

Bismut Ricci 平坦双扭曲积埃尔米特流形

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摘要

设 (M_1, g) 和 (M_2, h) 是两个埃尔米特流形, 双扭曲积埃尔米特流形 $(f_2 M_1 \times_{f_1} M_2, G)$ 是赋予了扭曲积埃尔米特度量 $G = f_2^2 g + f_1^2 h$ 的乘积流形 $M_1 \times M_2$, 其中 f_1 和 f_2 分别是 M_1 和 M_2 上的正值光滑函数。本文给出双扭曲积埃尔米特流形的 Bismut 联络, Bismut 曲率, Bismut Ricci 曲率和 Bismut 标量曲率的表达式, 并得到双扭曲积埃尔米特流形 Bismut Ricci 平坦的充要条件, 从而给出构造 Bismut Ricci 平坦埃尔米特流形的有效方法。

关键词

埃尔米特流形, 双扭曲积, Bismut 联络, Bismut Ricci 平坦

Bismut Ricci Flat Doubly Warped Product Hermitian Manifold

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Abstract

Let (M_1, g) and (M_2, h) be two Hermitian manifolds, the doubly warped product Hermitian manifold of $(f_2 M_1 \times_{f_1} M_2, G)$ is the product manifold $M_1 \times M_2$ endowed with the warped product Hermitian metric $G = f_2^2 g + f_1^2 h$, where f_1 and f_2 are positive smooth functions on M_1 and M_2 , respectively. In this paper, we obtain the formulae of Bismut connection, Bismut curvature, Bismut Ricci curvature and Bismut scalar curvature of doubly warped product Hermitian manifold. The necessary and sufficient condition for doubly warped product Hermitian manifold to be Bismut Ricci flat is given. Thus, we provide an effective way to construct Bismut Ricci flat Hermitian manifold.

Keywords

Hermitian Manifold, Doubly Warped Product, Bismut Connection, Bismut Ricci Flat

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1. 引言

扭曲积是构造具有特殊曲率流形的重要几何模型. 1969年, Bishop 和 O'Neil 为了构造一类具有负截面曲率的黎曼流形引入扭曲积的概念[1]. 2001年, Kozma, Peter 和 Varga 把扭曲积的概念推广到了实芬斯勒几何中, 并研究了扭曲积芬斯勒流形与分量流形上嘉当联络之间的关系[2]. 2016年, 何勇和钟春平在复芬斯勒几何中引入了扭曲积, 得到了双扭曲积复芬斯勒流形是复贝瓦尔德流形的充要条件[3]. 2018年, 何勇和张晓玲又把扭曲积的概念推广到了复几何中, 研究了双扭曲积埃尔米特流形[4]. 2022年, 倪琪慧和何勇等人给出了双扭曲积埃尔米特流形上 Levi-Civita 联络、Levi-Civita Ricci 曲率、Levi-Civita Ricci 曲率和 Levi-Civita 标量曲率的局部表达式, 得到了双扭曲积埃尔米特流形 Levi-Civita Ricci 平坦当且仅当其分量流形 Levi-Civita Ricci 平坦[5].

在复几何中, 陈联络 ∇^C , Levi-Civita 联络 ∇^{LC} 和 Bismut 联络 ∇^B 是非常经典的三种联络, 彼此之间存在紧密的联系. 当埃尔米特流形是凯勒流形时, 上述的三种联络退化为同一种联络. 1986年, Bismut 和 Freed 给出了 Bismut 联络的定义[6]. 1989年, Bismut 研究了 Bismut 联络的存在

性和唯一性[7]. 2020年, 王青松等人对所有具有平坦 Bismut 联络且紧致的埃尔米特流形进行了分类[8]. 2022年, 赵全庭和郑方阳研究了 Bismut 联络具有平行挠率张量的埃尔米特流形[9]. 2023年, 他们还给出了 4 维和 5 维 Bismut Kähler-like 埃尔米特流形的完整分类[10]. 同年, Barbaro 研究了与 Bismut 标量曲率有关的埃尔米特流形的 Yamabe 问题[11].

众所周知在微分几何中对不同的 Ricci 平坦流形分类是一个非常重要的研究内容. 1967年, Tani 首次在黎曼几何中提出了 Ricci 平坦的概念[12]. 2014年, 刘克峰和杨晓奎系统地研究了埃尔米特流形上的陈联络, Levi-Civita 联络和 Bismut 联络, 并给出了各联络所诱导的曲率之间的特殊性质[13]. 2023年, 赵全庭和郑方阳证明了 Bismut Ricci 平坦的 Bismut Kähler-like 埃尔米特流形一定是 Bismut 平坦[10]. 上述研究结果表明, 近几年, 众多学者对 Bismut Ricci 平坦埃尔米特流形产生了浓厚兴趣, 并得到了许多重要的研究成果. 因此, 研究 Bismut Ricci 平坦埃尔米特流形是非常有必要的.

受以上研究的启发, 我们想知道双扭曲积埃尔米特流形是否是 Bismut Ricci 平坦的? 因此, 本文将研究双扭曲积埃尔米特流形的 Bismut Ricci 平坦性.

2. 预备知识

设 (M, J, G) 是一个复 n 维埃尔米特流形, J 为复结构, G 为埃尔米特度量. 对于流形 M 上任意一个点 p , 复化切丛可分解为

$$T_p^{\mathbb{C}}M = T_p^{1,0}M \oplus T_p^{0,1}M,$$

其中 J 的特征值分别为 $\pm\sqrt{-1}$.

