f)=H(f)e^{-i\tau(-\Delta)^2}f,$$

其中

$$H(f)=-\|f\|_{L^2}^{-2}\left[\langle I(f),e^{-i\tau(-\Delta)^2}f\rangle+\frac{1}{2}\|I(f)\|_{L^2}^2\right].$$

现在, 我们给出四阶非线性Schrödinger方程(1)的新的低正则算法：

$$u^n=\Psi(u^{n-1})+J(u^{n-1}),\qquad n=0,1,2,...,\frac{T}{\tau}.\tag{2}$$

因此, 我们得到的主要定理为

定理 1. 设u^n是四阶非线性$Schrödinger$方程(1)满足(2)格式的数值解, 固定$T>0$. 对任意的$\gamma>\frac{d}{2}$, 假设$u_0(\boldsymbol{x})\in H^{\gamma+3}(\mathbb{T}^d)$, 存在常数$\tau_0$, $C>0$ 使得对任意的$0<\tau\le\tau_0$, 有

$$\|u(t_n)-u^n\|_{H^{\gamma+3}}\le C\tau, \qquad n=0,1,2,...,\frac{T}{\tau}.$$

此外,

$$|M(u^n)-M(u_0)|\le C\tau^3,$$

其中τ_0和$C>0$仅依赖于T和$\|u\|_{L^\infty((0,T);H^{\gamma+3})}$.

本文安排如下. 在第 2 节中, 我们给出了一些符号和一些有用的引理. 在第 3 节中, 我们给出一阶数值格式的主要构造过程. 在第 4 节中, 我们证明定理1.

2. 预备知识

在本节中, 我们介绍一些定义、性质和重要的估计. 为了方便记号, 我们使用$A\lesssim B$或者$B\gtrsim A$来表示如下含义：存在某绝对常数$C>0$, 使得$A\le CB$, 并使用$A\sim B$ 来表示$A\lesssim B\lesssim A$.

对于向量$\boldsymbol{\xi}:=(\xi_1,...,\xi_d)\in\mathbb{Z}^d$, $\boldsymbol{\xi}_1:=(\xi_{11},...,\xi_{1d})\in\mathbb{Z}^d$, $\boldsymbol{x}:=(x_1,...,x_d)\in\mathbb{T}^d$, 内积和模定义如下:

$$\boldsymbol{\xi}\cdot\boldsymbol{x}=\xi_1x_1+...+\xi_dx_d, \qquad |\boldsymbol{\xi}|^2=|\xi_1|^2+...+|\xi_d|^2.$$

定义$\langle\cdot,\cdot\rangle$为$L^2$内积

$$\langle f,g\rangle=\mathrm{Re}\int_{\mathbb{T}^d}f(\boldsymbol{x})\overline{g(\boldsymbol{x})}d\boldsymbol{x}.$$

在周期$\mathbb{T}^d$上, 我们定义函数f的Fourier变换为

$$\hat{f}_{\boldsymbol{\xi}}=\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}f(\boldsymbol{x})\,d\boldsymbol{x},$$

其Fourier逆变换为$f(\boldsymbol{x})=\sum_{\boldsymbol{\xi}\in\mathbb{Z}^d}e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\hat{f}_{\boldsymbol{\xi}}$.

对函数$f\in L^2(\mathbb{T}^d)$, 我们定义$f(\boldsymbol{x})$的Fourier 展开式为

$$f(\boldsymbol{x})=\sum_{\boldsymbol{\xi}\in\mathbb{Z}^d}\hat{f}_{\boldsymbol{\xi}}\,e^{i\boldsymbol{\xi}\cdot\boldsymbol{x}}.$$

Fourier变换常用性质

$$\|f\|_{L^2}^2=2\pi\sum_{\boldsymbol{\xi}\in\mathbb{Z}^d}|\hat{f}_{\boldsymbol{\xi}}|^2,$$

和

$$\widehat{fg}(\boldsymbol{\xi}) = \sum_{\boldsymbol{\xi},\boldsymbol{\xi}_1\in\mathbb{Z}^d} \hat{f}_{\boldsymbol{\xi}-\boldsymbol{\xi}_1}\hat{g}_{\boldsymbol{\xi}_1}.$$

