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Hans汉斯

广义BBM-Burgers方程扩散波的初边值问题

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摘　要

本文主要研究的是广义BBM-Burgers方程的初边值问题，利用能量估计的方法证明了广义BBM-Burgers方程的解关于扩散波的渐近稳定性。即对方程：$\begin{cases} u_t + f(u)_x = u_{xx} + u_{xxt} \\ u(x,t)\big|_{t=0} = u_0(x), u(0,t) = u_- \end{cases}$ 在本文中我们将证明在波的强度 $\delta := |u_+ - u_-|$ 及初值 $u_0(x)$ 适当小的情况下，广义BBM-Burgers方程的解整体存在且当时间t趋于无穷时收敛到非线性扩散波 $\bar{u}\left(\frac{x}{\sqrt{1+t}}\right)$。

关键词

广义BBM-Burgers方程，初边值问题，能量法，非线性扩散波

Initial Boundary Value Problem of Diffused Waves in Generalized BBM-Burgers Equation

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Abstract

This paper studies the initial boundary value problem of the generalized BBM-Burgers equation, and proves the asymptotic stability of the solution of the generalized BBM-Burgers equation with respect to the diffusion wave by using the method of energy estimation. To the equation:

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$$\begin{cases} u_t + f(u)_x = u_{xx} + u_{xxt} \\ u(x,t)|_{t=0} = u_0(x), u(0,t) = u_- \end{cases}$$
, in this paper, we will prove that the solution of the generalized BBM-Burgers equation exists as a whole and converges to the nonlinear diffusion wave $\bar{u}\left(x/\sqrt{1+t}\right)$ as time t approaches infinity when the wave intensity $\delta := |u_+ - u_-|$ and initial value $u_0(x)$ are appropriately small.

Keywords

Generalized BBM-Burgers Equation, Initial Boundary Value Problem, Energy Method, Nonlinear Diffusion Wave

1. 引言

本文考虑了以下广义 BBM-Burgers 方程

$$\begin{cases} u_t + f(u)_x = u_{xx} + u_{xxt}, \\ u(x,t)|_{t=0} = u_0(x), u(0,t) = u_-. \end{cases} \tag{1.1}$$

其中，$(x,t) \in R_+ \times R_+$，$f(u)$ 是一个充分光滑的凸函数，δ 表示色散系数且 $\delta \neq 0$，常数 $\mu > 0$ 为耗散系数。此外，我们假设 $u_0(x)$ 在边界 $x = 0$， $x = +\infty$ 上有：

$$u(0,0) = u_-, u(+\infty,0) = u_+, u_+ \neq u_-. \tag{1.2}$$

一些基本的波，比如稀疏波、扩张波、行波等均可描述上述初边值问题以及相应的大时间性态的渐近稳定性。本文主要研究的方向是非线性扩散波，广义 BBM-Burgers 方程最早是 Benjamin，Bona 和 Mahony [1]提出的，它是用来描述浅水波现象的 KdV 方程 $u_t + uu_x + u_{xxx} = 0$ 的一个改变，能够更好地解决热传导问题、热学中的双温问题等，因此对于该类问题具有很好的理论以及实际意义，对于 BBM-Burgers 方程

$$u_t + f(u) = u_{xx} + u_{xxt}. \tag{1.3}$$

解的存在性和大时间行为已经有广泛的研究了，Tang Y [2]研究从孤子解的演化角度去证明了 Burgers 方程在扰动作用下激波研究的一般途径；Yang Xiaojia [3]等人研究了 burgers 方程的两层高阶紧致有限差分隐式，利用傅里叶分析法对方案的稳定性进行了分析，最后通过数值实验验证了所提方案的准确性与可靠性。

读者可阅读[4]-[11]，下面介绍一些学者对于该问题的研究成果。

刘太平等人已经研究了 Burgers 方程的初边值问题[12]：

$$\begin{cases} u_t + f(u)_x = u_{xx}, x \in R^+, t > 0, \\ u(x,t)|_{x=0} = u_-, t > 0, \\ u(x,t)|_{t=0} = u_0(x) = \begin{cases} u_-, x = 0, \\ u_+, x \to \infty. \end{cases} \end{cases} \tag{1.4}$$

其中 $f(u)$ 为 R 上的光滑函数。

蒋米娜，徐艳玲[13]证明广义 BBM-Burgers 方程在假设条件 $u_- < u_+$ 下，依据特征速度 $f'(u_\pm)$ 的符号不同，将问题分成了五种情况，并且证明了解的整体存在性，分别找到了它们的渐近状态；陈琴，刘艳[14]研究了带有一般边界条件的广义 BBM-Burgers 方程的初边值问题，运用 L^2-能量法证明其强边界层解具有非线性稳定性。对于 BBM-Burgers 方程的研究成果，读者可参考文献[15]-[20]。

在前人研究的基础上，本文主要研究广义 BBM-Burgers 方程非线性扩散波的解的渐近稳定性，观察广义 BBM-Burgers 方程中含有热传导方程

$$\bar{u}_t - \bar{u}_{xx} = 0 \tag{1.5}$$

对于热传导方程，我们知道该方程的解是具有唯一的自相似解，即扩散波在本文中将证明广义 BBM-Burgers 方程的初边值问题在大时间里的解是非线性稳定的。我们将在扩散波附近定义一个扰动 $\phi = u - \bar{u}$，证明在波 $\delta := |u_+ - u_-|$ 足够小的

情况下，(1.1)的解收敛到(1.5)的自相似解。

定义如下的扰动

$$\begin{cases} \phi(x,t) = u(x,t) - \bar{u}(x,t), \\ \phi_0(x) = u_0(x) - \bar{u}(x,0), \phi(0,t) = 0. \end{cases} \tag{1.6}$$

则由(1.1) (1.5) (1.6)得到扰动方程为

$$\begin{cases} \phi_t - \phi_{xx} + f(\bar{u} + \phi)_x - (\phi + \bar{u})_{xxt} = 0, \\ \phi_0(x) = 0. \end{cases} \tag{1.7}$$

下面则是本文的定理。

定理 1：对于问题(1.6)假设存在足够小的波的强度 δ 和初值 $\Phi_0 = \|\phi_0\|_2^2$，且 δ 满足 $\|u_0(x) - \bar{u}(x,0)\|_{H^2(R_+)} + |u_+ - u_-| \le \delta < \varepsilon$，初边值问题(1.6)存在唯一解 $u(x,t)$，且满足

$$u(x,t) - \bar{u}(x,t) \in C\left([0,\infty), H^2(R_+)\right) \cap L^2\left([0,\infty), H^3(R_+)\right). \tag{1.8}$$

进一步则有

$$\sum_{k=0}^{2} \left\|\partial_x^k(\phi,\phi_x)(t)\right\|^2 + \sum_{k=0}^{2} \int_0^t \left\|\partial_x^k(\phi,\phi_x)(t)\right\|^2 \mathrm{d}t \le C(\Phi_0 + \delta). \tag{1.9}$$