在局部全纯坐标系 $z = (z^1, z^2, \dots, z^n)$ 下, $T_p^{1,0}M$ 和 $T_p^{0,1}M$ 分别由 $(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^2}, \dots, \frac{\partial}{\partial z^n})$ 和 $(\frac{\partial}{\partial \bar{z}^1}, \frac{\partial}{\partial \bar{z}^2}, \dots, \frac{\partial}{\partial \bar{z}^n})$ 张成, 分别简记为 $\{\frac{\partial}{\partial z^\alpha}\}$ 和 $\{\frac{\partial}{\partial \bar{z}^\alpha}\}$. 它们的对偶余标架场分别为 $\{dz^\alpha\}$ 和 $\{d\bar{z}^\alpha\}$ [14].

流形 M 上的一个埃尔米特度量 G 在局部坐标下表示为 $G = G_{\alpha\bar{\beta}}dz^\alpha \otimes d\bar{z}^\beta$, 其中矩阵 $G_{\alpha\bar{\beta}}$ 是正定的, 用 $G^{\bar{\beta}\gamma}$ 表示 $G_{\alpha\bar{\beta}}$ 的逆矩阵[15]. 本文采用爱因斯坦求和约定, 即 $G^{\bar{\beta}\gamma}G_{\alpha\bar{\beta}} = \delta_\alpha^\gamma$.

众所周知 ∇^B 是埃尔米特流形上唯一具有完全斜对称挠率的联络[9]. 对于 $\forall X, Y, Z \in T^{1,0}M$, Bismut 联络 ∇^B 表示为[16]

$$G(\nabla_X^B Y, Z) = G(\nabla_X^{LC} Y, Z) + \frac{1}{2}d\omega(JX, JY, JZ),$$

其中 Bismut 联络的挠率为[16]

$$H(X, Y, Z) = G(T^B(X, Y), Z) = \frac{1}{2}d\omega(JX, JY, JZ).$$

若埃尔米特流形的挠率是闭的, 即满足 $\partial\bar{\partial}\omega = 0$, 则称埃尔米特流形是多重闭流形[13]. 在局部全纯切丛上, Bismut 联络 ∇^B 的系数为[11]

$${}^b\Gamma_{\gamma\alpha}^{\beta} = G^{\bar{\delta}\beta} \frac{\partial G_{\alpha\bar{\delta}}}{\partial z^{\gamma}}, \quad (2.1)$$

$${}^b\Gamma_{\gamma\alpha}^{\beta} = G^{\bar{\delta}\beta} \left(\frac{\partial G_{\alpha\bar{\delta}}}{\partial z^{\gamma}} - \frac{\partial G_{\alpha\bar{\gamma}}}{\partial \bar{z}^{\delta}} \right). \quad (2.2)$$

Bismut 曲率张量为[13]

$$B(X, Y)Z = \nabla_X^B \nabla_Y^B Z - \nabla_Y^B \nabla_X^B Z - \nabla_{[X, Y]}^B Z,$$

$$B(X, Y, Z, W) = G(B(X, Y)Z, W),$$

Bismut 曲率系数分别为

$$B_{\alpha\bar{\gamma}\delta}^{\epsilon} = -\frac{\partial {}^b\Gamma_{\alpha\delta}^{\epsilon}}{\partial \bar{z}^{\gamma}} + \frac{\partial {}^b\Gamma_{\bar{\gamma}\delta}^{\epsilon}}{\partial z^{\alpha}} - {}^b\Gamma_{\alpha\delta}^{\sigma} {}^b\Gamma_{\bar{\gamma}\sigma}^{\epsilon} + {}^b\Gamma_{\bar{\gamma}\delta}^{\sigma} {}^b\Gamma_{\alpha\sigma}^{\epsilon}, \quad (2.3)$$

$$B_{\alpha\bar{\gamma}\delta\bar{\beta}} = G_{\epsilon\bar{\beta}} B_{\alpha\bar{\gamma}\delta}^{\epsilon}. \quad (2.4)$$

第一 Bismut Ricci 曲率张量为[13]

$$B^{(1)} = \sqrt{-1} B_{\alpha\bar{\beta}}^{(1)} dz^{\alpha} \wedge d\bar{z}^{\beta},$$

其中

$$B_{\alpha\bar{\gamma}}^{(1)} = G^{\bar{\beta}\delta} B_{\alpha\bar{\gamma}\delta\bar{\beta}}. \quad (2.5)$$

Bismut 标量曲率 B 为[11]

$$B = G^{\bar{\gamma}\alpha} B_{\alpha\bar{\gamma}}^{(1)}. \quad (2.6)$$

定义1. [17] 若埃尔米特流形 (M, G) 的第一 Bismut Ricci 曲率为0, 即 $B_{\alpha\bar{\gamma}}^{(1)} = 0$, 则称其 Bismut Ricci 平坦.

设 (M_1, g) 和 (M_2, h) 分别是复 m 维和复 n 维的埃尔米特流形, 则 $M = M_1 \times M_2$ 是一个复 $m+n$ 维的埃尔米特流形. 设 $\pi_1 : M \rightarrow M_1$ 和 $\pi_2 : M \rightarrow M_2$ 为自然投影. 对任意的 $z = (z_1, z_2) \in M$, $z_1 = (z^1, \dots, z^m) \in M_1$ 和 $z_2 = (z^{m+1}, \dots, z^{m+n}) \in M_2$, 有 $\pi_1(z) = z_1$ 和 $\pi_2(z) = z_2$ 成立.