定义Sobolev空间H^γ, $\gamma > 0$的范数为

$$\|f\|^2_{H^\gamma(\mathbb{T}^d)} = \sum_{\boldsymbol{\xi}\in\mathbb{Z}^d}(1+|\boldsymbol{\xi}|)^{2\gamma}|\hat{f}_{\boldsymbol{\xi}}|^2.$$

我们定义算子$(-\Delta)^{-1}$为

$$\widehat{(-\Delta)^{-1}f} = \begin{cases} |\boldsymbol{\xi}|^{-2}\hat{f}_{\boldsymbol{\xi}}, & \text{当} \quad \boldsymbol{\xi} \neq 0, \\ 0, & \text{当} \quad \boldsymbol{\xi} = 0. \end{cases}$$

现在, 我们给出要用到的一些重要的估计.

引理 2. （等距性质） 对$f \in H^\gamma$, $t \in \mathbb{R}$ 有

$$\|e^{-i(-\Delta)^2 t}f\|_{H^\gamma} = \|f\|_{H^\gamma}.$$

Proof. 根据H^γ范数的定义, 可知

$$\begin{aligned}\|e^{-i(-\Delta)^2 t}f\|_{H^\gamma} &= \sum_{\boldsymbol{\xi}\in\mathbb{Z}^d}(1+|\boldsymbol{\xi}|)^{2\gamma}|\widehat{e^{-i(-\Delta)^2 t}f}_{\boldsymbol{\xi}}|^2 \\ &= \sum_{\boldsymbol{\xi}\in\mathbb{Z}^d}(1+|\boldsymbol{\xi}|)^{2\gamma}|e^{-i|\boldsymbol{\xi}|^4 t}\hat{f}_{\boldsymbol{\xi}}|^2 \\ &= \sum_{\boldsymbol{\xi}\in\mathbb{Z}^d}(1+|\boldsymbol{\xi}|)^{2\gamma}|\hat{f}_{\boldsymbol{\xi}}|^2 \\ &= \|f\|_{H^\gamma}.\end{aligned}$$

□

引理 3. *(Kato-Ponce不等式)* 对任意的$\gamma > \frac{d}{2}$, $f, g \in H^\gamma$, 有下面不等式成立

$$\|fg\|_{H^\gamma} \leq \|f\|_{H^\gamma}\|g\|_{H^\gamma}.$$

Proof. 该引理的证明可参见文献[27]. □

3. 数值格式的构造

在本节中, 我们介绍数值格式的构造过程. 为了简化符号, 我们将省略所涉及的时空相关函数中的空间变量$\boldsymbol{x}$, 如$u(t) = u(t, \boldsymbol{x})$. 另外, 我们用$\tau > 0$表示时间步长, $t_n = n\tau$表示时间网格.

现在, 我们回顾文献[26]中对四阶非线性Schrödinger方程(1)算法的构造, 通过Duhamel公式, 有

$$u(t,\boldsymbol{x}) = e^{-it(-\Delta)^2}u_0 - i\mu\int_0^t e^{-i(t-s)(-\Delta)^2}(|u(s,x)|^2 u(s,\boldsymbol{x}))ds.$$

令扭转变量

$$v(t,\boldsymbol{x}) = e^{it(-\Delta)^2}u(t,\boldsymbol{x}), \tag{3}$$

则可得

$$v(t_n+\tau) = v(t_n) - i\mu\int_0^\tau e^{i(t_n+s)(-\Delta)^2}\left[|e^{-i(t_n+s)(-\Delta)^2}v(t_n+s)|^2 e^{-i(t_n+s)(-\Delta)^2}v(t_n+s)\right]ds.$$

通过Fourier展开, 可得

$$v(t_n+\tau) = \Phi^n(v(t_n)) + \mathcal{R}_1^n + \mathcal{R}_2^n, \tag{4}$$

其中

$$\Phi^n(v(t_n)) = v(t_n) - i\mu\tau e^{it_n(-\Delta)^2}\left[(e^{-it_n(-\Delta)^2}v(t_n))^2\cdot\cdot\varphi(2i\tau(-\Delta)^2)\overline{e^{-it_n(-\Delta)^2}v(t_n)}\right], \tag{5}$$

和

$$\begin{aligned}\mathcal{R}_1^n = -i\mu\int_0^\tau e^{i(t_n+s)(-\Delta)^2}\big[&|e^{-i(t_n+s)(-\Delta)^2}v(t_n+s)|^2 e^{-i(t_n+s)(-\Delta)^2}v(t_n+s)\\ &- |e^{-i(t_n+s)(-\Delta)^2}v(t_n)|^2 e^{-i(t_n+s)(-\Delta)^2}v(t_n)\big]ds,\end{aligned}$$

