Pure Mathematics 理论数学, 2023, 13(6), 1744-1752
Published Online June 2023 in Hans. https://www.hanspub.org/journal/pm
https://doi.org/10.12677/pm.2023.136178

低维Busemann-Petty测度问题的同伦形式

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收稿日期：2023年5月21日；录用日期：2023年6月22日；发布日期：2023年6月30日

摘 要

低维Busemann-Petty测度问题是指：对其有适当密度函数的Borel测度μ及n-维欧氏空间的两个中心对称凸体K 和L 来说，若它们被任意的n-i-维子空间所截，所得的n-i–维截面体的测度满足$\mu(K\cap\xi^{\perp})\leq\mu(L\cap\xi^{\perp})$，其中$\xi\in G(n,i)(1\leq i\leq n)$，那么其$n$-维凸体$K$和$L$的测度$\mu(K)\leq\mu(L)$是否成立？在已有文献中，获得了与Rubin及Zhang关于低维Busemann-Petty问题的n-维体积形式相一致的结论。本文证明了这个结论的同伦形式，即在上述条件下，对任意的$1\leq i\leq n$，有$\mu(K)\leq\ n^{i/2}\mu(L)$成立。

关键词

同伦形式，截面体，测度，Radon变换

An Isomorphic Version of the Lower Dimensional Busemann-Petty Problems for Measures

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文章引用: 朱先阳. 低维Busemann-Petty测度问题的同伦形式[J]. 理论数学, 2023, 13(6): 1744-1752.
DOI: 10.12677/pm.2023.136178

Received: May 21^{st}, 2023; accepted: Jun. 22^{nd}, 2023; published: Jun. 30^{th}, 2023

Abstract

The lower dimensional Busemann-Petty (LDBP) problem for arbitrary measures asks: For a given Borel measure μ with appropriate density and two origin-symmetric convex bodies K and L, does the assumption that $\mu(K\cap\xi^{\perp})\leq\mu(L\cap\xi^{\perp})$ holds for any $\xi\in G(n,i)(1\leq i<n)$ imply that $\mu(K)\leq\mu(L)$? It was proved that the problem has the same answer as Rubin and Zhang's solutions to the LDBP problem for volumes. In this paper we show an isomorphic version of this result. Namely, if the above conditions hold, then $\mu(K)\leq\ n^{i/2}\mu(L)$ for any $1\leq i\leq n$.

Keywords

Isomorphic Version, Intersection Body, Measures, Radon Transform

1. 引言

在n-维欧氏空间R^n中, 设$G(n,i)(1\leq i<n)$是i-维Grassmann流形, $L_{1,\mathrm{loc}}(R^n)$ 表示局部可积函数集, μ表示Borel测度, 其密度函数$f\in L_{1,\mathrm{loc}}(R^n)$是非负的偶函数. 对每一个$i$- 维子空间$\xi$而言, $\xi^{\perp}$就是ξ的中心正交补空间, 那么在$\xi^{\perp}$ 上，非负的偶函数$f\in L_{1,\mathrm{loc}}(R^n)$也是局部可积的, 则存在一个绝对连续的测度$\mu$, 对任意的紧致Borel集$K\subset\xi^{\perp}$(或者, $K\subset R^n$), 有

$$\mu(K)=\int_K f(x)dx, \tag{1}$$

其中dx表示空间$\xi^{\perp}$上的$(n-i)$-维Lebesgue测度(或者, 空间R^n上n-维Lebesgue测度).

低维Busemann-Petty测度问题是指: 当$n\geq 2$时, K和L是R^n上的两个中心对称凸体(即有非空内部的紧致凸子集), 对R^n中任意n-i–维线性空间$\xi^{\perp}$, 假设

$$\mu(K\cap\xi^{\perp})\leq\mu(L\cap\xi^{\perp})$$

成立，那么下式

$$\mu(K) \le \mu(L)?$$

是否成立？

Zvavitch [1]研究了上述问题中$i=1$的情况, 相应内容也可看文献 [2–4], 在f为一严格正的密度函数时，得出了$n \le 4$时结论成立，而$n \ge 5$时结论不成立. 当$n \ge 5$及$i=n-2$或$n-3$时, 从文献([5]定理5.1)中需要对球面上Radon变换的一个反转公式进行较为错综复杂的讨论，才得出结果. 文献 [6]考虑了两个不同加权的一般情况，从引理4.1获得了上述问题的结论是与Rubin及Zhang研究的关于n-维体积形式的低维Busemann-Petty问题相一致的. 对任意$x \in R^n$, 其$f(x) \equiv 1$时, 那么上述问题就由Zhang [7](或者 [5,8–10])提出, 并得出在$3 < n-i < n$时, 结论不成立, 在$i=n-2, n-3$且$n \ge 5$时, 仍然未解. 当$f(x) \equiv 1$且$i=1$ 时, 其问题就是文献 [11]中著名的Busemann-Petty 问题, 已经有彻底的答案, $n \le 4$ 结论肯定，$n \ge 5$结论否定. 了解其解答过程可参考文献 [1,12–22], 了解它的各种推广可阅读文献 [3,4,6–8,23–27].

