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The Initial Boundary Value Problems for a Class of Multidimensional Nonlinear Pseudo-Hyperbolic System of Higher Order*

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Abstract: In this paper, the initial boundary value problems for a class of multidimensional nonlinear pseudo-hyperbolic system of higher order have been studied by Galerkin method, combining with the priori estimate of solution and Sobolev imbedding theorems, the existence and uniqueness of the global strong solution have been proved.

Keywords: Nonlinear Pseudo-Hyperbolic System; Priori Estimate; Galerkin Method; Sobolev Imbedding Theorems; The Global Strong Solutions

一类高阶 *n* 维非线性伪双曲方程组的初边值问题*

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摘 要：本文研究了一类高阶 *n* 维非线性伪双曲方程组，利用 Galerkin 方法，通过解的先验估计，结合 Sobolev 嵌入定理，证明了初边值问题整体强解的存在性和唯一性。

关键词：非线性伪双曲方程组；先验估计；Galerkin 方法；Sobolev 嵌入定理；整体强解

1. 引言

非线性伪双曲方程是从动物神经传播，粘弹性杆纵振动等生物、力学中提出的一类重要的非线性偏微分方程，它的研究具有重要的理论与实际意义。文[1]研究了一类高阶非线性伪双曲方程组，但空间变量是一维的. 文[2]研究了一类多维非线性伪双曲方程 $u_{tt}-\Delta u_t=f(u)$ 的整体解，文[3]研究了一类高阶 n 维非线性伪双曲方程的初边值问题.关于高阶 n 维非线性伪双曲方程组的研究在已有文献中还未见到，本文研究了一类高阶 n 维非线性伪双曲方程组的初边值问题，得到了整体强解的存在性和唯一性。

设 Ω 是 R^n 中具有充分光滑边界的有界域，$T>0$，考虑如下的初边值问题：

$$
(\mathrm{I})\begin{cases}
\boldsymbol{u}_{tt}+(-1)^M\Delta^M\boldsymbol{u}+(-1)^M\Delta\boldsymbol{u}_t=\boldsymbol{f}(\boldsymbol{u}_t),x\in\Omega,t\in[0,T], & (1.1)\\
D^\gamma\boldsymbol{u}(x,t)\big|_{\partial\Omega\times[0,T]}=\mathbf{0}, \quad 0\le|\gamma|\le M-1, & (1.2)\\
\boldsymbol{u}(x,0)=\boldsymbol{\varphi}(x), \quad x\in\Omega, & (1.3)\\
\boldsymbol{u}_t(x,0)=\boldsymbol{\psi}(x), \quad x\in\Omega, & (1.4)
\end{cases}
$$

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其中 M 为正整数，$\boldsymbol{\varphi}(x)\in H^{2M}(\Omega)\cap H_0^M(\Omega)$，$\boldsymbol{\psi}(x)\in H^{2M}(\Omega)\cap H_0^M(\Omega)$，$H^{2M}(\Omega),H_0^M(\Omega)$ 为相应的 Sobplev 空间，其意义见[4]，$\boldsymbol{u}=\boldsymbol{u}(x,t)=(u_1,u_2,\cdots,u_N)^{\mathrm{T}}$，$\gamma=(\gamma_1,\gamma_2,\cdots,\gamma_n)$ 为多重指标，$\gamma_i\ (i=1,2,\cdots,n)$ 为非负整数，$|\gamma|=\gamma_1+\gamma_2+\cdots+\gamma_n$，$\boldsymbol{f}(\boldsymbol{u}_t)=\left(\boldsymbol{f}_1(\boldsymbol{u}_t),\boldsymbol{f}_2(\boldsymbol{u}_t),\cdots,\boldsymbol{f}_N(\boldsymbol{u}_t)\right)^{\mathrm{T}}$，假设 $\boldsymbol{f}$ 满足：$\boldsymbol{f}(\boldsymbol{0})=\boldsymbol{0}$，$\boldsymbol{f}\in C^1$ 且 Jacobian 矩阵 $\dfrac{\partial\boldsymbol{f}}{\partial\boldsymbol{u}}$ 半有界，即 $\exists k_0>0$，$\forall\boldsymbol{\xi}\in R^n$ 满足：

$$\left(\frac{\partial\boldsymbol{f}}{\partial\boldsymbol{u}}\boldsymbol{\xi},\boldsymbol{\xi}\right)\le k_0(\boldsymbol{\xi},\boldsymbol{\xi}),\tag{1.5}$$

$$|\boldsymbol{f}(\boldsymbol{u})|\le a_1+b_1|\boldsymbol{u}|^{\frac{p}{2}}\quad (a_1,b_1\text{ 为正常数}),\tag{1.6}$$