记号：

在本文中，回忆一些关于勒贝格空间和分数阶索伯列夫空间的一些必要的基础知识，对于更多细节的地方，可以参考[21] [22] [23]。$L^p = L^p(R_+)$ 表示为一般的 Lebesgue 空间，其定义的范数为

$$\|v\|_{L^p} = \left(\int_0^{+\infty} |v(x)|^p \mathrm{d}x\right)^{\frac{1}{p}}, 1 \le p < +\infty, \quad \|v\|_{L^\infty} = \text{esssuup}|v(x)|.$$

$u \in W_{l,2} = H^l(\Omega)$，$H^l$ 表示一般的 Sobolev 空间，u 的范数为

$$\|u\|_{H^l} = \left(\sum_{|\alpha| \le l} \int \left|D^\alpha u\right|^2 \mathrm{d}x\right)^{\frac{1}{2}}$$

在不会导致混淆的情况下，我们将省略积分区域 R_+，并且 $C > 0$，$O(1)$，来表示一般常数。

2. 基础知识

引理 2.1：假设 $\bar{u}(x,t)$ 为问题(1.5)的自相似解，则 $\bar{u}(x,t)$ 满足：

$$\left|\partial_x^k\partial_t^l\bar{u}\right| = E\left|u_+ - u_-\right|(1+t)^{-\frac{k}{2}-l}\omega(x,t). \tag{2.1}$$

其中 $k,l = 1,2,\cdots$， $\omega(x,t) = \exp\left\{-\frac{\sigma x^2}{1+t}\right\}$。 σ 是正常数，这里的 σ 仅与 u_+ 和 u_- 有关。

为了得到定理 1.1 的证明，我们下面将先给出扰动方程(1.7)的局部存在性以及先验估计，根据先验估计的证明相关引理，再利用靴带原理，将解延拓到全局，最终证明广义 BBM-Burgers 方程的初边值问题在大时间里的解是非线性稳定的，即证明定理 1.1。

引理 2.2 (Sobolev 不等式)对于任意函数 $f(x) \in H^1(R)$，有

$$\|f\|_{L^\infty} \le C\|f\|^{\frac{1}{2}}\|f_x\|^{\frac{1}{2}}. \tag{2.2}$$

引理 2.3 (Young 不等式)设 $1 < p,q < \infty$，满足 $\frac{1}{p}+\frac{1}{q}=1$，则有

$$ab < \frac{a^p}{p}+\frac{b^q}{q}(a,b>0). \tag{2.3}$$

特别地，当 $p = q = 2$ 时，我们称之为 Cauchy 不等式。

3. 定理 1.1 的证明

1) 命题 3.1 (局部存在性)考虑初始时间为 t 的扰动方程

$$\begin{cases}\phi_t - \phi_{xx} + f(\bar{u}+\phi)_x - (\phi+\bar{u})_{xxt} = 0, \\ \phi_0(x) = 0.\end{cases} \tag{3.1}$$

若 $\phi(x,t) \in H^2$，且有 $\|\phi(t)\|_2 \le M$ 存在适当小的 t_0，使得方程在 $[t, t+t_0]$ 上有唯一解。

2) 命题 3.2 (先验估计)在定理 1 的条件下，若使得 $\phi(x,t) \in X(x,T)$ 为问题(3.1)的解，则我们作出如下的先验假设：

$$\sup_{0\le t\le T}\sum_{k=0}^{2}\left\|\partial^k\phi(t)\right\|_{L^2}^2 \le C\varepsilon. \tag{3.2}$$

其中 C 为正常数，ε 是依赖于初值以及波的强度的一个充分小的正常数。根据以上先验假设，我们用 Sobolev 不等式 $\|f\|_{L^\infty} \le \left(\|f\|_{L^2}\right)^{\frac{1}{2}}\left(\|\partial_x f\|_{L^2}\right)^{\frac{1}{2}}$ 可以得到相应函数的 L^∞ 范数，则我们可以得到如下的先验估计：

$$\sum_{k=0}^{2}\left\|\partial^k\phi(t)\right\|_{L^2}^2 + \sum_{k=0}^{2}\int_0^{+\infty}\left\|\partial^k\phi(x,t)\right\|_{L^2}^2 \mathrm{d}x \le C\left\|\phi_0\right\|_2^2. \tag{3.3}$$

3) 定理 1.1 的证明

为了证明本文的主要结果，首先证明一些基本的引理。下面将证明 $\phi(x,t)$ 的低阶、高阶估计。证明过程由下面几个引理组成。

4. 先验估计的证明

引理 4.1：在先验假设的情况下，对于充分小的正数 δ，有

$$\|\phi\|^2 + \|\phi_x\|^2 + \int_0^t\left(\|\phi\|^2 + \|\phi_x\|^2\right)\mathrm{d}t \le C\left(\|\phi_0\|_2^2 + \delta\right). \tag{4.1.1}$$

证明：在扰动方程(1.6)两边同时乘以 ϕ 后，在 $[0,t]\times[0,+\infty)$ 上积分，

$$\int_0^t\int_0^{+\infty}\phi\phi_t\mathrm{d}x\mathrm{d}t-\int_0^t\int_0^{+\infty}\phi\phi_{xx}\mathrm{d}x\mathrm{d}t+\int_0^t\int_0^{+\infty}\phi f\left(\bar{u}+\phi\right)_x\mathrm{d}x\mathrm{d}t-\int_0^t\int_0^{+\infty}\phi\left(\phi+\bar{u}\right)_{xxt}\mathrm{d}x\mathrm{d}t=0. \tag{4.1.2}$$

将(4.1.2)整理可得

$$\int_0^t\int_0^{+\infty}\phi\phi_t\mathrm{d}x\mathrm{d}t-\int_0^t\int_0^{+\infty}\phi\phi_{xx}\mathrm{d}x\mathrm{d}t+\int_0^t\int_0^{+\infty}-\phi_{xxt}\phi\mathrm{d}x\mathrm{d}t=\int_0^t\int_0^{+\infty}\bar{u}_{xxt}\phi\mathrm{d}x\mathrm{d}t-\int_0^t\int_0^{+\infty}f\left(\phi+\bar{u}\right)_x\phi\mathrm{d}x\mathrm{d}t. \tag{4.1.3}$$