设 $d\pi_1 : T^{1,0}(M) \rightarrow T^{1,0}M_1$, $d\pi_2 : T^{1,0}(M) \rightarrow T^{1,0}M_2$ 分别是由 π_1 和 π_2 诱导的全纯切映射. 对于任意的 $v = (v_1, v_2) \in T_z^{1,0}(M)$, $v_1 = (v^1, \dots, v^m) \in T_{z_1}^{1,0}M_1$, $v_2 = (v^{m+1}, \dots, v^{m+n}) \in T_{z_2}^{1,0}M_2$, 有 $d\pi_1(z, v) = (z_1, v_1)$ 和 $d\pi_2(z, v) = (z_2, v_2)$ 成立.

定义2. [4] 设 (M_1, g) 和 (M_2, h) 是两个埃尔米特流形. 设 $f_1 : M_1 \rightarrow (0, +\infty)$ 和 $f_2 : M_2 \rightarrow (0, +\infty)$ 是两个光滑函数. 双扭曲积埃尔米特流形 $(f_2 M_1 \times_{f_1} M_2, G)$ 是赋予了如下埃尔米特度量 $G : M \rightarrow (0, +\infty)$ 的乘积流形 $M = M_1 \times M_2$:

$$G(z, v) = (f_2 \circ \pi_2)^2(z)g(\pi_1(z), d\pi_1(v)) + (f_1 \circ \pi_1)^2(z)h(\pi_2(z), d\pi_2(v)), \quad (2.7)$$

其中 $z = (z_1, z_2) \in M$, $v = (v_1, v_2) \in T_z^{1,0}M$, f_1 和 f_2 被称为扭曲函数. (M_1, g) 和 (M_2, h) 被称为 $(f_2 M_1 \times_{f_1} M_2, G)$ 的分量流形.

若 $f_1 \equiv 1$ 与 $f_2 \equiv 1$ 有且仅有一个成立, 则称 $(f_2 M_1 \times_{f_1} M_2, G)$ 是单扭曲积埃尔米特流形; 若 $f_1 \equiv 1$ 且 $f_2 \equiv 1$ 都成立, 则称 $(f_2 M_1 \times_{f_1} M_2, G)$ 是乘积埃尔米特流形; 若 f_1 和 f_2 都不为常数, 则称 $(f_2 M_1 \times_{f_1} M_2, G)$ 是非平凡的双扭曲积埃尔米特流形.

在本文中, 对小写希腊字母指标, 小写拉丁字母指标, 带撇号的小写拉丁字母指标的取值范围约定如下: $1 \leq \alpha, \beta, \gamma, \delta, \epsilon, \sigma \leq m+n$, $1 \leq i, j, k, s, t, p, l \leq m$ 和 $m+1 \leq i', j', k', s', t', p', l' \leq m+n$. 与 (M_1, g) 和 (M_2, h) 有关的几何量, 分别在其上方加指标 1 和 2 以示区别, 如 ${}^1\Gamma_{jk}^i$ 和 ${}^2\Gamma_{j'k'}^{i'}$ 分别表示埃尔米特流形 (M_1, g) 和 (M_2, h) 上的 Bismut 联络系数.

设 $(f_2 M_1 \times_{f_1} M_2, G)$ 是埃尔米特流形 (M_1, g) 和 (M_2, h) 的双扭曲积, 记

$$g_{i\bar{j}} = \frac{\partial^2 g}{\partial v^i \partial \bar{v}^j}, \quad h_{i'\bar{j}'} = \frac{\partial^2 h}{\partial v^{i'} \partial \bar{v}^{j'}}. \quad (2.8)$$

则 G 的基本张量矩阵为[4]

$$(G_{\alpha\bar{\beta}}) = \left(\frac{\partial^2 G}{\partial v^\alpha \partial \bar{v}^\beta} \right) = \begin{pmatrix} f_2^2 g_{i\bar{j}} & 0 \\ 0 & f_1^2 h_{i'\bar{j}'} \end{pmatrix}, \quad (2.9)$$

其逆矩阵 $(G^{\bar{\beta}\alpha})$ 为[4]

$$(G^{\bar{\beta}\alpha}) = \begin{pmatrix} f_2^{-2} g^{\bar{j}i} & 0 \\ 0 & f_1^{-2} h^{\bar{j}'i'} \end{pmatrix}. \quad (2.10)$$

3. 双扭曲积埃尔米特流形的 Bismut 标量曲率

本节将给出双扭曲积埃尔米特流形的 Bismut 联络, Bismut 曲率, Bismut Ricci 曲率以及 Bismut 标量曲率的局部表达式.

命题1. 设 $(f_2 M_1 \times_{f_1} M_2, G)$ 是一个双扭曲积埃尔米特流形, 则 Bismut 联络 ${}^b\Gamma_{\gamma\alpha}^\beta$ 系数为

$${}^b\Gamma_{jk}^i = {}^1\Gamma_{jk}^i, \quad (3.1)$$

$${}^b\Gamma_{j'k'}^{i'} = {}^2\Gamma_{j'k'}^{i'}, \quad (3.2)$$

$${}^b\Gamma_{j'k}^i = 2f_2^{-1} \frac{\partial f_2}{\partial z^{j'}} \delta_k^i, \quad (3.3)$$

$${}^b\Gamma_{jk'}^{i'} = 2f_1^{-1} \frac{\partial f_1}{\partial z^j} \delta_{k'}^{i'}, \quad (3.4)$$

$${}^b\Gamma_{jk}^{i'} = {}^b\Gamma_{j'k'}^i = {}^b\Gamma_{jk'}^i = {}^b\Gamma_{j'k}^{i'} = 0. \quad (3.5)$$

证明. 令 (2.1) 中的 $\beta = i, \gamma = j$ 和 $\alpha = k$, 有

$${}^b\Gamma_{jk}^i = G^{\delta i} \frac{\partial G_{k\bar{\delta}}}{\partial z^j} = G^{\bar{s}i} \frac{\partial G_{k\bar{s}}}{\partial z^j} + G^{\bar{s}'i} \frac{\partial G_{k\bar{s}'}}{\partial z^j}. \quad (3.6)$$

将 (2.9) 和 (2.10) 代入 (3.6) 中, 并根据 (2.1) 得

$${}^b\Gamma_{jk}^i = f_2^{-2} g^{\bar{s}i} \frac{\partial f_2^2 g_{k\bar{s}}}{\partial z^j} = g^{\bar{s}i} \frac{\partial g_{k\bar{s}}}{\partial z^j} = {}^b\Gamma_{jk}^i.$$

同理可得命题1中的其余等式. □

根据 (2.2), (2.9) 和 (2.10) 易得下面的命题2.