其中，当 $2M<n$ 时，$2\le p<\dfrac{2n}{n-2M}$；当 $2M\ge n$ 时，$2\le p<+\infty$。记 $\boldsymbol{u}\cdot\boldsymbol{v}=\sum\limits_{i=1}^{N}u_iv_i$，$(\boldsymbol{u},\boldsymbol{v})=\int_\Omega\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}x$，$|\boldsymbol{u}|^2_{L^2(\Omega)}=(\boldsymbol{u},\boldsymbol{u})$，$[\boldsymbol{u},\boldsymbol{v}]=\int_0^t(\boldsymbol{u},\boldsymbol{v})\,\mathrm{d}t$，$\|\boldsymbol{u}\|^2_{L^2(\Omega)}=[\boldsymbol{u},\boldsymbol{u}]$，$|\boldsymbol{u}|=\left(\sum\limits_{i=1}^{N}u_i^2\right)^{\frac{1}{2}}$。

2. 解的存在性

在证明解的存在性之前先来证明一个引理。

引理 1.1：若 $\{\omega_j(x)\,|\,j=1,2,\cdots\}$ 为问题

$$\begin{cases}(-1)^M\Delta^M w_j=\lambda_j w_j,\\ D^\gamma w_j(x)\big|_{\partial\Omega}=0,\ 0\le|\gamma|\le M-1,\end{cases}\tag{A}$$

的特征函数系，这里 Ω 为 R^n 中具有充分光滑边界的有界域，M 为正整数，$\Delta=\sum\limits_{i=1}^{n}\dfrac{\partial^2}{\partial x_i^2}$ 为拉普拉斯算子，则 $\{\omega_j(x)\,|\,j=1,2,\cdots\}$ 构成 $L^2(\Omega)$ 中的正交基底且构成 $H_0^M(\Omega)$ 的正交基底，也构成 $H^{2M}(\Omega)\cap H_0^M(\Omega)$ 的正交基底。

证明：考虑特征值问题

$$\begin{cases}(-1)^M\Delta^M w=\lambda w,\\ D^\gamma w(x)\big|_{\partial\Omega}=0,\ 0\le|\gamma|\le M-1,\end{cases}\tag{A}$$

记 $L=(-1)^M\Delta^M$，$\Delta=\sum\limits_{i=1}^{n}\dfrac{\partial^2}{\partial x_i^2}$ 为 n 维拉普拉斯算子，$\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$ 为多重指标，$\alpha_i\ (i=1,2,\cdots,n)$ 为非负整数，$|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_n$，$H^{2M}(\Omega)=\left\{u(x)\,|\,u(x)\in L^2(\Omega),D^\alpha u(x)\in L^2(\Omega),0\le|\alpha|\le 2M\right\}$，$H_0^M(\Omega)$ 为 $C_0^\infty(\Omega)$ 按 $H^M(\Omega)$ 范数的完备化空间，这里 $H^M(\Omega)$ 范数为 $\|u(x)\|_{H^M(\Omega)}=\left\{\sum\limits_{0\le|\alpha|\le M}\int\limits_\Omega\left(D^\alpha u(x)\right)^2\mathrm{d}x\right\}^{\frac{1}{2}}$，$C_0^\infty(\Omega)$ 为在 Ω 中具有紧支集的无穷次可微函数组成的集合，记 $\nabla=\left(\dfrac{\partial}{\partial x_1},\cdots,\dfrac{\partial}{\partial x_n}\right)$ 为梯度算子。

$$(u(x),v(x))=\int\limits_\Omega u(x)v(x)\,\mathrm{d}x,$$

$$[u(x),v(x)]=\int\limits_\Omega\nabla^M u(x)\nabla^M v(x)\,\mathrm{d}x,$$

$$\{u(x),v(x)\}=\int\limits_\Omega\Delta^M u(x)\Delta^M v(x)\,\mathrm{d}x.$$

$$\forall u(x),v(x)\in H_0^M(\Omega),$$

$$\left(Lu(x),v(x)\right)=\int_\Omega(-1)^M\Delta^M u(x)v(x)\mathrm{d}x=\int_\Omega\nabla^M u(x)\nabla^M v(x)\mathrm{d}x$$
$$=\int_\Omega u(x)(-1)^M\Delta^M v(x)\mathrm{d}x=\left(u(x),Lv(x)\right),$$