则(4.1.3)通过分部积分可得

$$\int_0^t\int_0^{+\infty}\phi\phi_t\mathrm{d}x\mathrm{d}t=\frac{1}{2}\int_0^{+\infty}\phi^2\mathrm{d}x-\frac{1}{2}\int_0^{+\infty}\phi_0^2\mathrm{d}x. \tag{4.1.4}$$

$$\begin{aligned}\int_0^t\int_0^{+\infty}-\phi\phi_{xx}\mathrm{d}x\mathrm{d}t&=-\int_0^t\int_0^{+\infty}\phi\mathrm{d}\left(\phi_x\right)x\mathrm{d}t\\&=-\int_0^t\phi\phi_x\Big|_0^{+\infty}\mathrm{d}t+\int_0^t\int_0^{+\infty}\phi_x^2\mathrm{d}x\mathrm{d}t\\&=\int_0^t\int_0^{+\infty}\phi_x^2\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{4.1.5}$$

$$\begin{aligned}\int_0^t\int_0^{+\infty}-\phi_{xxt}\phi\mathrm{d}x\mathrm{d}t&=-\int_0^t\int_0^{+\infty}\phi\mathrm{d}\left(\phi_{xt}\right)\mathrm{d}t\\&=-\int_0^t\phi\phi_{xt}\Big|_0^{+\infty}\mathrm{d}t+\int_0^t\int_0^{+\infty}\phi_{xt}\phi_x\mathrm{d}x\mathrm{d}t\\&=\int_0^t\int_0^{+\infty}\phi_{xt}\phi_x\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{4.1.6}$$

其中

$$\int_0^t\int_0^{+\infty}\phi_x\phi_{xt}\mathrm{d}x\mathrm{d}t=\int_0^t\int_0^{+\infty}\phi_x\mathrm{d}\left(\phi_x\right)\mathrm{d}x=\int_0^{+\infty}\phi_x^2\Big|_0^t\mathrm{d}x-\int_0^t\int_0^{+\infty}\phi_x\phi_{xt}\mathrm{d}x\mathrm{d}t. \tag{4.1.7}$$

所以

$$\int_0^t\int_0^{+\infty}\phi_x\phi_{xt}\mathrm{d}x\mathrm{d}t=\frac{1}{2}\int_0^{+\infty}\phi_x^2\Big|_0^t\mathrm{d}x=\frac{1}{2}\int_0^{+\infty}\phi_x^2\,|\,\mathrm{d}x-\frac{1}{2}\int_0^{+\infty}\phi_{x0}^2\,|\,\mathrm{d}x. \tag{4.1.8}$$

将(4.1.8)代入到(4.1.6)中可得

$$\int_0^t\int_0^{+\infty}-\phi\phi_{xxt}\mathrm{d}x\mathrm{d}t=\int_0^t\int_0^{+\infty}\phi_x\phi_{xt}\mathrm{d}x\mathrm{d}t=\frac{1}{2}\int_0^{+\infty}\phi_x^2\mathrm{d}x-\frac{1}{2}\int_0^{+\infty}\phi_{x0}^2\mathrm{d}x. \tag{4.1.9}$$

再将(4.1.4) (4.1.5) (4.1.9)代入到(4.1.3)中并整理可得

$$\begin{aligned}&\frac{1}{2}\int_0^{+\infty}\phi^2\mathrm{d}x+\int_0^t\int_0^{+\infty}\phi_x^2\mathrm{d}x\mathrm{d}t+\frac{1}{2}\int_0^{+\infty}\phi_x^2\mathrm{d}x\\&=\frac{1}{2}\int_0^{+\infty}\phi_0^2\mathrm{d}x+\frac{1}{2}\int_0^{+\infty}\phi_{x0}^2\mathrm{d}x+\int_0^t\int_0^{+\infty}\bar{u}_{xxt}\phi\mathrm{d}x\mathrm{d}t-\int_0^t\int_0^{+\infty}f\left(\phi+\bar{u}\right)_x\phi\mathrm{d}x\mathrm{d}t\end{aligned} \tag{4.1.10}$$

因为$\bar{u}_x$是有界的，所以对方程(4.1.10)的右侧进行估计

$$\begin{aligned}\int_0^t\int_0^{+\infty}\bar{u}_{xxt}\phi\mathrm{d}x\mathrm{d}t&=\int_0^t\int_0^{+\infty}\phi\mathrm{d}\left(\bar{u}_{xt}\right)\mathrm{d}t=\int_0^t\phi\bar{u}_{xt}\Big|_0^{+\infty}\mathrm{d}t-\int_0^t\int_0^{+\infty}\phi_x\bar{u}_{xt}\mathrm{d}x\mathrm{d}t\\&=-\int_0^t\int_0^{+\infty}\phi_x\bar{u}_{xt}\mathrm{d}x\mathrm{d}t\le C\delta+C\delta\int_0^t\int_0^{+\infty}\phi_x^2\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{4.1.11}$$

$$\int_0^t\int_0^{+\infty}f\left(\phi+\bar{u}\right)_x\phi\mathrm{d}x\mathrm{d}t=\int_0^t\int_0^{+\infty}\left[f\left(\phi+\bar{u}\right)-f\left(\bar{u}\right)\right]_x\phi\mathrm{d}x\mathrm{d}t+\int_0^t\int_0^{+\infty}f\left(\bar{u}\right)_x\phi\mathrm{d}x\mathrm{d}t. \tag{4.1.12}$$

则有

$$\begin{aligned}&\int_0^t\int_0^{+\infty}\left[f\left(\phi+\bar{u}\right)-f\left(\bar{u}\right)\right]_x\phi\mathrm{d}x\mathrm{d}t\\&=\int_0^t\int_0^{+\infty}f''\left(\xi\right)\phi\bar{u}_x\mathrm{d}x\mathrm{d}t+\int_0^t\int_0^{+\infty}\phi\phi_x f'\left(\phi+\bar{u}\right)\mathrm{d}x\mathrm{d}t\\&=\int_0^t\int_0^{+\infty}f''\left(\xi\right)\phi\bar{u}_x\mathrm{d}x\mathrm{d}t-\int_0^t\int_0^{+\infty}f''\left(\phi+\bar{u}\right)\left(\phi_x+\bar{u}_x\right)\phi^2\mathrm{d}x\mathrm{d}t\\&=\int_0^t\int_0^{+\infty}f''\left(\xi\right)\phi\bar{u}_x\mathrm{d}x\mathrm{d}t-\int_0^t\int_0^{+\infty}f''\left(\phi+\bar{u}\right)\phi_x\phi^2\mathrm{d}x\mathrm{d}t-\int_0^t\int_0^{+\infty}f''\left(\phi+\bar{u}\right)\bar{u}_x\phi^2\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{4.1.13}$$

$$\begin{aligned}&\int_0^t\int_0^{+\infty}\left[f(\phi+\bar{u})-f(\bar{u})\right]_x\phi\mathrm{d}x\mathrm{d}t+\int_0^t\int_0^{+\infty}f''(\phi+\bar{u})\bar{u}_x\phi^2\mathrm{d}x\mathrm{d}t\\&\le C\delta+C(\delta+\varepsilon)\int_0^t\int_0^{+\infty}\phi^2\mathrm{d}x\mathrm{d}t+C\varepsilon\int_0^t\int_0^{+\infty}\phi_x^2\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{4.1.14}$$

$$\int_0^t\int_0^{+\infty}f(\bar{u})_x\phi\mathrm{d}x\mathrm{d}t=\int_0^t\int_0^{+\infty}f'(\bar{u})_x\bar{u}_x\phi\mathrm{d}x\mathrm{d}t\le C\delta+C\delta\int_0^t\int_0^{+\infty}\phi^2\mathrm{d}x\mathrm{d}t.\tag{4.1.15}$$