并且

$$\mathcal{R}_2^n = -i\mu\sum_{\boldsymbol{\xi}\in\mathbb{Z}^d}\sum_{\substack{\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3\in\mathbb{Z}^d\\ \boldsymbol{\xi}=\boldsymbol{\xi}_1+\boldsymbol{\xi}_2+\boldsymbol{\xi}_3}}\bar{\hat{v}}_1\hat{v}_2\hat{v}_3\, e^{i\boldsymbol{\xi}\cdot\boldsymbol{x}}e^{it_n\alpha}\int_0^\tau e^{is2|\boldsymbol{\xi}_1|^4}(e^{is\beta}-1)ds. \tag{6}$$

其中我们简记$\bar{\hat{v}}_1 = \bar{\hat{v}}_{\boldsymbol{\xi}_1}(t_n)$, $\hat{v}_2 = \hat{v}_{\boldsymbol{\xi}_2}(t_n)$ 和$\hat{v}_3 = \hat{v}_{\boldsymbol{\xi}_3}(t_n)$, 且$\alpha = |\boldsymbol{\xi}|^4 + |\boldsymbol{\xi}_1|^4 - |\boldsymbol{\xi}_2|^4 - |\boldsymbol{\xi}_3|^4$. 利用$\boldsymbol{\xi} = \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 + \boldsymbol{\xi}_2 + \boldsymbol{\xi}_3$, 则$\alpha$可以分解成

$$\alpha = 2|\boldsymbol{\xi}_1|^4 + \sum_{\substack{j,k=1\\ j\neq k}}^3 |\boldsymbol{\xi}_j|^2\boldsymbol{\xi}_j\bar{\boldsymbol{\xi}}_k + \sum_{\substack{j,k,h=1\\ j\neq k\neq h}}^3 |\boldsymbol{\xi}_j|^2\boldsymbol{\xi}_k\bar{\boldsymbol{\xi}}_h + \sum_{\substack{j,k,h=1\\ j\neq k\neq h}}^3 \boldsymbol{\xi}_j^2\bar{\boldsymbol{\xi}}_k\bar{\boldsymbol{\xi}}_h = 2|\boldsymbol{\xi}_1|^4 + \beta,$$

其中

$$\beta = \sum_{\substack{j,k=1\\ j\neq k}}^3 |\boldsymbol{\xi}_j|^2\boldsymbol{\xi}_j\bar{\boldsymbol{\xi}}_k + \sum_{\substack{j,k,h=1\\ j\neq k\neq h}}^3 |\boldsymbol{\xi}_j|^2\boldsymbol{\xi}_k\bar{\boldsymbol{\xi}}_h + \sum_{\substack{j,k,h=1\\ j\neq k\neq h}}^3 \boldsymbol{\xi}_j^2\bar{\boldsymbol{\xi}}_k\bar{\boldsymbol{\xi}}_h = O(\sum_{\substack{j,k=1\\ j\neq k}}^3 |\boldsymbol{\xi}_j|^3|\boldsymbol{\xi}_k|).$$

下面, 我们不加证明地罗列出一些估计, 相关证明可见[26]. 首先, 我们给出局部误差项估计.

引理 4. *(局部误差) 存在常数C仅依赖于$\|v\|_{L^\infty((0,T);H^{\gamma+3})}$, 使得*

$$\|v(t_{n+1}) - \Phi^n(v(t_n))\|_{H^\gamma} \leq C\tau^2,$$

其次, 我们介绍稳定性估计.

引理 5. *(稳定性) 设$f, g \in H^\gamma$, 对于$\gamma > \frac{d}{2}$, 则有*

$$\|\Phi^n(f) - \Phi^n(g)\|_{H^\gamma} \leq (1 + C\tau)\|f - g\|_{H^\gamma} + C\tau\|f - g\|_{H^\gamma}^3,$$

在上述一阶收敛的基础上, 我们希望构造一种可以保持解的物理性态的算法, 解在满足一阶收敛的前提下, 还能够保持几乎质量守恒性, 因此我们要对算法进行修正. 定义

$$I^n(v) = \Phi^n(v) - v, \tag{7}$$

这里的Φ^n由(5)给出. 再定义

$$J^n(v) = H^n(v)v, \tag{8}$$

其中

$$H^n(v) = -\|v\|_{L^2}^{-2}\left[\langle I^n(v), v\rangle + \frac{1}{2}\|I^n(v)\|_{L^2}^2\right].$$

有了上面的修正, 我们构造出满足几乎质量守恒的数值格式

$$v^{n+1} = v^n + I^n(v^n) + J^n(v^n). \tag{9}$$

将扭转变量(3)反过来代入到(9), 即可得到四阶非线性Schrödinger方程(1) 的一阶收敛数值算法(2).