由上式知微分算子 $L=(-1)^M\Delta^M$ 的共轭算子 $L^*=L$，即算子 L 是自共轭的。

$\forall u(x),v(x)\in H_0^M(\Omega)$，微分算子 L 对应的双线性形式为

$$\left(Lu(x),v(x)\right)=\int_\Omega\nabla^M u(x)\nabla^M v(x)\mathrm{d}x$$

$$\left(Lu(x),u(x)\right)=\int_\Omega\nabla^M u(x)\nabla^M u(x)\mathrm{d}x=\int_\Omega\left|\nabla^M u(x)\right|^2\mathrm{d}x=\left|\nabla^M u(x)\right|_{L^2(\Omega)}^2$$

$\left|\nabla^M u(x)\right|_{L^2(\Omega)}$ 为 $H_0^M(\Omega)$ 中的等价模。

事实上，$\forall u(x)\in H_0^M(\Omega)$, $\left|u(x)\right|_{H^M(\Omega)}=\left\{\left|u(x)\right|_{L^2(\Omega)}^2+\left|\nabla u(x)\right|_{L^2(\Omega)}^2+\cdots+\left|\nabla^M u(x)\right|_{L^2(\Omega)}^2\right\}^{\frac{1}{2}}$, 由 *Poincare′* 不等式，

$$\left|u(x)\right|_{L^2(\Omega)}\le C\left|\nabla u(x)\right|_{L^2(\Omega)},$$
$$\left|\nabla u(x)\right|_{L^2(\Omega)}\le C\left|\nabla^2 u(x)\right|_{L^2(\Omega)},$$
$$\vdots$$
$$\left|\nabla^{M-1}u(x)\right|_{L^2(\Omega)}\le C\left|\nabla^M u(x)\right|_{L^2(\Omega)},$$

这里 C 在不同位置代表不同正常数，因此 $\left|u(x)\right|_{H^M(\Omega)}\le C\left|\nabla^M u(x)\right|_{L^2(\Omega)}$，而 $\left|\nabla^M u(x)\right|_{L^2(\Omega)}\le C\left|u(x)\right|_{H^M(\Omega)}$ 显然成立。

$$\left(Lu(x),u(x)\right)=\left|\nabla^M u(x)\right|_{L^2(\Omega)}^2>C\left|u(x)\right|_{H^M(\Omega)}^2>\frac{C}{2}\left|u(x)\right|_{H^M(\Omega)}^2$$

(这里 C 为某一正常数)，算子 L 在 $H_0^M(\Omega)$ 上是自共轭，严格强制的。

由[5, p. 252]定理 7.23，特征问题 (A) 具有特征函数 $\left\{\omega_j(x)\right\}_{j=1}^{+\infty}$ 且对 $j=1,2,\cdots,\omega_j(x)\in C^\infty\left(\bar{\Omega}\right)$，$\left\{\omega_j(x)\right\}_{j=1}^{+\infty}$ 构成 $L^2(\Omega)$ 的一组正交基，$\omega_j(x)\in H^{2M}(\Omega)\cap H_0^M(\Omega)$ $(j=1,2,\cdots)$，$\omega_j(x)$ 对应的特征值 λ_j $(j=1,2,\cdots)$ 都是实的且 $\lim\limits_{j\to\infty}\lambda_j=+\infty$，由于微分算子 $L=(-1)^M\Delta^M$ 是自共轭的且 L 对应的双线性形式在 $H_0^M(\Omega)$ 上是严格强制的，同样由[5, p. 252]定理 7.23，特征值 $\left\{\lambda_j\right\}_{j=1}^{+\infty}$ 都是正数，下面说明特征函数系 $\left\{\omega_j(x)\right\}_{j=1}^{+\infty}$ 构成 $H_0^M(\Omega)$ 的一组正交基。