将(4.1.13)和(4.1.15)代入到(4.1.12)中得

$$\int_0^t\int_0^{+\infty}f(\phi+\bar{u})_x\phi\mathrm{d}x\mathrm{d}t\le C\delta+C(\delta+\varepsilon)\int_0^t\int_0^{+\infty}\phi^2\mathrm{d}x\mathrm{d}t+C\varepsilon\int_0^t\int_0^{+\infty}\phi_x^2\mathrm{d}x\mathrm{d}t.\tag{4.1.16}$$

再将(4.1.11)和(4.1.16)代入到(4.1.10)中得

$$\begin{aligned}&\frac{1}{2}\int_0^{+\infty}\phi^2\mathrm{d}x+\int_0^t\int_0^{+\infty}\phi_x^2\mathrm{d}x\mathrm{d}t+\frac{1}{2}\int_0^{+\infty}\phi_x^2\mathrm{d}x+\int_0^t\int_0^{+\infty}f''(\phi+\bar{u})\bar{u}_x\phi^2\mathrm{d}x\mathrm{d}t\\&\le C\left(\|\phi_0\|_2^2+\delta\right)+C(\delta+\varepsilon)\int_0^t\int_0^{+\infty}\left(\phi^2+\phi_x^2\right)\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{4.1.17}$$

由于 $\bar{u}_x$ 是有界的，且 f 是光滑函数，所以

$$\int_0^t\int_0^{+\infty}f''(\phi+\bar{u})\bar{u}_x\phi^2\mathrm{d}x\mathrm{d}t\ge C\int_0^t\int_0^{+\infty}\phi^2\mathrm{d}x\mathrm{d}t.\tag{4.1.18}$$

将(4.1.18)代入到(4.1.17)中并且整理可得

$$\int_0^{+\infty}\phi^2\mathrm{d}x+\int_0^{+\infty}\phi_x^2\mathrm{d}x+\int_0^t\int_0^{+\infty}\left(\phi^2+\phi_x^2\right)\mathrm{d}x\mathrm{d}t\le C(\Phi_0+\delta).\tag{4.1.19}$$

即引理 4.1 证毕。

引理 4.2：在先验假设的情况下，对于充分小的正数 δ，有

$$\|\phi_x\|^2+\|\phi_{xx}\|^2+\int_0^t\left(\|\phi_x\|^2+\|\phi_{xx}\|^2\right)\mathrm{d}t\le C\left(\|\phi_0\|_2^2+\delta\right).$$

证明：在扰动方程(1.6)两边关于 x 求导，再在两边同时乘以 ϕ_x 后，在 $[0,t]\times[0,+\infty)$ 上积分，

$$\int_0^t\int_0^{+\infty}\phi_x\phi_{xt}\mathrm{d}x\mathrm{d}t-\int_0^t\int_0^{+\infty}\phi_x\phi_{xxx}\mathrm{d}x\mathrm{d}t+\int_0^t\int_0^{+\infty}\phi_x f(\bar{u}+\phi)_{xx}\mathrm{d}x\mathrm{d}t-\int_0^t\int_0^{+\infty}\phi_x(\phi+\bar{u})_{xxxt}\mathrm{d}x\mathrm{d}t=0.\tag{4.2.1}$$

下面计算(4.2.1)的各项

$$\begin{aligned}\int_0^t\int_0^{+\infty}\phi_x\phi_{xt}\mathrm{d}x\mathrm{d}t&=\int_0^t\int_0^{+\infty}\phi_x\mathrm{d}(\phi_x)\mathrm{d}x\\&=\frac{1}{2}\int_0^{+\infty}\phi_x^2\mathrm{d}x-\frac{1}{2}\int_0^{+\infty}\phi_{x0}^2\mathrm{d}x.\end{aligned}\tag{4.2.2}$$

$$\begin{aligned}\int_0^t\int_0^{+\infty}-\phi_x\phi_{xxx}\mathrm{d}x\mathrm{d}t&=-\int_0^t\int_0^{+\infty}\phi_x\mathrm{d}(\phi_{xx})\mathrm{d}t\\&=-\int_0^t\phi_x\phi_{xx}\Big|_0^{+\infty}\mathrm{d}t+\int_0^t\int_0^{+\infty}\phi_{xx}^2\mathrm{d}x\mathrm{d}t\\&=\int_0^t\int_0^{+\infty}\phi_{xx}^2\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{4.2.3}$$

$$\begin{aligned}-\int_0^t\int_0^{+\infty}\phi_x\phi_{xxxt}\mathrm{d}x\mathrm{d}t&=-\int_0^t\int_0^{+\infty}\phi_x\mathrm{d}(\phi_{xxt})\mathrm{d}t\\&=-\int_0^t\phi_x\phi_{xxt}\Big|_0^{+\infty}\mathrm{d}t+\int_0^t\int_0^{+\infty}\phi_{xx}\phi_{xxt}\mathrm{d}x\mathrm{d}t\\&=\int_0^t\int_0^{+\infty}\phi_{xx}\mathrm{d}(\phi_{xx})\mathrm{d}x\\&=\frac{1}{2}\int_0^{+\infty}\phi_{xx}^2\Big|_0^t\mathrm{d}x\\&=\frac{1}{2}\int_0^{+\infty}\phi_{xx}^2\mathrm{d}x-\frac{1}{2}\int_0^{+\infty}\phi_{xx0}^2\mathrm{d}x.\end{aligned}\tag{4.2.4}$$

将(4.2.2)~(4.2.4)代入到(4.2.1)中整理可得

$$\begin{aligned}&\frac{1}{2}\int_0^{+\infty}\phi_x^2\mathrm{d}x+\frac{1}{2}\int_0^{+\infty}\phi_{xx}^2\mathrm{d}x+\int_0^t\int_0^{+\infty}\phi_{xx}^2\mathrm{d}x\mathrm{d}t\\&=\frac{1}{2}\int_0^{+\infty}\phi_{x0}^2\mathrm{d}x+\frac{1}{2}\int_0^{+\infty}\phi_{xx0}^2\mathrm{d}x-\int_0^t\int_0^{+\infty}\phi_x f\left(\bar{u}+\phi\right)_{xx}\mathrm{d}x\mathrm{d}t-\int_0^t\int_0^{+\infty}\phi_x\bar{u}_{xxxt}\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{4.2.5}$$