命题2. 设 $(f_2 M_1 \times_{f_1} M_2, G)$ 是一个双扭曲积埃尔米特流形, 则 Bismut 联络系数 ${}^b\Gamma_{\gamma\alpha}^\beta$ 为

$${}^b\Gamma_{jk}^i = {}^b\Gamma_{jk}^i, \quad (3.7)$$

$${}^b\Gamma_{j'k'}^{i'} = {}^b\Gamma_{j'k'}^{i'}, \quad (3.8)$$

$${}^b\Gamma_{j'k}^i = 2f_2^{-1} \frac{\partial f_2}{\partial \bar{z}^{j'}} \delta_k^i, \quad (3.9)$$

$${}^b\Gamma_{jk'}^{i'} = 2f_1^{-1} \frac{\partial f_1}{\partial \bar{z}^j} \delta_{k'}^{i'}, \quad (3.10)$$

$${}^b\Gamma_{jk}^{i'} = -2f_1^{-2} f_2 h^{\bar{s}'i'} \frac{\partial f_2}{\partial \bar{z}^{s'}} g_{k\bar{j}}, \quad (3.11)$$

$${}^b\Gamma_{j'k'}^i = -2f_2^{-2} f_1 g^{\bar{s}i} \frac{\partial f_1}{\partial \bar{z}^s} h_{k'\bar{j}'}, \quad (3.12)$$

$${}^b\Gamma_{jk'}^i = {}^b\Gamma_{j'k}^{i'} = 0. \quad (3.13)$$

命题3. 设 $(f_2 M_1 \times_{f_1} M_2, G)$ 是一个双扭曲积埃尔米特流形, 则 Bismut 曲率系数 $B_{\alpha\gamma\delta}^\epsilon$ 为

$$B_{j\bar{i}k}^i = B_{j\bar{i}k}^i, \quad (3.14)$$

$$B_{j'\bar{i}'k'}^{i'} = B_{j'\bar{i}'k'}^{i'}, \quad (3.15)$$

$$B_{j'\bar{i}'k}^i = 2f_2^{-2} g^{\bar{s}i} \frac{\partial f_1}{\partial \bar{z}^s} h_{k'\bar{i}'} \frac{\partial f_1}{\partial z^j} - 2f_2^{-2} f_1 \left(\frac{\partial g^{\bar{s}i}}{\partial z^j} \frac{\partial f_1}{\partial \bar{z}^s} + g^{\bar{s}i} \frac{\partial^2 f_1}{\partial \bar{z}^s \partial z^j} + g^{\bar{s}l} \frac{\partial f_1}{\partial \bar{z}^s} g^{\bar{s}i} \frac{\partial g_{l\bar{s}}}{\partial z^j} \right) h_{k'\bar{i}'}, \quad (3.16)$$

$$B_{j'\bar{i}k}^{i'} = 2f_1^{-2} \frac{\partial f_2}{\partial z^{j'}} \frac{\partial f_2}{\partial \bar{z}^{s'}} h^{\bar{s}'i'} g_{k\bar{i}} - 2f_1^{-2} f_2 \left(\frac{\partial h^{\bar{s}'i'}}{\partial z^{j'}} \frac{\partial f_2}{\partial \bar{z}^{s'}} + h^{\bar{s}'i'} \frac{\partial^2 f_2}{\partial \bar{z}^{s'} \partial z^{j'}} \right) + h^{\bar{s}'l'} \frac{\partial f_2}{\partial \bar{z}^{s'}} h^{\bar{s}'i'} \frac{\partial h_{l'\bar{s}'}}{\partial z^{j'}} g_{k\bar{i}}, \quad (3.17)$$

$$B_{j'\bar{i}k}^i = B_{j'\bar{i}'k}^{i'} = B_{j\bar{i}k'}^i = B_{j'\bar{i}'k'}^{i'} = B_{j'\bar{i}k}^i = B_{j'\bar{i}'k'}^{i'} = 0, \quad (3.18)$$

$$B_{j\bar{i}k}^i = B_{j'\bar{i}'k}^{i'} = B_{j\bar{i}k'}^i = B_{j'\bar{i}'k'}^{i'} = B_{j'\bar{i}k}^i = B_{j'\bar{i}'k'}^{i'} = 0. \quad (3.19)$$