反证：注意到 $\left|\nabla^M u(x)\right|_{L^2(\Omega)}$ 为 $H_0^M(\Omega)$ 的等价模，在 $H_0^M(\Omega)$ 中可定义内积 $[\bullet,\bullet]$ (因在 $H_0^M(\Omega)$ 中由该内积诱导的范数与范数 $\left|u(x)\right|_{H^M(\Omega)}$ 等价，按内积空间定义可验证 $[\bullet,\bullet]$ 确实为空间 $H_0^M(\Omega)$ 中一个内积)。若 $\left\{\omega_j(x)\right\}_{j=1}^{+\infty}$ 按此内积不构成 $H_0^M(\Omega)$ 的一组正交基，则在 $H_0^M(\Omega)$ 中由 $\left\{\omega_j(x)\right\}_{j=1}^{+\infty}$ 张成的闭线性子空间有非零的正交补空间 V，在 V 中取一非零元素 $v(x)\in V$，因此按此内积有

$$\forall j=1,2,3,\cdots,\left[\omega_j(x),v(x)\right]=0\Rightarrow\int_\Omega\nabla^M\omega_j(x)\nabla^M v(x)\mathrm{d}x=0,$$
$$\Rightarrow\int_\Omega(-1)^M\Delta^M\omega_j(x)v(x)\mathrm{d}x=0\Rightarrow\int_\Omega\lambda_j\omega_j(x)v(x)\mathrm{d}x=0.$$

由 $\lambda_j>0,\Rightarrow\int_\Omega\omega_j(x)v(x)\mathrm{d}x=0$，即 $\left(\omega_j(x),v(x)\right)=0$ $(j=1,2,\cdots)$. 由 $\left\{\omega_j(x)\right\}_{j=1}^{+\infty}$ 构成 $L^2(\Omega)$ 的一组正交基 $\Rightarrow v(x)=0$，与 $v(x)$ 为非零元素矛盾。故 $\left\{\omega_j(x)\right\}_{j=1}^{+\infty}$ 构成 $H_0^M(\Omega)$ 中的一组正交基。

下面证明特征函数系 $\left\{\omega_j(x)\right\}_{j=1}^{+\infty}$ 构成 $H^{2M}(\Omega)\cap H_0^M(\Omega)$ 中的一组正交基。

由[6, pp. 75-76]结论知，$\left|\Delta^M u(x)\right|_{L^2(\Omega)}$ 为空间 $H^{2M}(\Omega)\cap H_0^M(\Omega)$ 中的等价模，在 $H^{2M}(\Omega)\cap H_0^M(\Omega)$ 中可以定义内积 $\{\bullet,\bullet\}$，若按此内积 $\left\{\omega_j(x)\right\}_{j=1}^{+\infty}$ 不构成 $H^{2M}(\Omega)\cap H_0^M(\Omega)$ 中一正交基底，则在 $H^{2M}(\Omega)\cap H_0^M(\Omega)$ 中由

$\left\{\omega_j(x)\right\}_{j=1}^{+\infty}$ 张成闭线性子空间有非零正交补 V'，在 V' 中取一非零元素 $v(x)\in H^{2M}(\Omega)\cap H_0^M(\Omega)$，则 $\left\{\omega_j(x),v(x)\right\}=0$，即

$$\int_\Omega \Delta^M\omega_j(x)\Delta^M v(x)\mathrm{d}x=0,\ \int_\Omega(-1)^M\Delta^M\omega_j(x)(-1)^M\Delta^M v(x)\mathrm{d}x=0,\ \overset{(A)\text{式}}{\Longrightarrow}\int_\Omega\lambda_j\omega_j(x)(-1)^M\Delta^M v(x)\mathrm{d}x=0,$$

由

$$\lambda_j>0(j=1,2,\cdots)\Rightarrow\int_\Omega\omega_j(x)(-1)^M\Delta^M v(x)\mathrm{d}x=0,j=1,2,3,\cdots,$$
$$\Rightarrow\int_\Omega\omega_j(x)(-1)^M\Delta^M v(x)\mathrm{d}x=0,j=1,2,3,\cdots,\Rightarrow\int_\Omega\nabla^M\omega_j(x)\nabla^M v(x)\mathrm{d}x=0,$$
$$\Rightarrow\int_\Omega(-1)^M\Delta^M\omega_j(x)v(x)\mathrm{d}x=0,\Rightarrow\int_\Omega\lambda_j\omega_j(x)v(x)\mathrm{d}x=0,$$

由

$$\lambda_j>0(j=1,2,3,\cdots)\Rightarrow\int_\Omega\omega_j(x)v(x)\mathrm{d}x=0(j=1,2,\cdots),$$
$$\Rightarrow(\omega_j(x),v(x))=0\ (j=1,2,\cdots).$$

由 $\left\{\omega_j(x)\right\}_{j=1}^{+\infty}$ 构成 $L^2(\Omega)$ 一组正交基知 $v(x)=0$，与 $v(x)$ 为一非零元素矛盾。因此 $\left\{\omega_j(x)\right\}_{j=1}^{+\infty}$ 构成 $H^{2M}(\Omega)\cap H_0^M(\Omega)$ 的一组正交基。引理证毕！