下面对(4.2.5)的右侧进行估计

$$\begin{aligned}\int_0^t\int_0^{+\infty}\phi_x\bar{u}_{xxxt}\mathrm{d}x\mathrm{d}t&=\int_0^t\phi_x\bar{u}_{xxt}\Big|_0^{+\infty}\mathrm{d}t-\int_0^t\int_0^{+\infty}\phi_{xx}\bar{u}_{xxt}\mathrm{d}x\mathrm{d}t\\&=-\int_0^t\int_0^{+\infty}\phi_{xx}\bar{u}_{xxt}\mathrm{d}x\mathrm{d}t\\&\le C\delta+C\delta\int_0^t\int_0^{+\infty}\phi_{xx}^2\mathrm{d}x.\end{aligned}\tag{4.2.6}$$

$$\begin{aligned}&-\int_0^t\int_0^{+\infty}\phi_x f\left(\bar{u}+\phi\right)_{xx}\mathrm{d}x\mathrm{d}t\\&=-\int_0^t\phi_x f\left(\bar{u}+\phi\right)_x\Big|_0^{+\infty}\mathrm{d}t+\int_0^t\int_0^{+\infty}\phi_{xx} f\left(\bar{u}+\phi\right)_x\mathrm{d}x\mathrm{d}t\\&=\int_0^t\int_0^{+\infty}\phi_{xx} f\left(\bar{u}+\phi\right)_x\mathrm{d}x\mathrm{d}t-\int_0^t\int_0^{+\infty}\phi_{xx} f\left(\bar{u}\right)_x\mathrm{d}x\mathrm{d}t+\int_0^t\int_0^{+\infty}\phi_{xx} f\left(\bar{u}\right)_x\mathrm{d}x\mathrm{d}t\\&=\int_0^t\int_0^{+\infty}\phi_{xx} f'\left(\bar{u}+\phi\right)\left(\phi_x+\bar{u}_x\right)\mathrm{d}x\mathrm{d}t-\int_0^t\int_0^{+\infty}\phi_{xx} f'\left(\bar{u}\right)\bar{u}_x\mathrm{d}x\mathrm{d}t+\int_0^t\int_0^{+\infty}\phi_{xx} f'\left(\bar{u}\right)\bar{u}_x\mathrm{d}x\mathrm{d}t\\&=I_1-I_2+I_3.\end{aligned}\tag{4.2.7}$$

下面对(4.2.7)的各项进行计算

$$\begin{aligned}I_1-I_2&=\int_0^t\int_0^{+\infty}\phi_{xx} f'\left(\bar{u}+\phi\right)\left(\phi_x+\bar{u}_x\right)\mathrm{d}x\mathrm{d}t-\int_0^t\int_0^{+\infty}\phi_{xx} f'\left(\bar{u}\right)\bar{u}_x\mathrm{d}x\mathrm{d}t\\&=\int_0^t\int_0^{+\infty}\phi_{xx} f'\left(\bar{u}+\phi\right)\phi_x\mathrm{d}x\mathrm{d}t+\int_0^t\int_0^{+\infty}\phi_{xx} f'\left(\bar{u}+\phi\right)\bar{u}_x\mathrm{d}x\mathrm{d}t-\int_0^t\int_0^{+\infty}\phi_{xx} f'\left(\bar{u}\right)\bar{u}_x\mathrm{d}x\mathrm{d}t,\end{aligned}\tag{4.2.8}$$

用泰勒展开式以及分部积分可将(4.2.8)写成

利

$$\begin{aligned}I_1-I_2&=\int_0^t\int_0^{+\infty}f''\left(\xi\right)\phi\phi_{xx}\bar{u}_x\mathrm{d}x\mathrm{d}t+\int_0^t\int_0^{+\infty}f'\left(\bar{u}+\phi\right)\phi_x\phi_{xx}\mathrm{d}x\mathrm{d}t\\&=\int_0^t f''\left(\xi\right)\phi\phi_x\bar{u}_x\Big|_0^{+\infty}\mathrm{d}t-\int_0^t\int_0^{+\infty}\left(f''\left(\xi\right)\phi\bar{u}_x\right)_x\phi_x\mathrm{d}x\mathrm{d}t+\int_0^t\int_0^{+\infty}f'\left(\bar{u}+\phi\right)\phi_x\phi_{xx}\mathrm{d}x\mathrm{d}t\\&=-\int_0^t\int_0^{+\infty}\left(f'''\left(\xi\right)\phi\bar{u}_x+f''\left(\xi\right)\phi_x\bar{u}_x+f''\left(\xi\right)\phi\bar{u}_{xx}\right)\phi_x\mathrm{d}x\mathrm{d}t\\&\quad-\frac{1}{2}\int_0^t\int_0^{+\infty}f''\left(\bar{u}+\phi\right)\phi_x^2\mathrm{d}x\mathrm{d}t-\frac{1}{2}\int_0^t\int_0^{+\infty}f''\left(\bar{u}+\phi\right)\bar{u}_x\phi_x\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{4.2.9}$$

将 $-\frac{1}{2}\int_0^t\int_0^{+\infty}f''\left(\bar{u}+\phi\right)\bar{u}_x\phi_x\mathrm{d}x\mathrm{d}t$ 移到等式(4.2.5)左边，则

$$I_1-I_2\le C\delta+C\delta\int_0^t\int_0^{+\infty}\phi_x^2\mathrm{d}x\mathrm{d}t+C\delta+C\delta\int_0^t\int_0^{+\infty}\phi^2\mathrm{d}x\mathrm{d}t.\tag{4.2.10}$$

$$I_3=-\int_0^t\int_0^{+\infty}\phi_{xx} f'\left(\bar{u}\right)\bar{u}_x\mathrm{d}x\mathrm{d}t\le C\delta+C\delta\int_0^t\int_0^{+\infty}\phi_{xx}^2\mathrm{d}x\mathrm{d}t.\tag{4.2.11}$$

将(4.2.6) (4.2.10)~(4.2.11)代入到(4.2.5)中并且整理可得

$$\begin{aligned}&\frac{1}{2}\int_0^{+\infty}\phi_x^2\mathrm{d}x+\frac{1}{2}\int_0^{+\infty}\phi_{xx}^2\mathrm{d}x+\int_0^t\int_0^{+\infty}\phi_{xx}^2\mathrm{d}x\mathrm{d}t+\frac{1}{2}\int_0^t\int_0^{+\infty}f''\left(\bar{u}+\phi\right)\bar{u}_x\phi_x\mathrm{d}x\mathrm{d}t\\&\le C\delta+C\delta\int_0^t\int_0^{+\infty}\phi_{xx}^2\mathrm{d}x+C\delta\int_0^t\int_0^{+\infty}\phi_x^2\mathrm{d}x\mathrm{d}t+C\delta\int_0^t\int_0^{+\infty}\phi^2\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{4.2.12}$$