从这些论文结果可知, 低维Busemann-Petty测度问题在绝大部分维数下是不成立的, 因此提出它的同伦形式如下.

低维Busemann-Petty测度问题的同伦形式是指: μ是一任意测度, 其密度函数f是偶的非负连续的, K 和L是R^n 上的两个中心对称凸体, 若

$$\mu(K \cap \xi^{\perp}) \le \mu(L \cap \xi^{\perp}), \quad \forall \xi \in G(n,i), \quad 1 \le i < n$$

成立, 是否存在一个全局常数$\mathcal{L}$使得

$$\mu(K) \le \mathcal{L}\mu(L)?$$

成立?

在论文中我们给出了一个结果, 与测度μ无关, 也与所选凸体K和L无关, 仅与空间维数有关的常数$\mathcal{L}=n^{i/2}$.

2. 常数$\mathcal{L}=n^{i/2}$的低维Busemann-Petty测度问题的同伦形式

本论文将利用球面Radon变换和它的对偶变换的概念及相关结论, 请参阅以下文献 [7,8,10,13,22,28–30].

符号$C(S^{n-1})$表示定义在单位球面S^{n-1}上的所有连续函数构成的空间, 而$C_e(S^{n-1})$是$C(S^{n-1})$ 中所有偶的连续函数构成的子空间, $C_e^{\infty}(S^{n-1})$则代表$C_e(S^{n-1})$中无穷可微函数构成的子集, 类似地, $C(G(n,i))$表示$G(n,i)$ 上的连续函数空间. 当$f \in C(S^{n-1})$, $g \in C(G(n,i))(1 \le i \le n-1)$时, 定义$i$-维球面Radon变换$R_i f$和它的对偶变换$R_i^t g$如下：

$$(R_i f)(\xi) = \int_{S^{n-1}\cap\xi} f(u)d\sigma_i(u); \tag{2}$$

$$(R_i^t g)(u) = \int_{\xi \in G(n,i)} g(\xi) d\upsilon_i(\xi), \tag{3}$$

其中σ_i是S^{i-1}上的Haar概率测度(这里S^{i-1}其实就是$S^{n-1} \cap \xi$), υ_i 表示齐性空间$\{\xi \in G(n,i) : u \in \xi\}$上的Haar 概率测度.

在$R_i f$和$R_i^t g$之间存在着互相关联的对偶性([29–31]):

$$\int_{G(n,i)} (R_i f)(\xi) g(\xi) d\xi = \int_{S^{n-1}} f(u)(R_i^t g)(u) du. \tag{4}$$

因为这种对偶性, 当μ和ν分别是S^{n-1}和$G(n,i)$的有限Borel测度时, 就可以把上面的定义拓展到测度的i-维球面Radon 变换$R_i\mu$和它的对偶变换$R_i^t\nu$,

$$\int_{G(n,i)} (R_i\mu)(\xi) g(\xi) d\xi = \int_{S^{n-1}} (R_i^t g)(u) d\mu(u), \ \ g \in C(G(n,i)); \tag{5}$$

$$\int_{S^{n-1}} (R_i^t \nu)(u) f(u) du = \int_{G(n,i)} (R_i f)(\xi) d\nu(\xi), \ \ f \in C(S^{n-1}). \tag{6}$$

通常把(5)及(6)简化为下式,

$$(R_i\mu, g) = (\mu, R_i^t g) = \mu(R_i^t g);$$

$$(R_i^t\nu, f) = (\nu, R_i f) = \nu(R_i f).$$

若K是R^n中的关于原点的星体, 即任意一条经过原点的直线与R^n中的紧致子集K的边界相交, 有且仅有异于原点的两个交点, 则定义K的径向函数$\rho_K = \rho(K, \cdot)$为

$$\rho_K(x) = \max\{\lambda > 0 : \lambda x \in K\}, \quad x \in R^n \backslash \{0\}.$$

引理2.1 假设测度μ的密度函数$f \in L_{1,\mathrm{loc}}(\xi^\perp)$, $\xi^\perp$是$\xi \in G(n,i)$的正交补, 那么对任意凸体$L \subset R^n$有,

$$\mu(L \cap \xi^\perp) = (n-i)\omega_{n-i} R_{n-i} \left(\int_0^{\rho_L} f(r\cdot) r^{n-i-1} dr \right)(\xi^\perp), \tag{7}$$

其中ω_j表示R^j中j-单位球的体积.