设 $\left\{\omega_j(x)\middle|j=1,2,\cdots\right\}$ 为问题 $\begin{cases}(-1)^M\Delta^M w_j=\lambda_j w_j,\\ D^\gamma w_j(x)\big|_{\partial\Omega}=0,\ \ 0\le|\gamma|\le M-1,\end{cases}$ 的特征函数系，由引理1.1知 $\left\{\omega_j(x)\middle|j=1,2,\cdots\right\}$ 构成 $L^2(\Omega)$ 中的正交基底且构成 $H_0^M(\Omega)$ 的正交基底，也构成 $H^{2M}(\Omega)\cap H_0^M(\Omega)$ 的正交基底。对任何固定的 $m\in Z^+$ (Z^+ 为自然数集)，在由 $\left\{\omega_1(x),\cdots,\omega_m(x)\right\}$ 所张成的有限维空间中用下列方式确定非线性问题(I)的近似解

$$\begin{cases}u_{mi}=u_{mi}(x,t)=\sum_{j=1}^m g_{mij}(t)\omega_j(x), & (2.1)\\ u_{mi}(x,0)=\varphi_{mi}(x)=\sum_{j=1}^m\xi_{mij}\omega_j(x), & (2.2)\\ u_{mit}(x,0)=\psi_{mi}(x)=\sum_{j=1}^m\eta_{mij}\omega_j(x), & (2.3)\end{cases}$$

其中 $i=1,2,\cdots,N$。记 $\boldsymbol{u}_m=(u_{m1},u_{m2},\cdots,u_{mN})^{\mathrm{T}}$，$\boldsymbol{\varphi}_m=(\boldsymbol{\varphi}_{m1},\boldsymbol{\varphi}_{m2},\cdots,\boldsymbol{\varphi}_{mN})^{\mathrm{T}}$，$\boldsymbol{\psi}_m=(\boldsymbol{\psi}_{m1},\boldsymbol{\psi}_{m2},\cdots,\boldsymbol{\psi}_{mN})^{\mathrm{T}}$，使其满足

$$(\mathrm{II})\begin{cases}(u_{mitt},\omega_k)+\left((-1)^M\Delta^M u_{mit},\omega_k\right)+\left((-1)^M\Delta^M u_{mi},\omega_k\right)=\left(f_i(\boldsymbol{u}_{mt}),\omega_k\right),\\ u_{mi}(x,0)=\varphi_{mi}(x),\\ u_{mit}(x,0)=\psi_{mi}(x),\end{cases}\qquad(2.4)$$
$$i=1,2,\cdots,N;k=1,2,\cdots,m.$$

由于 $\boldsymbol{\varphi}(x)\in H^{2M}(\Omega)\cap H_0^M(\Omega),\boldsymbol{\psi}(x)\in H^{2M}(\Omega)\cap H_0^M(\Omega)$，知 $\boldsymbol{\varphi}(x)=\left(\varphi_1(x),\cdots,\varphi_N(x)\right)^{\mathrm{T}}$，$\boldsymbol{\psi}(x)=\left(\psi_1(x),\cdots,\psi_N(x)\right)^{\mathrm{T}}$ 中的分量函数 $\varphi_i\in H^{2M}(\Omega)\cap H_0^M(\Omega)$，$\psi_i\in H^{2M}(\Omega)\cap H_0^M(\Omega)\ (i=1,2,\cdots,N)$。而 $\left\{\omega_j(x)\right\}_{j=1}^{+\infty}$ 构成 $H^{2M}(\Omega)\cap H_0^M(\Omega)$ 的一组正交基，可选适当的 ξ_{mij}，$\eta_{mij}\ (j=1,2,\cdots,m)$ 使当 $m\to+\infty$ 时，$\varphi_{mi}\to\varphi_i$，$\psi_{mi}\to\psi_i$ 在 $H^{2M}(\Omega)\cap H_0^M(\Omega)$ 中强收敛，由常微分方程理论知(II)存在局部解 $\boldsymbol{u}_{mi}(x,t)(i=1,2,\cdots,N)$。