由于 $\bar{u}_x$ 是有界的，且 f 是光滑函数，所以

$$\frac{1}{2}\int_0^t\int_0^{+\infty}f''\left(\bar{u}+\phi\right)\bar{u}_x\phi_x\mathrm{d}x\mathrm{d}t\ge C\int_0^t\int_0^{+\infty}\phi_x^2\mathrm{d}x\mathrm{d}t.\tag{4.2.13}$$

将(4.2.13)代入到(4.2.12)中，并且运用已得证的引理 4.1 可得

$$\int_0^{+\infty}\phi_x^2\mathrm{d}x+\int_0^{+\infty}\phi_{xx}^2\mathrm{d}x+\int_0^t\int_0^{+\infty}\phi_{xx}^2\mathrm{d}x\mathrm{d}t+\int_0^t\int_0^{+\infty}\phi_x^2\mathrm{d}x\mathrm{d}t\le C\left(\Phi_0+\delta\right). \tag{4.2.14}$$

即引理 4.2 证毕。

引理 4.3：在先验假设的情况下，对于充分小的正数 δ，有

$$\left\|\phi_{xx}\right\|^2+\left\|\phi_{xxx}\right\|^2+\int_0^t\left(\left\|\phi_{xx}\right\|^2+\left\|\phi_{xxx}\right\|^2\right)\mathrm{d}t\le C\left(\left\|\phi_0\right\|_2^2+\delta\right).$$

证明：在扰动方程(1.6)两边关于 x 求导，再在两边同时乘以 ϕ_{xx} 后，在 $[0,t]\times[0,+\infty)$ 上积分，

$$\int_0^t\int_0^{+\infty}\phi_{xx}\phi_{xxt}\mathrm{d}x\mathrm{d}t-\int_0^t\int_0^{+\infty}\phi_{xx}\phi_{xxxx}\mathrm{d}x\mathrm{d}t+\int_0^t\int_0^{+\infty}\phi_{xx}f\left(\bar{u}+\phi\right)_{xxx}\mathrm{d}x\mathrm{d}t-\int_0^t\int_0^{+\infty}\phi_{xx}\left(\phi+\bar{u}\right)_{xxxxt}\mathrm{d}x\mathrm{d}t=0. \tag{4.3.1}$$

下面则对(4.3.1)的各项进行计算

$$\int_0^t\int_0^{+\infty}\phi_{xx}\phi_{xxt}\mathrm{d}x\mathrm{d}t=\frac{1}{2}\int_0^{+\infty}\phi_{xx}^2\Big|_0^t\mathrm{d}x=\frac{1}{2}\int_0^{+\infty}\phi_{xx}^2\mathrm{d}x-\frac{1}{2}\int_0^{+\infty}\phi_{xx0}^2\mathrm{d}x. \tag{4.3.2}$$

$$-\int_0^t\int_0^{+\infty}\phi_{xx}\phi_{xxxx}\mathrm{d}x\mathrm{d}t=-\int_0^t\phi_{xx}\phi_{xxx}\Big|_0^{+\infty}\mathrm{d}t+\int_0^t\int_0^{+\infty}\phi_{xxx}^2\mathrm{d}x\mathrm{d}t=\int_0^t\int_0^{+\infty}\phi_{xxx}^2\mathrm{d}x\mathrm{d}t. \tag{4.3.3}$$

$$\begin{aligned}-\int_0^t\int_0^{+\infty}\phi_{xx}\phi_{xxxxt}\mathrm{d}x\mathrm{d}t&=-\int_0^t\phi_{xx}\phi_{xxxt}\Big|_0^{+\infty}\mathrm{d}t+\int_0^t\int_0^{+\infty}\phi_{xxx}\phi_{xxxt}\mathrm{d}x\mathrm{d}t\\&=\int_0^t\int_0^{+\infty}\phi_{xxx}\phi_{xxxt}\mathrm{d}x\mathrm{d}t=\frac{1}{2}\int_0^{+\infty}\phi_{xxx}^2\Big|_0^t\mathrm{d}x\\&=\frac{1}{2}\int_0^{+\infty}\phi_{xxx}^2\mathrm{d}x-\frac{1}{2}\int_0^{+\infty}\phi_{xxx0}^2\mathrm{d}x.\end{aligned} \tag{4.3.4}$$

将(4.3.2)~(4.3.4)代入到(4.3.1)中可得

$$\begin{aligned}&\frac{1}{2}\int_0^{+\infty}\phi_{xx}^2\mathrm{d}x+\frac{1}{2}\int_0^{+\infty}\phi_{xxx}^2\mathrm{d}x+\int_0^t\int_0^{+\infty}\phi_{xxx}^2\mathrm{d}x\mathrm{d}t\\&=\frac{1}{2}\int_0^{+\infty}\phi_{xxx0}^2\mathrm{d}x+\frac{1}{2}\int_0^{+\infty}\phi_{xx0}^2\mathrm{d}x-\int_0^t\int_0^{+\infty}\phi_{xx}f\left(\bar{u}+\phi\right)_{xxx}\mathrm{d}x\mathrm{d}t+\int_0^t\int_0^{+\infty}\phi_{xx}\bar{u}_{xxxxt}\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{4.3.5}$$

对等式(4.3.5)的右侧进行估计

$$\int_0^t\int_0^{+\infty}\phi_{xx}\bar{u}_{xxxxt}\mathrm{d}x\mathrm{d}t\le C\delta+C\delta\int_0^t\int_0^{+\infty}\phi_{xx}^2\mathrm{d}x\mathrm{d}t. \tag{4.3.6}$$

$$\begin{aligned}-\int_0^t\int_0^{+\infty}\phi_{xx}f\left(\bar{u}+\phi\right)_{xxx}\mathrm{d}x\mathrm{d}t&=-\int_0^t\phi_{xx}f\left(\bar{u}+\phi\right)_{xx}\Big|_0^{+\infty}\mathrm{d}t+\int_0^t\int_0^{+\infty}\phi_{xxx}f\left(\bar{u}+\phi\right)_{xx}\mathrm{d}x\mathrm{d}t\\&=\int_0^t\int_0^{+\infty}\phi_{xxx}f\left(\bar{u}+\phi\right)_{xx}\mathrm{d}x\mathrm{d}t\\&=\int_0^t\int_0^{+\infty}\phi_{xxx}\left[f'(\bar{u}+\phi)(\bar{u}_x+\phi_x)\right]_x\mathrm{d}x\mathrm{d}t\\&=\int_0^t\int_0^{+\infty}\phi_{xxx}f''(\bar{u}+\phi)(\bar{u}_x+\phi_x)^2\mathrm{d}x\mathrm{d}t\\&\quad+\int_0^t\int_0^{+\infty}\phi_{xxx}f'(\bar{u}+\phi)(\bar{u}_{xx}+\phi_{xx})\mathrm{d}x\mathrm{d}t\\&=I_1+I_2.\end{aligned} \tag{4.3.7}$$