证明 根据极坐标公式和定义(2), 有

$$\begin{aligned}
\mu(L \cap \xi^\perp) &= \int_{L \cap \xi^\perp} f(x) dx \\
&= \int_{S^{n-1} \cap \xi^\perp} \int_0^{\rho_L(u)} f(ru) r^{n-i-1} dr du \\
&= (n-i)\omega_{n-i} \int_{S^{n-1} \cap \xi^\perp} \int_0^{\rho_L(u)} f(ru) r^{n-i-1} dr d\sigma_i(u) \\
&= (n-i)\omega_{n-i} R_{n-i} \left(\int_0^{\rho_L} f(r\cdot) r^{n-i-1} dr \right)(\xi^\perp),
\end{aligned}$$

在(7)中, 取$f=1$, 则有

$$\mathrm{vol}_{n-i}(L\cap\xi^{\perp})=\omega_{n-i}(R_{n-i}\rho_L^{n-i})(\xi^{\perp}). \tag{8}$$

R^n中的一个原点对称星体K称为是$(n-i)$-截面体([7, 28]), 如果在$G(n,n-i)$上存在非负的Borel测度μ, 有如下等式成立,

$$\int_{S^{n-1}}\rho_K^i(u)f(u)du=(R_{n-i}f,\mu)=\mu(R_{n-i}f),\quad \forall f\in C(S^{n-1}). \tag{9}$$

上式也等价于

$$\rho_K^i=R_{n-i}^t\mu.$$

用符号$\mathcal{I}_{n-i}^n$表示所有$(n-i)$-截面体构成的集合类, 在$i=1$时, $\mathcal{I}_{n-1}^n$就是Lutwak [19] 定义的截面体概念, 一般用符号$\mathcal{I}^n$表示.

本论文主要结论的证明需要Milman [10]建立的关于上述概念的一个迭代性质.

引理2.2 若$K_1\in\mathcal{I}_{n-i_1}^n$, $K_2\in\mathcal{I}_{n-i_2}^n(i_1,i_2<n)$, 且满足$i_3=i_1+i_2<n$, 那么依据等式$\rho_{K_3}^{i_3}=\rho_{K_1}^{i_1}\rho_{K_2}^{i_2}$ 确定的原点对称星体K_3一定是一个$(n-i_3)$- 截面体, 即$K_3\in\mathcal{I}_{n-i_3}^n$.

我们还需要下面的基本不等式, $i=1$的情况在论文 [1,3]中已经证明.

引理2.3 设整数i满足$1\le i\le n-1$, $\omega,a,b\in(0,\infty)$是任意正实数, 及$\alpha:R^+\to R^+$是可测函数, 那么有结论

$$\frac{\omega}{a^i}\int_0^a t^{n-1}\alpha(t)dt-\omega\int_0^a t^{n-i-1}\alpha(t)dt\le\frac{\omega}{a^i}\int_0^b t^{n-1}\alpha(t)dt-\omega\int_0^b t^{n-i-1}\alpha(t)dt, \tag{10}$$

假设上式涉及到的积分都存在. 如果另外非负函数$\alpha(t)$除去R^+中某个可数点集外, $\alpha(t)>0$成立, 那么当且仅当$a=b$时(10)式是一个恒等式.

证明 当$a\ne b$时, 要证明的不等式(10)等价于

$$a^i\int_0^b t^{n-i-1}\alpha(t)dt-a^i\int_0^a t^{n-i-1}\alpha(t)dt\le\int_0^a t^{n-1}\alpha(t)dt-\int_0^b t^{n-1}\alpha(t)dt.$$

也就是

$$\int_a^b\left(\frac{a}{t}\right)^i t^{n-1}\alpha(t)dt\le\int_a^b t^{n-1}\alpha(t)dt. \tag{11}$$

因为$\alpha\ge 0$, 且在(11)中容易得到若$0<a<b$则有$a/t\le 1$, 若$a>b>0$则有$a/t\ge 1$, 所以无论$a<b$还是$a>b$, 不等式(11)显然成立.

当$a=b$时, (10)式立即变成了一个恒等式. 反之, 假设

$$\int_0^a t^{n-1}\alpha(t)dt-a^i\int_0^a t^{n-i-1}\alpha(t)dt=\int_0^b t^{n-1}\alpha(t)dt-a^i\int_0^b t^{n-i-1}\alpha(t)dt.$$

即是

$$\int_0^a t^{n-1}\alpha(t)dt + a^i\int_0^b t^{n-i-1}\alpha(t)dt = \int_0^b t^{n-1}\alpha(t)dt + a^i\int_0^a t^{n-i-1}\alpha(t)dt,$$

由于函数$\int_0^s t^{n-1}\alpha(t)dt$和$a^i\int_0^s t^{n-i-1}\alpha(t)dt$ 在$s>0$时的单调性(因为非负函数$\alpha(t)$除去$t\in R^+$中某个可数点集外, 有$\alpha(t)\neq 0$), 所以$a=b$.