4. 定理1的证明

本节我们通过局部误差分析和稳定性分析对一阶收敛结果给出一个严格的证明. 由引理2可知, 扭转变量(3)在Sobolev空间是等距的, 则有

$$\|u(t_n) - u^n\|_{H^\gamma} = \|e^{-it_n(-\Delta)^2}v(t_n) - e^{-it_n(-\Delta)^2}v^n\|_{H^\gamma} = \|v(t_n) - v^n\|_{H^\gamma}.$$

因此, 我们只需要证明对$v(t_n)$和v^n的一阶收敛定理和几乎质量守恒性.

首先, 我们给出$I^n(v)$的收敛阶.

引理 6. *设$v \in H^\gamma$, 当$\gamma > \frac{d}{2}$时, 存在常数C仅依赖于$\|v\|_{L^\infty((0,T);H^{\gamma+3})}$, 下面不等式成立*

$$\|I^n(v)\|_{L^2} \leq C\tau.$$

Proof. 回顾$I^n(v)$的定义(7)可知,

$$I^n(v) = -i\mu\tau e^{it_n(-\Delta)^2}\left[(e^{-it_n(-\Delta)^2}v)^2 \cdot \varphi(2i\tau(-\Delta)^2)\overline{e^{-it_n(-\Delta)^2}v}\right].$$

因此,

$$\begin{aligned}\|I^n(v)\|_{L^2} \lesssim & \tau\|(e^{-it_n(-\Delta)^2}v)^2\|_{L^\infty} \cdot \|\varphi(2i\tau(-\Delta)^2)\overline{e^{-it_n(-\Delta)^2}v}\|_{L^2} \\ \lesssim & \tau\|v\|_{H^\gamma}^3.\end{aligned}$$

所以, 引理6得证. $\square$

其次, 在给出$\langle I^n(v), v\rangle$的收敛阶之前, 我们先要给出一个要用到的不等式.

引理 7. *存在常数C仅依赖于$\|v\|_{L^\infty((0,T);H^{\gamma+3})}$, 使得*

$$\|\mathcal{R}_2^n\|_{H^{-\gamma}} \leq C\tau^2.$$

Proof. 根据$\mathcal{R}_2^n$的定义(6), 可得

$$\begin{aligned}|\mathcal{R}_2^n| \lesssim & \tau^2 \sum_{\boldsymbol{\xi}\in\mathbb{Z}^d} \sum_{\substack{\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3\in\mathbb{Z}^d \\ \boldsymbol{\xi}=\boldsymbol{\xi}_1+\boldsymbol{\xi}_2+\boldsymbol{\xi}_3}} e^{i\boldsymbol{\xi}\cdot\boldsymbol{x}} e^{it_n\alpha} \Big(\sum_{\substack{j,k=1 \\ j\neq k}}^3 |\boldsymbol{\xi}_j|^3|\boldsymbol{\xi}_k| \cdot \bar{\hat{v}}_j\hat{v}_k\hat{v}_h\Big) \\ \lesssim & \tau^2 \sum_{\boldsymbol{\xi}\in\mathbb{Z}^d} \sum_{\substack{\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3\in\mathbb{Z}^d \\ \boldsymbol{\xi}=\boldsymbol{\xi}_1+\boldsymbol{\xi}_2+\boldsymbol{\xi}_3}} e^{i\boldsymbol{\xi}\cdot\boldsymbol{x}} e^{it_n\alpha} (|\boldsymbol{\xi}_j|^3\bar{\hat{v}}_j \cdot |\boldsymbol{\xi}_k|\hat{v}_k \cdot \hat{v}_h).\end{aligned}$$