$$\begin{aligned}I_1&=\int_0^t\int_0^{+\infty}\phi_{xxx}f''(\bar{u}+\phi)\bar{u}_x^2+\phi_{xxx}f''(\bar{u}+\phi)\phi_x^2+\phi_{xxx}f''(\bar{u}+\phi)2\bar{u}_x\phi_x\mathrm{d}x\mathrm{d}t\\&\quad-\int_0^t\int_0^{+\infty}f\left(\bar{u}\right)_{xx}\phi_{xxx}\mathrm{d}x\mathrm{d}t+\int_0^t\int_0^{+\infty}f\left(\bar{u}\right)_{xx}\phi_{xxx}\mathrm{d}x\mathrm{d}t\\&=\int_0^t\int_0^{+\infty}\phi_{xxx}\bar{u}_x^2f'''(\xi)\phi\mathrm{d}x\mathrm{d}t+\int_0^t\int_0^{+\infty}\left[\phi_{xxx}f''(\bar{u}+\phi)\phi_x^2+\phi_{xxx}f''(\bar{u}+\phi)2\bar{u}_x\phi_x\right]\mathrm{d}x\mathrm{d}t\\&\quad-\int_0^t\int_0^{+\infty}f'(\bar{u})\bar{u}_{xx}\phi_{xxx}\mathrm{d}x\mathrm{d}t+\int_0^t\int_0^{+\infty}f\left(\bar{u}\right)_{xx}\phi_{xxx}\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{4.3.8}$$

所以对其估计为

$$\begin{aligned} I_1 &= \int_0^t\int_0^{+\infty}\phi_{xxx}\bar{u}_x^2 f'''(\xi)\phi\mathrm{d}x\mathrm{d}t + \int_0^t\int_0^{+\infty}\left[\phi_{xxx} f''(\bar{u}+\phi)\phi_x^2 + \phi_{xxx} f''(\bar{u}+\phi)2\bar{u}_x\phi_x\right]\mathrm{d}x\mathrm{d}t \\ &\quad -\int_0^t\int_0^{+\infty} f'(\bar{u})\bar{u}_{xx}\phi_{xxx}\mathrm{d}x\mathrm{d}t + \int_0^t\int_0^{+\infty} f(\bar{u})_{xx}\phi_{xxx}\mathrm{d}x\mathrm{d}t \\ &\le C\delta + C\delta\int_0^t\int_0^{+\infty}\left(\phi_{xxx}^2+\phi_x^2+\phi^2\right)\mathrm{d}x\mathrm{d}t \end{aligned} \tag{4.3.9}$$

$$\begin{aligned} I_2 &= \int_0^t\int_0^{+\infty} f'(\bar{u}+\phi)(\bar{u}_{xx}+\phi_{xx})\mathrm{d}(\phi_{xx})\mathrm{d}t, \\ &= \int_0^t f'(\bar{u}+\phi)(\bar{u}_{xx}+\phi_{xx})\phi_{xx}\Big|_0^{+\infty}\mathrm{d}t - \int_0^t\int_0^{+\infty}\left[f'(\bar{u}+\phi)(\bar{u}_{xx}+\phi_{xx})\right]_x\phi_{xx}\mathrm{d}x\mathrm{d}t \\ &= -\int_0^t\int_0^{+\infty}\left[f'(\bar{u}+\phi)(\bar{u}_{xx}+\phi_{xx})\right]_x\phi_{xx}\mathrm{d}x\mathrm{d}t \\ &= -\int_0^t\int_0^{+\infty}\left[f''(\bar{u}+\phi)(\bar{u}_x+\phi_x)(\bar{u}_{xx}+\phi_{xx}) + f'(\bar{u}+\phi)(\bar{u}_{xxx}+\phi_{xxx})\right]\phi_{xx}\mathrm{d}x\mathrm{d}t \\ &= -\int_0^t\int_0^{+\infty}\left[f''(\bar{u}+\phi)(\bar{u}_x+\phi_x)\bar{u}_{xx} + f''(\bar{u}+\phi)\phi_x\phi_{xx} + f'(\bar{u}+\phi)(\bar{u}_{xxx}+\phi_{xxx})\right]\phi_{xx}\mathrm{d}x\mathrm{d}t \\ &\quad -\int_0^t\int_0^{+\infty} f''(\bar{u}+\phi)\bar{u}_x\phi_{xx}^2\mathrm{d}x\mathrm{d}t \end{aligned} \tag{4. 3.10}$$

将 $-\int_0^t\int_0^{+\infty} f''(\bar{u}+\phi)\bar{u}_x\phi_{xx}^2\mathrm{d}x\mathrm{d}t$ 移至(4.3.5)的左侧，则对(4.3.10)的估计为

$$\begin{aligned} I_2 &= -\int_0^t\int_0^{+\infty}\left[f''(\bar{u}+\phi)(\bar{u}_x+\phi_x)\bar{u}_{xx} + f''(\bar{u}+\phi)\phi_x\phi_{xx}\right]\phi_{xx}\mathrm{d}x\mathrm{d}t \\ &\quad + \int_0^t\int_0^{+\infty}\phi_{xx} f'(\bar{u}+\phi)(\bar{u}_{xxx}+\phi_{xxx})\mathrm{d}x\mathrm{d}t \\ &\le C\delta + C\delta\int_0^t\int_0^{+\infty}\left(\phi^2+\phi_x^2+\phi_{xx}^2+\phi_{xxx}^2\right)\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{4.3.11}$$

将(4.3.6) (4.3.10)以及(4.3.11)代入到(4.3.5)中可得

$$\begin{aligned} &\frac{1}{2}\int_0^{+\infty}\phi_{xx}^2\mathrm{d}x + \frac{1}{2}\int_0^{+\infty}\phi_{xxx}^2\mathrm{d}x + \int_0^t\int_0^{+\infty}\phi_{xxx}^2\mathrm{d}x\mathrm{d}t + \int_0^t\int_0^{+\infty} f''(\bar{u}+\phi)\bar{u}_x\phi_{xx}^2\mathrm{d}x\mathrm{d}t \\ &\le C\delta + C\delta\int_0^t\int_0^{+\infty}\left(\phi^2+\phi_x^2+\phi_{xx}^2+\phi_{xxx}^2\right)\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{4.3.12}$$

由于 $\bar{u}_x$ 是有界的，且 f 是光滑函数，所以

$$\int_0^t\int_0^{+\infty} f''(\bar{u}+\phi)\bar{u}_x\phi_{xx}^2\mathrm{d}x\mathrm{d}t \ge C\int_0^t\int_0^{+\infty}\phi_{xx}^2\mathrm{d}x\mathrm{d}t. \tag{4.3.13}$$