证明. 首先证明 (3.14) 式. 令 (2.3) 中 $\epsilon = i, \alpha = j, \gamma = t$ 和 $\delta = k$, 有

$$\begin{aligned} B_{j\bar{t}k}^i &= -\frac{\partial {}^b\Gamma_{jk}^i}{\partial \bar{z}^t} + \frac{\partial {}^b\Gamma_{\bar{t}k}^i}{\partial z^j} - {}^b\Gamma_{jk}^\sigma {}^b\Gamma_{t\sigma}^i + {}^b\Gamma_{\bar{t}k}^\sigma {}^b\Gamma_{j\sigma}^i \\ &= -\frac{\partial {}^b\Gamma_{jk}^i}{\partial \bar{z}^t} + \frac{\partial {}^b\Gamma_{\bar{t}k}^i}{\partial z^j} - {}^b\Gamma_{jk}^l {}^b\Gamma_{\bar{t}l}^i - {}^b\Gamma_{jk}^{l'} {}^b\Gamma_{\bar{t}l'}^i + {}^b\Gamma_{\bar{t}k}^l {}^b\Gamma_{jl}^i + {}^b\Gamma_{\bar{t}k}^{l'} {}^b\Gamma_{j'l'}^i. \end{aligned} \quad (3.20)$$

将 (3.1), (3.5), (3.7), (3.11) 和 (3.13) 代入 (3.20), 并根据 (2.3) 可得

$$\begin{aligned} B_{j\bar{t}k}^i &= -\frac{\partial {}^1\Gamma_{jk}^i}{\partial \bar{z}^t} + \frac{\partial {}^1\Gamma_{\bar{t}k}^i}{\partial z^j} - {}^1\Gamma_{jk}^l {}^1\Gamma_{\bar{t}l}^i + {}^1\Gamma_{\bar{t}k}^l {}^1\Gamma_{jl}^i \\ &= {}^1B_{j\bar{t}k}^i. \end{aligned}$$

同理可得 (3.15) 式.

接下来证明 (3.16) 式. 令 (2.3) 中 $\epsilon = i, \alpha = j, \gamma = t'$ 和 $\delta = k'$, 有

$$\begin{aligned} B_{j\bar{t}'k'}^i &= -\frac{\partial {}^b\Gamma_{jk'}^i}{\partial \bar{z}^{t'}} + \frac{\partial {}^b\Gamma_{\bar{t}'k'}^i}{\partial z^j} - {}^b\Gamma_{jk'}^\sigma {}^b\Gamma_{\bar{t}'\sigma}^i + {}^b\Gamma_{\bar{t}'k'}^\sigma {}^b\Gamma_{j\sigma}^i \\ &= -\frac{\partial {}^b\Gamma_{jk'}^i}{\partial \bar{z}^{t'}} + \frac{\partial {}^b\Gamma_{\bar{t}'k'}^i}{\partial z^j} - {}^b\Gamma_{jk'}^l {}^b\Gamma_{\bar{t}'l}^i - {}^b\Gamma_{jk'}^{l'} {}^b\Gamma_{\bar{t}'l'}^i + {}^b\Gamma_{\bar{t}'k'}^l {}^b\Gamma_{jl}^i + {}^b\Gamma_{\bar{t}'k'}^{l'} {}^b\Gamma_{j'l'}^i. \end{aligned} \quad (3.21)$$

将 (3.1), (3.4), (3.5), (3.8), (3.9) 和 (3.12) 代入 (3.21), 可得

$$\begin{aligned} B_{j\bar{t}'k'}^i &= \frac{\partial(-2f_2^{-2}f_1g^{\bar{s}i}\frac{\partial f_1}{\partial \bar{z}^s}h_{k'\bar{t}'})}{\partial z^j} - 2f_1^{-1}\frac{\partial f_1}{\partial z^{j'}}\delta_{k'}^{l'} \cdot (-2f_2^{-2}f_1g^{\bar{s}i}\frac{\partial f_1}{\partial \bar{z}^s}h_{l'\bar{t}'}) \\ &\quad - 2f_2^{-2}f_1g^{\bar{s}l}\frac{\partial f_1}{\partial \bar{z}^s}h_{k'\bar{t}'} \cdot g^{\bar{s}i}\frac{\partial g_{l\bar{s}}}{\partial z^j} \\ &= -2f_2^{-2}\frac{\partial f_1}{\partial z^j}g^{\bar{s}i}\frac{\partial f_1}{\partial \bar{z}^s}h_{k'\bar{t}'} - 2f_2^{-2}f_1\frac{\partial g^{\bar{s}i}}{\partial z^j}\frac{\partial f_1}{\partial \bar{z}^s}h_{k'\bar{t}'} - 2f_2^{-2}f_1g^{\bar{s}i}\frac{\partial^2 f_1}{\partial \bar{z}^s\partial z^j}h_{k'\bar{t}'} \\ &\quad + 4f_2^{-2}\frac{\partial f_1}{\partial z^j}g^{\bar{s}i}\frac{\partial f_1}{\partial \bar{z}^s}h_{k'\bar{t}'} - 2f_2^{-2}f_1g^{\bar{s}l}\frac{\partial f_1}{\partial \bar{z}^s}g^{\bar{s}i}\frac{\partial g_{l\bar{s}}}{\partial z^j}h_{k'\bar{t}'} \\ &= 2f_2^{-2}\frac{\partial f_1}{\partial \bar{z}^s}g^{\bar{s}i}\frac{\partial f_1}{\partial z^j}h_{k'\bar{t}'} - 2f_2^{-2}f_1\left(\frac{\partial g^{\bar{s}i}}{\partial z^j}\frac{\partial f_1}{\partial \bar{z}^s} + g^{\bar{s}i}\frac{\partial^2 f_1}{\partial \bar{z}^s\partial z^j} + g^{\bar{s}l}\frac{\partial f_1}{\partial \bar{z}^s}g^{\bar{s}i}\frac{\partial g_{l\bar{s}}}{\partial z^j}\right)h_{k'\bar{t}'}. \end{aligned}$$

同理可得 (3.17) 式.