为得到整体解，下面作近似解的先验估计：引理 2.1：若条件(1.5)，(1.6)成立，则有估计式

$$\left|\boldsymbol{u}_m\right|_{L^2(\Omega)}^2+\left|\boldsymbol{u}_{mt}\right|_{L^2(\Omega)}^2+\left|\nabla^M \boldsymbol{u}_m\right|_{L^2(\Omega)}^2+\left\|\nabla^M \boldsymbol{u}_{mt}\right\|_{L^2(\Omega)}^2 \le \text{const}\ \left(0\le t\le T\right).$$

证明：方程组(2.4)两边同乘 $g'_{mik}(t)$，

$$\left(u_{mitt}, g'_{mik}(t)\omega_k\right)+\left((-1)^M \Delta^M u_{mi}, g'_{mik}\omega_k\right)+\left((-1)^M \Delta^M u_{mit}, g'_{mik}(t)\omega_k\right)=\left(f_i(\boldsymbol{u}_{mt}), g'_{mik}(t)\omega_k\right),$$

关于 k 从 1 到 m 作和得

$$\left(u_{mitt}, u_{mit}\right)+\left((-1)^M u_{mi}, u_{mit}\right)+\left((-1)^M \Delta^M u_{mit}, u_{mit}\right)=\left(f_i(\boldsymbol{u}_{mt}), u_{mit}\right),$$

关于 i 从 1 到 N 作和得

$$\left(\boldsymbol{u}_{mtt}, \boldsymbol{u}_{mt}\right)+\left((-1)^M \Delta^M \boldsymbol{u}_m, \boldsymbol{u}_{mt}\right)+\left((-1)^M \Delta^M \boldsymbol{u}_{mt}, \boldsymbol{u}_{mt}\right)=\left(\boldsymbol{f}(\boldsymbol{u}_{mt}), \boldsymbol{u}_{mt}\right)$$

而由条件(1.5)知

$$\left(\boldsymbol{f}(\boldsymbol{u}_{mt}), \boldsymbol{u}_{mt}\right)=\left(\boldsymbol{f}(\boldsymbol{u}_{mt})-\boldsymbol{f}(\boldsymbol{0}), \boldsymbol{u}_{mt}\right)=\left(\frac{\partial \boldsymbol{f}(\boldsymbol{u}_{mt})}{\partial \boldsymbol{u}_{mt}}\Big|_{\theta \boldsymbol{u}_{mt}} \boldsymbol{u}_{mt}, \boldsymbol{u}_{mt}\right)\le k_0\left(\boldsymbol{u}_{mt}, \boldsymbol{u}_{mt}\right)\quad \left(0<\theta<1\right),$$

首先利用 Green 公式，然后两端从 0 到 t 积分，得

$$\frac{1}{2}\left(\boldsymbol{u}_{mt}, \boldsymbol{u}_{mt}\right)-\frac{1}{2}\left(\boldsymbol{\psi}_m, \boldsymbol{\psi}_m\right)+\frac{1}{2}\left(\nabla^M \boldsymbol{u}_m, \nabla^M \boldsymbol{u}_m\right)-\frac{1}{2}\left(\nabla^M \boldsymbol{\varphi}_m, \nabla^M \boldsymbol{\varphi}_m\right)+\left[\nabla^M \boldsymbol{u}_{mt}, \nabla^M \boldsymbol{u}_{mt}\right]\le k_0\left[\boldsymbol{u}_{mt}, \boldsymbol{u}_{mt}\right],$$

从而有

$$\frac{1}{2}\left(\boldsymbol{u}_{mt}, \boldsymbol{u}_{mt}\right)+\frac{1}{2}\left(\nabla^M \boldsymbol{u}_m, \nabla^M \boldsymbol{u}_m\right)+\left[\nabla^M \boldsymbol{u}_{mt}, \nabla^M \boldsymbol{u}_{mt}\right]\le k_0\left[\boldsymbol{u}_{mt}, \boldsymbol{u}_{mt}\right]+\frac{1}{2}\left(\boldsymbol{\psi}_m, \boldsymbol{\psi}_m\right)+\frac{1}{2}\left(\nabla^M \boldsymbol{\varphi}_m, \nabla^M \boldsymbol{\varphi}_m\right),$$

两边加上 $\left[\boldsymbol{u}_m, \boldsymbol{u}_{mt}\right]+\left[\nabla^M \boldsymbol{u}_m, \nabla^M \boldsymbol{u}_{mt}\right]$，左边将它化为

$$\frac{1}{2}(\boldsymbol{u}_m, \boldsymbol{u}_m)-\frac{1}{2}(\boldsymbol{\varphi}_m, \boldsymbol{\varphi}_m)+\frac{1}{2}\left(\nabla^M \boldsymbol{u}_m, \nabla^M \boldsymbol{u}_m\right)-\frac{1}{2}\left(\nabla^M \boldsymbol{\varphi}_m, \nabla^M \boldsymbol{\varphi}_m\right),$$