将(4.3.13)代入到(4.3.12)中，并且运用已得证的引理 4.1 和引理 4.2 可得

$$\int_0^{+\infty}\phi_{xx}^2\mathrm{d}x + \int_0^{+\infty}\phi_{xxx}^2\mathrm{d}x + \int_0^t\int_0^{+\infty}\phi_{xxx}^2\mathrm{d}x\mathrm{d}t + \int_0^t\int_0^{+\infty}\phi_{xx}^2\mathrm{d}x\mathrm{d}t \le C(\Phi_0+\delta). \tag{4.3.14}$$

则引理 4.3 证毕。

由此引理全部证明完毕，故先验估计得证。

参考文献

[1] Benjamin, T.B., Bona, J.L. and Mahony, J.J. (1972) Model Equations for Long Waves in Nonlinear Dispersive Systems. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 272, 47-78. https://doi.org/10.1098/rsta.1972.0032

[2] Tang, Y., Yan, R.-J. and Zhang, W.-K. (2000) Perturbative Influence on the Shockwave of Burgers Equation. *Chinese Physics Letters*, **17**, 157-158. https://doi.org/10.1088/0256-307X/17/3/001

[3] Yang, X., Ge, Y. and Bin, L. (2021) A Class of Compact Finite Difference Schemes for Solving the 2D and 3D Burgers' Equations. *Mathematics and Computers in Simulation*, **185**, 510-534. https://doi.org/10.1016/j.matcom.2021.01.009

[4] Zhao, H. and Admas, R.A. (2000) Existence and Convergence of Solutions for the Generalized BBM-Burgers Equations with Dissipative Term 2: The Multidimensional Case. *Applicable Analysis*, **75**, 107-135. https://doi.org/10.1080/00036810008840838

[5] Bisognin, V. and Menzala, G.P. (1994) Asymptotic Behaviour in Time of KdV Type Equations with Time Dependent Coefficients. *Applied Mathematics Letters*, **7**, 85-89. https://doi.org/10.1016/0893-9659(94)90099-X

[6] Sachdev, L.P., Enflo, O.B., Rao, S.C., *et al*. (2003) Large-Time Asymptotics for Periodic Solutions of Some Generalized Burgers Equations. *Studies in Applied Mathematics*, **110**, 181-204. https://doi.org/10.1111/1467-9590.00236

[7] Marti, J.-A. (2003) Nonlinear Algebraic Analysis of Delta Shock Wave Solutions to Burgers' Equation. *Pacific Journal of Mathematics*, **210**, 165-187. https://doi.org/10.2140/pjm.2003.210.165

[8] Öziş, T., Aksan, E. and Özdeş, A. (2003) A Finite Element Approach for Solution of Burgers' Equation. *Applied Mathematics and Computation*, **139**, 417-428. https://doi.org/10.1016/S0096-3003(02)00204-7

[9] Wang, W.-Q. (2004) A Class of Alternating Group Method of Burgers' Equation. *Applied Mathematics and Mechanics*, **25**, 236-244. https://doi.org/10.1007/BF02437325

[10] Abbasbandy, S. and Darvishi, M.T. (2004) A Numerical Solution of Burgers' Equation by Time Discretization of Adomian's Decomposition Method. *Applied Mathematics and Computation*, **170**, 95-102. https://doi.org/10.1016/j.amc.2004.10.060

[11] Glasman, E.A., Golovin, A.A. and Nepomnyashchy, A.A. (2004) Instabilities of Wavy Patterns Governed by Coupled Burgers Equations. *SIAM Journal on Applied Mathematics*, **65**, 230-251. https://doi.org/10.1137/S0036139903432913

[12] Mei, M. (1999) L_q-Decay Rates of Solutions for Benjamin-Bona-Mahony-Burgers Equations. *Journal of Differential Equations*, **158**, 314-340. https://doi.org/10.1006/jdeq.1999.3638

[13] 蒋咪娜, 徐艳玲. 广义 BBM-Burgers 方程初边值问题解的渐近行为[J]. 华中师范大学学报(自然科学版), 2004, 38(1): 5-9+17. https://doi.org/10.19603/J.Cnki.1000-1190.2004.01.002

[14] 陈琴, 刘艳. 带一般边界 BBM-Burgers 方程强边界层解的稳定性[J]. 哈尔滨商业大学学报(自然科学版), 2012, 28(2): 205-209. https://doi.org/10.19492/J.Cnki.1672-0946.2012.02.020

[15] 李志深. 高维广义 BBM 方程组的初边值问题[J]. 应用数学, 1990(4): 71-80.

[16] Xu, H. and Yang, X. (2018) Global Existence and Temporal Decay Estimate to the Generalized BBM-Burgers Equation with Dissipative Term. *Wuhan University Journal of Natural Sciences*, **23**, 475-479. https://doi.org/10.1007/s11859-018-1351-9

[17] Yin, H. and Zhao, H. (2009) Nonlinear Stability of Boundary Layer Solutions Forgeneralized Benjamin-Bona-Mahony-Burgers Equation in the Half Space. *Kinetic and Related Models*, **2**, 521-550. https://doi.org/10.3934/krm.2009.2.521

[18] Wu, G. (2017) Existence and Analytic Regularity of Certain Solutions for the Generalized BBM-Burgers Equation in R^n. *Annales Polonici Mathematici*, **119**, 69-78. https://doi.org/10.4064/ap3995-3-2017

[19] Li, T. and Yin, H. (2013) Convergence Rate to Strong Boundary Layer Solutions Forgeneralized BBM-Burgers Equations with Non-Convex Flux. *Communications on Pure Applied Analysis*, **13**, 835-858. https://doi.org/10.3934/cpaa.2014.13.835

[20] Hayashi, N., Kaikina, E.I. and Naumkin, P.I. (2007) Large Time Asymptotics for the BBM-Burgers Equation. *Annales Henri Poincaré*, **8**, 485-511. https://doi.org/10.1007/s00023-006-0314-4

[21] Brezis, H. and Mironescu, P. (2001) Composition in Fractional Sobolev Spaces. *Discrete & Continuous Dynamical Systems*, **7**, 241-246. https://doi.org/10.3934/dcds.2001.7.241

[22] Demengel, F. and Demengel, G. (2012) Functional Spaces for the Theory of Elliptic Partial Differential Equations. Springer, London. https://doi.org/10.1007/978-1-4471-2807-6

[23] Di Nezza, E., Palatucci, G. and Valdinoci, E. (2012) Hitchhiker's Guide to the Fractional Sobolev Spaces. *Bulletin des Sciences Mathematiques*, **136**, 521-573. https://doi.org/10.1016/j.bulsci.2011.12.004