因此, 由Sobolev不等式可得

$$\begin{aligned}\|\mathcal{R}_2^n\|_{H^{-\gamma}} \lesssim & \tau^2\||\nabla|^3 v \cdot |\nabla|v \cdot v\|_{H^{-\gamma}} \\ \lesssim & \tau^2\||\nabla|^3 v \cdot |\nabla|v \cdot v\|_{L^1} \\ \lesssim & \tau^2\||\nabla|^3 v\|_{L^2}\,\||\nabla|v\|_{L^2}\,\||v\|_{L^\infty} \\ \lesssim & \tau^2\|v\|_{H^{\gamma+3}}^3.\end{aligned}$$

所以, 该引理得证. $\square$

引理 8. *设$v \in H^{\gamma+3}$, 当$\gamma > \frac{d}{2}$时, 存在常数C仅依赖于$\|v\|_{L^\infty((0,T);H^{\gamma+3})}$, 下面不等式成立*

$$|\langle I^n(v), v\rangle| \leq C\tau^2.$$

Proof. 回顾$I^n(v)$的定义(7)得

$$
\begin{aligned}
I^n(v) =& -i\mu \sum_{\boldsymbol{\xi}\in\mathbb{Z}^d} \sum_{\substack{\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3\in\mathbb{Z}^d\\ \boldsymbol{\xi}=\boldsymbol{\xi}_1+\boldsymbol{\xi}_2+\boldsymbol{\xi}_3}} \bar{\hat{v}}_1\hat{v}_2\hat{v}_3\, e^{i\boldsymbol{\xi}\cdot\boldsymbol{x}} e^{it_n\alpha}\int_0^\tau e^{is2|\boldsymbol{\xi}_1|^4}ds\\
=& -i\mu \sum_{\boldsymbol{\xi}\in\mathbb{Z}^d} \sum_{\substack{\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3\in\mathbb{Z}^d\\ \boldsymbol{\xi}=\boldsymbol{\xi}_1+\boldsymbol{\xi}_2+\boldsymbol{\xi}_3}} \bar{\hat{v}}_1\hat{v}_2\hat{v}_3\, e^{i\boldsymbol{\xi}\cdot\boldsymbol{x}} \int_0^\tau e^{i(t_n+s)\alpha}ds - \mathcal{R}_2^n\\
=& -i\mu\int_0^\tau e^{i(t_n+s)(-\Delta)^2}\left[|e^{-i(t_n+s)(-\Delta)^2}v|^2 e^{-i(t_n+s)(-\Delta)^2}v\right]ds - \mathcal{R}_2^n.
\end{aligned}
$$

因此, 我们有

$$
\langle I^n(v), v\rangle = \langle -i\mu\int_0^\tau e^{i(t_n+s)(-\Delta)^2}\left[|e^{-i(t_n+s)(-\Delta)^2}v|^2 e^{-i(t_n+s)(-\Delta)^2}v\right]ds, v\rangle + \langle -\mathcal{R}_2^n, v\rangle.
$$

由内积的定义, 可得

$$
\langle -i\mu\int_0^\tau e^{i(t_n+s)(-\Delta)^2}\left[|e^{-i(t_n+s)(-\Delta)^2}v|^2 e^{-i(t_n+s)(-\Delta)^2}v\right]ds, v\rangle = 0.
$$

结合引理7, 可得,

$$
|\langle I^n(v), v\rangle| = |\langle -\mathcal{R}_2^n, v\rangle| \lesssim \|\mathcal{R}_2^n\|_{H^{-\gamma}}\|v\|_{H^\gamma} \lesssim \tau^2\|v\|^4_{H^{\gamma+3}}.
$$

因此, 证明了引理的结论. □

接下来, 结合局部误差估计和稳定性结果, 我们给出定理1的证明.

Proof. 首先, 由数值解的构造(9), 我们有

$$
\begin{aligned}
v^{n+1} - v(t_{n+1}) =& v^n + I^n(v^n) + J^n(v^n) - v(t_{n+1})\\
=& \Phi^n(v^n) - \Phi^n(v(t_n)) + \Phi^n(v(t_n)) - v(t_{n+1}) + J^n(v^n).
\end{aligned}
$$