右边将它估计为

$$\left[\boldsymbol{u}_m, \boldsymbol{u}_{mt}\right]+\left[\nabla^M \boldsymbol{u}_m, \nabla^M \boldsymbol{u}_{mt}\right]\le\left[\boldsymbol{u}_m, \boldsymbol{u}_m\right]+\left[\boldsymbol{u}_{mt}, \boldsymbol{u}_{mt}\right]+\frac{1}{2\varepsilon}\left[\nabla^M \boldsymbol{u}_m, \nabla^M \boldsymbol{u}_m\right]+\frac{\varepsilon}{2}\left[\nabla^M \boldsymbol{u}_{mt}, \nabla^M \boldsymbol{u}_{mt}\right],$$

从而有

$$\begin{aligned}&\frac{1}{2}\left(\boldsymbol{u}_m, \boldsymbol{u}_m\right)+\left(\nabla^M \boldsymbol{u}_m, \nabla^M \boldsymbol{u}_m\right)+\frac{1}{2}\left(\boldsymbol{u}_{mt}, \boldsymbol{u}_{mt}\right)+\left[\nabla^M \boldsymbol{u}_{mt}, \nabla^M \boldsymbol{u}_{mt}\right]\le\frac{1}{2}\left(\boldsymbol{\psi}_m, \boldsymbol{\psi}_m\right)+\frac{1}{2}\left(\nabla^M \boldsymbol{\varphi}_m, \nabla^M \boldsymbol{\varphi}_m\right)\\&+\frac{1}{2}\left(\nabla^M \boldsymbol{\varphi}_m, \nabla^M \boldsymbol{\varphi}_m\right)+k_0\left[\boldsymbol{u}_{mt}, \boldsymbol{u}_{mt}\right]+\left[\boldsymbol{u}_m, \boldsymbol{u}_m\right]+\left[\boldsymbol{u}_{mt}, \boldsymbol{u}_{mt}\right]+\frac{1}{2\varepsilon}\left[\nabla^M \boldsymbol{u}_m, \nabla^M \boldsymbol{u}_m\right]+\frac{\varepsilon}{2}\left[\nabla^M \boldsymbol{u}_{mt}, \nabla^M \boldsymbol{u}_{mt}\right],\end{aligned}$$

所以，

$$\begin{aligned}&\frac{1}{2}\left(\boldsymbol{u}_m, \boldsymbol{u}_m\right)+\left(\nabla^M \boldsymbol{u}_m, \nabla^M \boldsymbol{u}_m\right)+\frac{1}{2}\left(\boldsymbol{u}_{mt}, \boldsymbol{u}_{mt}\right)+\left[\nabla^M \boldsymbol{u}_{mt}, \nabla^M \boldsymbol{u}_{mt}\right]\le\frac{1}{2}\left(\boldsymbol{\psi}_m, \boldsymbol{\psi}_m\right)+\frac{1}{2}\left(\nabla^M \boldsymbol{\varphi}_m, \nabla^M \boldsymbol{\varphi}_m\right)+\frac{1}{2}\left(\nabla^M \boldsymbol{\varphi}_m, \nabla^M \boldsymbol{\varphi}_m\right)\\&+\left[\boldsymbol{u}_m, \boldsymbol{u}_m\right]+\left(k_0+1\right)\left[\boldsymbol{u}_{mt}, \boldsymbol{u}_{mt}\right]+\frac{1}{2\varepsilon}\left[\nabla^M \boldsymbol{u}_m, \nabla^M \boldsymbol{u}_m\right]+\frac{\varepsilon}{2}\left[\nabla^M \boldsymbol{u}_{mt}, \nabla^M \boldsymbol{u}_{mt}\right],\end{aligned}$$