若K和L是R^n中的两个原点对称凸体, 则定义它们之间的Banach-Mazur距离为

$$d_{BM}(K,L)=\inf\{d>0:\exists T\in GL(n):K\subset TL\subset dK\},$$

且记

$$d_IK=\min\{d_{BM}(K,L):L\text{为截面体}\}.$$

定理2.4 设$f\in L_{1,\mathrm{loc}}(R^n)$是一个偶的非负连续函数, μ是密度函数为f的有限Borel测度, 且$K,L\subset R^n$是两个原点对称凸体, 满足条件

$$\mu(K\cap\xi^\perp)\leq\mu(L\cap\xi^\perp),\quad \xi\in G(n,i),\quad 1\leq i\leq n-1, \tag{12}$$

那么有

$$\mu(K)\leq d_I^i(K)\mu(L).$$

证明 根据(7)式重写(12)为

$$R_{n-i}\Big(\int_0^{\rho_K(\cdot)}f(t\cdot)t^{n-i-1}dt\Big)(\xi^\perp)\leq R_{n-i}\Big(\int_0^{\rho_L(\cdot)}f(t\cdot)t^{n-i-1}dt\Big)(\xi^\perp). \tag{13}$$

通过观察, 一方面, 对满足$Q\subset K\subset d_I(K)Q$(由于截面体的线性变换像仍然是一个截面体)的截面体$Q\in\mathcal{I}^n$, 连续地利用引理2.2, 则$Q\in\mathcal{I}^n_{n-i}$. 再在$G(n,n-i)$上关于测度μ 对(13)式积分(对应于$(n-i)$-截面体Q), 利用(6), (9), 得到

$$\int_{S^{n-1}}\rho_Q^i(u)\int_0^{\rho_K(u)}f(tu)t^{n-i-1}dtdu\leq\int_{S^{n-1}}\rho_Q^i(u)\int_0^{\rho_L(u)}f(tu)t^{n-i-1}dtdu; \tag{14}$$

另一方面, 在引理2.3中, 取$\omega=\rho_Q^i(u)$, $a=\rho_K(u)$, $b=\rho_L(u)$和$\alpha(t)=f(tu)$, 得出下式

$$\begin{aligned}&\frac{\rho_Q^i(u)}{\rho_K^i(u)}\int_0^{\rho_K(u)}t^{n-1}f(tu)dt-\rho_Q^i(u)\int_0^{\rho_K(u)}t^{n-i-1}f(tu)dt\\ \leq&\frac{\rho_Q^i(u)}{\rho_K^i(u)}\int_0^{\rho_L(u)}t^{n-1}f(tu)dt-\rho_Q^i(u)\int_0^{\rho_L(u)}t^{n-i-1}f(tu)dt\end{aligned} \tag{15}$$

对(15)在S^{n-1}上积分, 依据(14)和$Q\subset K\subset d_I(K)Q$, 我们获得了

$$\int_{S^{n-1}}\frac{\rho_Q^i(u)}{\rho_K^i(u)}\int_0^{\rho_K(u)}t^{n-1}f(tu)dtdu\leq\int_{S^{n-1}}\frac{\rho_Q^i(u)}{\rho_K^i(u)}\int_0^{\rho_L(u)}t^{n-1}f(tu)dtdu,$$

即

$$\frac{1}{d_I^i(K)}\int_{S^{n-1}}\int_0^{\rho_K(u)} t^{n-1}f(tu)dtdu \le \int_{S^{n-1}}\int_0^{\rho_L(u)} t^{n-1}f(tu)dtdu. \tag{16}$$

这个不等式其实就是下式的极坐标形式

$$\mu(K) \le d_I^i(K)\mu(L).$$

显然地, 欧氏单位球B_2^n是个截面体, 依据John定理, 及对任意的原点对称凸体$K \subset R^n$有$d_{BM}(K, B_2^n) \le \sqrt{n}$成立的事实, 与定理2.4一道可以构建如下推论.

推论2.5 设$f \in L_{1,\text{loc}}(R^n)$是一偶的非负连续函数, μ是有密度函数为f的有限的Borel测度, K和L 是R^n 中两个原点对称凸体，如果满足下式

$$\mu(K \cap \xi^{\perp}) \le \mu(L \cap \xi^{\perp}), \quad \xi \in G(n, i), \quad 1 \le i \le n-1,$$

那么有

$$\mu(K) \le n^{i/2}\mu(L).$$

注：定理2.4和推论2.5中$i = 1$的情形在文献 [3]中已经证明.

基金项目

国家自然科学基金资助项目(10801140)；铜仁学院博士科研启动基金项目(trxyDH2220)。

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