再证明 (3.18) 的第一个等式. 令 (2.3) 中 $\epsilon = i, \alpha = j', \gamma = t$ 和 $\delta = k$, 有

$$\begin{aligned} B_{j'\bar{t}k}^i &= -\frac{\partial {}^b\Gamma_{j'k}^i}{\partial \bar{z}^t} + \frac{\partial {}^b\Gamma_{\bar{t}k}^i}{\partial z^{j'}} - {}^b\Gamma_{j'k}^\sigma {}^b\Gamma_{\bar{t}\sigma}^i + {}^b\Gamma_{\bar{t}k}^\sigma {}^b\Gamma_{j'\sigma}^i \\ &= -\frac{\partial {}^b\Gamma_{j'k}^i}{\partial \bar{z}^t} + \frac{\partial {}^b\Gamma_{\bar{t}k}^i}{\partial z^{j'}} - {}^b\Gamma_{j'k}^l {}^b\Gamma_{\bar{t}l}^i - {}^b\Gamma_{j'k}^{l'} {}^b\Gamma_{\bar{t}l'}^i + {}^b\Gamma_{\bar{t}k}^l {}^b\Gamma_{j'l}^i + {}^b\Gamma_{\bar{t}k}^{l'} {}^b\Gamma_{j'l'}^i. \end{aligned} \quad (3.22)$$

将 (3.3), (3.5), (3.7), (3.11), (3.12) 和 (3.13) 代入 (3.22), 可得

$$\begin{aligned} B_{j\bar{i}k}^i &= -\frac{\partial(2f_2^{-1}\frac{\partial f_2}{\partial z^{j'}}\delta_k^i)}{\partial \bar{z}^t} + \frac{\partial[g^{\bar{s}i}(\frac{\partial g_{k\bar{s}}}{\partial \bar{z}^t} - \frac{\partial g_{k\bar{t}}}{\partial \bar{z}^s})]}{\partial \bar{z}^{j'}} - (2f_2^{-1}\frac{\partial f_2}{\partial z^{j'}}\delta_k^i) \cdot [g^{\bar{s}i}(\frac{\partial g_{l\bar{s}}}{\partial \bar{z}^t} - \frac{\partial g_{l\bar{t}}}{\partial \bar{z}^s})] \\ &\quad + [g^{\bar{s}l}(\frac{\partial g_{k\bar{s}}}{\partial \bar{z}^t} - \frac{\partial g_{k\bar{t}}}{\partial \bar{z}^s})] \cdot (2f_2^{-1}\frac{\partial f_2}{\partial z^{j'}}\delta_l^i) \\ &= -2f_2^{-1}\frac{\partial f_2}{\partial z^{j'}}g^{\bar{s}i}(\frac{\partial g_{k\bar{s}}}{\partial \bar{z}^t} - \frac{\partial g_{k\bar{t}}}{\partial \bar{z}^s}) + 2f_2^{-1}\frac{\partial f_2}{\partial z^{j'}}g^{\bar{s}i}(\frac{\partial g_{k\bar{s}}}{\partial \bar{z}^t} - \frac{\partial g_{k\bar{t}}}{\partial \bar{z}^s}) \\ &= 0 \end{aligned}$$

同理可得 (3.18) 中的其余等式以及 (3.19) 中各式. \square

注1. f_1 是流形 M_1 上的正值光滑函数, 故 f_1 只能对 z^j 求偏导数; 同理, f_2 只能对 $z^{j'}$ 求偏导数. $g_{k\bar{s}}$ 是 M_1 上的埃尔米特度量, 故 $g_{k\bar{s}}$ 只能对 z^j 求偏导数; 同理, $h_{k'\bar{s}'}$ 只能对 $z^{j'}$ 求偏导数.

命题4. 设 $(f_2 M_1 \times_{f_1} M_2, G)$ 是一个双扭曲积埃尔米特流形, 则 Bismut 曲率系数 $B_{\alpha\bar{\gamma}\delta\bar{\beta}}$ 为

$$B_{j\bar{i}k\bar{p}} = f_2^2 B_{j\bar{i}k\bar{p}}^1, \quad (3.23)$$

$$B_{j'\bar{i}'k'\bar{p}'} = f_1^2 B_{j'\bar{i}'k'\bar{p}'}^2, \quad (3.24)$$

$$B_{j\bar{i}'k'\bar{p}} = 2\frac{\partial f_1}{\partial \bar{z}^p}\frac{\partial f_1}{\partial z^j}h_{k'\bar{i}'} - 2f_1(g_{i\bar{p}}\frac{\partial g^{\bar{s}i}}{\partial z^j}\frac{\partial f_1}{\partial \bar{z}^s} + \frac{\partial^2 f_1}{\partial \bar{z}^p\partial z^j} + g^{\bar{p}i}\frac{\partial f_1}{\partial \bar{z}^p}\frac{\partial g_{i\bar{p}}}{\partial z^j})h_{k'\bar{i}'}, \quad (3.25)$$

$$B_{j'\bar{i}k\bar{p}'} = 2\frac{\partial f_2}{\partial z^{j'}}\frac{\partial f_2}{\partial \bar{z}^{p'}}g_{k\bar{i}} - 2f_2(h_{i'\bar{p}'}\frac{\partial h^{\bar{s}'i'}}{\partial z^{j'}}\frac{\partial f_2}{\partial \bar{z}^{s'}} + \frac{\partial^2 f_2}{\partial \bar{z}^{p'}\partial z^{j'}} + h^{\bar{p}'i'}\frac{\partial f_2}{\partial \bar{z}^{p'}}\frac{\partial h_{i'\bar{p}'}}{\partial z^{j'}})g_{k\bar{i}}, \quad (3.26)$$

$$B_{j\bar{i}k\bar{p}} = B_{j\bar{i}'k'\bar{p}} = B_{j\bar{i}k\bar{p}} = B_{j\bar{i}k\bar{p}'} = B_{j'\bar{i}'k'\bar{p}'} = B_{j\bar{i}k\bar{p}'} = 0, \quad (3.27)$$

$$B_{j'\bar{i}k\bar{p}'} = B_{j\bar{i}'k'\bar{p}} = B_{j'\bar{i}'k'\bar{p}'} = B_{j\bar{i}k\bar{p}'} = B_{j'\bar{i}'k'\bar{p}'} = B_{j\bar{i}'k'\bar{p}} = 0. \quad (3.28)$$