根据引理4和引理5, 可得

$$
\begin{aligned}
\|v(t_{n+1}) - v^{n+1}\|_{H^\gamma} \le& \|\Phi^n(v^n) - \Phi^n(v(t_n))\|_{H^\gamma} + \|\Phi^n(v(t_n)) - v(t_{n+1})\|_{H^\gamma} + \|J^n(v^n)\|_{H^\gamma}\\
\le& (1+C\tau)\|v(t_n) - v^n\|_{H^\gamma} + C\tau\|v(t_n) - v^n\|^3_{H^\gamma} + C\tau^2 + \|J^n(v^n)\|_{H^\gamma}.
\end{aligned}
$$

根据J^n的定义(8), 有

$$
\|J^n(v^n)\|_{H^\gamma} \le |H^n(v^n)|\|v^n\|_{H^\gamma} \le |H^n(v^n)|(\|v^n - v(t_n)\|_{H^\gamma} + \|v(t_n)\|_{H^\gamma}).
$$

由引理6和引理8, 可得

$$|H^n(v^n)| \leq \|v^n\|_{L^2}^{-2}(\tau^2\|v^n\|_{H^{\gamma+3}}^4 + \frac{1}{2}\tau^2\|v^n\|_{H^\gamma}^6) \leq C\tau^2(1+\|v^n - v(t_n)\|_{H^\gamma}^4). \tag{10}$$

因此, 我们有

$$\|J^n(v^n)\|_{H^\gamma} \leq C\tau^2(1+\|v^n - v(t_n)\|_{H^\gamma}^5). \tag{11}$$

结合(11), 可得

$$\|v(t_{n+1}) - v^{n+1}\|_{H^\gamma} \leq C\tau^2 + (1+C\tau)\|v^n - v(t_n)\|_{H^\gamma} + C\tau\|v^n - v(t_n)\|_{H^\gamma}^5.$$

利用迭代法和Gronwall's不等式, 可得

$$\|v(t_{n+1}) - v^{n+1}\|_{H^\gamma} \leq C\tau^2\sum_{j=0}^{n}(1+C\tau)^j \leq C\tau, \quad n = 0,1,2,...,\frac{T}{\tau}-1,$$

其中C仅依赖于$\|v\|_{L^\infty((0,T);H^{\gamma+3})}$. 上式成立, 就意味着一阶收敛且先验估计满足

$$\|v^n\|_{H^\gamma} \leq C, \qquad n = 0,1,2,...,\frac{T}{\tau}-1.$$

因为$v^n = v^n + I^n(v^n) + J^n(v^n)$, 有

$$\begin{aligned}\langle v^{n+1}, v^{n+1}\rangle =& \langle v^n + I^n(v^n) + J^n(v^n), v^n + I^n(v^n) + J^n(v^n)\rangle \\ =& \|v^n\|_{L^2}^2 + 2\langle I^n(v^n), J^n(v^n)\rangle + \|J^n(v^n)\|_{L^2}^2 \\ &+ 2\langle I^n(v^n), v^n\rangle + \langle J^n(v^n), v^n\rangle + \|I^n(v^n)\|_{L^2}^2.\end{aligned}$$

根据$I^n(v^n)$和$J^n(v^n)$的定义可得

$$2\langle I^n(v^n), v^n\rangle + \langle J^n(v^n), v^n\rangle + \|I^n(v^n)\|_{L^2}^2 = 0,$$

则

$$\langle v^{n+1}, v^{n+1}\rangle = \|v^n\|_{L^2}^2 + 2\langle I^n(v^n), J^n(v^n)\rangle + \|J^n(v^n)\|_{L^2}^2.$$

一方面, 利用引理8和(10), 我们可得

$$\langle I^n(v^n), J^n(v^n)\rangle \leq |H^n(v^n)|\,|\langle I^n(v^n), v^n\rangle| \lesssim C\tau^4.$$

另一方面, 结合(10)和(11), 有

$$\|J^n(v^n)\|_{L^2}^2 \leq |H^n(v^n)|^2\,\|v^n\|_{L^2}^2 \leq C\tau^4.$$

综上可得

$$\left|\|v^{n+1}\|_{L^2}^2 - \|v^n\|_{L^2}^2\right| \le C\tau^4,$$

即

$$|M(v^{n+1}) - M(v^n)| \le C\tau^4. \tag{12}$$

对(12)进行迭代可得

$$|M(v^n) - M(u_0)| \le C\tau^3,$$

C仅依赖于$\|v\|_{L^\infty((0,T);H^{\gamma+3})}$.

因此, 我们得到了定理1的证明. □

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