证明. 首先证明 (3.23) 式. 令 (2.4) 中 $\alpha = j, \gamma = t, \delta = k$ 和 $\beta = p$, 则有

$$B_{j\bar{i}k\bar{p}} = G_{\epsilon\bar{p}}B_{j\bar{i}k}^\epsilon = G_{i\bar{p}}B_{j\bar{i}k}^i + G_{i'\bar{p}}B_{j\bar{i}k}^{i'}. \quad (3.29)$$

将 (2.9), (3.14) 和 (3.19) 的第一个等式代入 (3.29) 中, 并根据 (2.4) 可得

$$B_{j\bar{i}k\bar{p}} = f_2^2 g_{i\bar{p}} B_{j\bar{i}k}^i = f_2^2 B_{j\bar{i}k\bar{p}}^1.$$

同理可得 (3.24) 式.

接下来证明 (3.25) 式. 令 (2.4) 中 $\alpha = j, \gamma = t', \delta = k'$ 和 $\beta = p$, 则有

$$B_{j\bar{i}'k'\bar{p}} = G_{\epsilon\bar{p}}B_{j\bar{i}'k'}^\epsilon = G_{i\bar{p}}B_{j\bar{i}'k'}^i + G_{i'\bar{p}}B_{j\bar{i}'k'}^{i'}. \quad (3.30)$$

将 (2.9), (3.16) 和 (3.19) 代入 (3.30) 中, 可得

$$\begin{aligned} B_{j\bar{t}'k'\bar{p}} &= f_2^2 g_{i\bar{p}} \cdot [2f_2^{-2} g^{\bar{s}i} \frac{\partial f_1}{\partial \bar{z}^s} h_{k'\bar{t}'} \frac{\partial f_1}{\partial z^j} - 2f_2^{-2} f_1 (\frac{\partial g^{\bar{s}i}}{\partial z^j} \frac{\partial f_1}{\partial \bar{z}^s} + g^{\bar{s}i} \frac{\partial^2 f_1}{\partial \bar{z}^s \partial z^j} + g^{\bar{s}l} \frac{\partial f_1}{\partial \bar{z}^s} g^{\bar{s}i} \frac{\partial g_{l\bar{s}}}{\partial z^j}) h_{k'\bar{t}'}] \\ &= 2 \frac{\partial f_1}{\partial \bar{z}^p} \frac{\partial f_1}{\partial z^j} h_{k'\bar{t}'} - 2f_1 (g_{i\bar{p}} \frac{\partial g^{\bar{s}i}}{\partial z^j} \frac{\partial f_1}{\partial \bar{z}^s} + \frac{\partial^2 f_1}{\partial \bar{z}^p \partial z^j} + g^{\bar{p}l} \frac{\partial f_1}{\partial \bar{z}^p} \frac{\partial g_{l\bar{p}}}{\partial z^j}) h_{k'\bar{t}'} \end{aligned}$$

同理可得 (3.26) 式.

再证明 (3.27) 的第一个等式. 令 (2.4) 中 $\alpha = j'$, $\gamma = t$, $\delta = k$ 和 $\beta = p$, 则有

$$B_{j'\bar{t}k\bar{p}} = G_{\epsilon\bar{p}} B_{j'\bar{t}k}^\epsilon = G_{i\bar{p}} B_{j'\bar{t}k}^i + G_{i'\bar{p}} B_{j'\bar{t}k}^{i'} \quad (3.31)$$

将 (2.9), (3.17) 和 (3.18) 的第一个等式代入 (3.31), 可得

$$B_{j'\bar{t}k\bar{p}} = 0.$$

同理可得 (3.31) 的其余等式和 (3.32) 中各式. □

命题5. 设 $(f_2 M_1 \times_{f_1} M_2, G)$ 是一个双扭曲积埃尔米特流形, 则 Bismut Ricci 曲率系数 $B_{\alpha\bar{\gamma}}^{(1)}$ 为

$$B_{j\bar{t}}^{(1)} = B_{j\bar{t}}^{(1)}, \quad (3.32)$$

$$B_{j'\bar{t}'}^{(1)} = B_{j'\bar{t}'}^{(1)}, \quad (3.33)$$

$$B_{j\bar{t}'}^{(1)} = 0, \quad (3.34)$$

$$B_{j'\bar{t}}^{(1)} = 0. \quad (3.35)$$

证明. 首先给出 (3.32) 的证明过程. 令 (2.5) 中 $\alpha = j$ 和 $\gamma = t$, 则有

$$B_{j\bar{t}}^{(1)} = G^{\bar{\beta}\delta} B_{j\bar{t}\delta\bar{\beta}} = G^{\bar{p}k} B_{j\bar{t}k\bar{p}} + G^{\bar{p}'k} B_{j\bar{t}k\bar{p}'} + G^{\bar{p}k'} B_{j\bar{t}k'\bar{p}} + G^{\bar{p}'k'} B_{j\bar{t}k'\bar{p}'}. \quad (3.36)$$

将 (2.10), (3.23) 和 (3.27) 代入 (3.36), 并根据 (2.5) 可得

$$B_{j\bar{t}}^{(1)} = f_2^{-2} g^{\bar{p}k} f_2^2 B_{j\bar{t}k\bar{p}} = B_{j\bar{t}}^{(1)}.$$

同理可得 (3.33) 式.

接着给出 (3.34) 的证明过程. 令 (2.5) 中 $\alpha = j'$ 和 $\gamma = t$, 则有

$$B_{j'\bar{t}'}^{(1)} = G^{\bar{\beta}\delta} B_{j'\bar{t}'\delta\bar{\beta}} = G^{\bar{p}k} B_{j'\bar{t}'k\bar{p}} + G^{\bar{p}'k} B_{j'\bar{t}'k\bar{p}'} + G^{\bar{p}k'} B_{j'\bar{t}'k'\bar{p}} + G^{\bar{p}'k'} B_{j'\bar{t}'k'\bar{p}'}. \quad (3.37)$$

将 (2.10), (3.26), (3.27) 和 (3.28) 代入 (3.37), 可得

$$B_{j'\bar{t}'}^{(1)} = 0.$$

同理可得 (3.35). □

定理1. 设 $(f_2 M_1 \times_{f_1} M_2, G)$ 是一个双扭曲积埃尔米特流形, 则埃尔米特度量 G 沿着一个非零向量 $v = (v^i, v^{i'}) \in T_z^{1,0} M$ 的 Bismut 标量曲率为

$$B_G = f_2^{-2} B_g(v_1) + f_1^{-2} B_h(v_2), \quad (3.38)$$

其中 $B_g(v_1)$ 和 $B_h(v_2)$ 分别表示 (M_1, g) 和 (M_2, h) 的 Bismut 标量曲率.

证明. 根据(2.6), 则有

$$B_G = G^{\bar{\gamma}\alpha} B_{\alpha\bar{\gamma}}^{(1)} = G^{\bar{t}j} B_{j\bar{t}}^{(1)} + G^{\bar{t}j'} B_{j'\bar{t}}^{(1)} + G^{\bar{t}'j} B_{j\bar{t}'}^{(1)} + G^{\bar{t}'j'} B_{j'\bar{t}'}^{(1)}. \quad (3.39)$$

将 (2.10) 和 (3.32)-(3.35) 代入 (3.39), 我们有

$$\begin{aligned} B_G &= G^{\bar{t}j} B_{j\bar{t}}^{(1)} + G^{\bar{t}'j'} B_{j'\bar{t}'}^{(1)} \\ &= f_2^{-2} g^{\bar{t}j} B_{j\bar{t}}^{(1)} + f_1^{-2} h^{\bar{t}'j'} B_{j'\bar{t}'}^{(1)} \\ &= f_2^{-2} B_g(v_1) + f_1^{-2} B_h(v_2). \end{aligned}$$

证毕. □

4. Bismut Ricci 平坦双扭曲积埃尔米特流形

设 M_1 和 M_2 是两个 Bismut Ricci 平坦埃尔米特流形, 自然地会考虑双扭曲积埃尔米特流形 $(f_2 M_1 \times_{f_1} M_2, G)$ 是否也是 Bismut Ricci 平坦的.

定理2. 设 $(f_2 M_1 \times_{f_1} M_2, G)$ 是一个双扭曲积埃尔米特流形, 则 $(f_2 M_1 \times_{f_1} M_2, G)$ Bismut Ricci 平坦当且仅当 (M_1, g) 和 (M_2, h) 均 Bismut Ricci 平坦.

证明. 根据定义1, $(f_2 M_1 \times_{f_1} M_2, G)$ Bismut Ricci 平坦当且仅当 $B_{\alpha\bar{\gamma}}^{(1)} = 0$, 即等价于

$$\left\{ \begin{array}{l} B_{j\bar{t}}^{(1)} = 0, \end{array} \right. \quad (4.1)$$

$$\left\{ \begin{array}{l} B_{j'\bar{t}'}^{(1)} = 0, \end{array} \right. \quad (4.2)$$

$$\left\{ \begin{array}{l} B_{j'\bar{t}}^{(1)} = 0, \end{array} \right. \quad (4.3)$$

$$\left\{ \begin{array}{l} B_{j\bar{t}'}^{(1)} = 0. \end{array} \right. \quad (4.4)$$

将 (3.32)-(3.35) 分别代入到 (4.1)-(4.4) 中, 上述方程组等价于

$$\left\{ \begin{array}{l} B_{j\bar{t}}^{(1)} = 0, \end{array} \right. \quad (4.5)$$

$$\left\{ \begin{array}{l} B_{j'\bar{t}'}^{(1)} = 0. \end{array} \right. \quad (4.6)$$

(4.5) 和 (4.6) 等价于 (M_1, g) 和 (M_2, h) 分别 Bismut Ricci 平坦. □

注2. 定理2给出了一种构造 Bismut Ricci 平坦埃尔米特流形的有效方法.

5. 结论

本文研究了双扭曲积埃尔米特流形的 Bismut 联络, 给出了双扭曲积埃尔米特流的 Bismut 联络, Bismut 曲率, Bismut Ricci 曲率和 Bismut 标量曲率的局部坐标表达式, 得到了双扭曲积埃尔米特流形 Bismut Ricci 平坦当且仅当其分量流形均 Bismut Ricci 平坦. 通过双扭曲积的方法, 提供了一种构造 Bismut Ricci 平坦埃尔米特流形的新的途径, 这为研究 Bismut Ricci 平坦埃尔米特流形提供了一个新的思路.

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