取 ε 充分小，使得 $1-\frac{\varepsilon}{2}\ge\frac{1}{2}$，则

$$\begin{aligned}&\frac{1}{2}\left(\boldsymbol{u}_m, \boldsymbol{u}_m\right)+\left(\nabla^M \boldsymbol{u}_m, \nabla^M \boldsymbol{u}_m\right)+\frac{1}{2}\left(\boldsymbol{u}_{mt}, \boldsymbol{u}_{mt}\right)+\frac{1}{2}\left[\nabla^M \boldsymbol{u}_{mt}, \nabla^M \boldsymbol{u}_{mt}\right]\le\frac{1}{2}\left(\nabla^M \boldsymbol{\varphi}_m, \nabla^M \boldsymbol{\varphi}_m\right)\\&+\frac{1}{2}\left(\boldsymbol{\psi}_m, \boldsymbol{\psi}_m\right)+\frac{1}{2}\left(\nabla^M \boldsymbol{\varphi}_m, \nabla^M \boldsymbol{\varphi}_m\right)+\left(k_0+1\right)\left[\boldsymbol{u}_{mt}, \boldsymbol{u}_{mt}\right]+\left[\boldsymbol{u}_m, \boldsymbol{u}_m\right]+\frac{1}{2\varepsilon}\left[\nabla^M \boldsymbol{u}_m, \nabla^M \boldsymbol{u}_m\right],\end{aligned}$$

由于当 $m \to +\infty$ 时，$\boldsymbol{\varphi}_m \to \boldsymbol{\varphi}$，$\boldsymbol{\psi}_m \to \boldsymbol{\psi}$ 在 $H^{2M}(\Omega) \cap H_0^M(\Omega)$ 中强收敛，所以当 $m \to +\infty$ 时，$(\boldsymbol{\psi}_m, \boldsymbol{\psi}_m) \to (\boldsymbol{\psi}, \boldsymbol{\psi})$，$(\nabla^M \boldsymbol{\varphi}_m, \nabla^M \boldsymbol{\varphi}_m) \to (\nabla^M \boldsymbol{\varphi}, \nabla^M \boldsymbol{\varphi})$，$\frac{1}{2}(\boldsymbol{\psi}_m, \boldsymbol{\psi}_m) + (\nabla^M \boldsymbol{\varphi}_m, \nabla^M \boldsymbol{\varphi}_m)$ 能用一与 m 无关的正常数来控制，由 Gronwall 不等式得

$$|\boldsymbol{u}_m|_{L^2(\Omega)}^2 + |\boldsymbol{u}_{mt}|_{L^2(\Omega)}^2 + |\nabla^M \boldsymbol{u}_m|_{L^2(\Omega)}^2 + \|\nabla^M \boldsymbol{u}_{mt}\|_{L^2(\Omega)}^2 \le \text{const}$$

引理 2.1 证毕！

引理 2.2：若条件(1.5)，(1.6)成立，则有估计式

$$|\boldsymbol{u}_{mtt}|_{L^2(\Omega)}^2 + |\nabla^M \boldsymbol{u}_{mt}|_{L^2(\Omega)}^2 + \|\nabla^M \boldsymbol{u}_{mtt}\|_{L^2(\Omega)}^2 \le \text{const}\ (0 \le t \le T).$$

证明：方程组(2.4)两边关于 t 求导得

$$(u_{mittt}, \omega_k) + \left((-1)^M \Delta^M u_{mit}, \omega_k\right) + \left((-1)^M \Delta^M u_{mitt}, \omega_k\right) = \left(\frac{\mathrm{d}}{\mathrm{d}t} f_i(\boldsymbol{u}_{mt}), \omega_k\right),$$

两边同乘以 $g''_{mik}(t)$，得

$$(u_{mittt}, g''_{mik}(t)\omega_k) + \left((-1)^M \Delta^M u_{mit}, g''_{mik}(t)\omega_k\right) + \left((-1)^M \Delta^M u_{mitt}, g''_{mik}(t)\omega_k\right) = \left(\frac{\mathrm{d}}{\mathrm{d}t} f_i(\boldsymbol{u}_{mt}), g''_{mik}(t)\omega_k\right),$$

关于 k 从 1 到 m 作和得

$$(u_{mittt}, u_{mitt}) + \left((-1)^M \Delta^M u_{mit}, u_{mitt}\right) + \left((-1)^M \Delta^M u_{mitt}, u_{mitt}\right) = \left(\frac{\mathrm{d}}{\mathrm{d}t} f_i(\boldsymbol{u}_{mt}), u_{mitt}\right),$$

关于 i 从 1 到 N 作和得

$$(\boldsymbol{u}_{mttt}, \boldsymbol{u}_{mtt}) + \left((-1)^M \Delta^M \boldsymbol{u}_{mt}, \boldsymbol{u}_{mtt}\right) + \left((-1)^M \Delta^M \boldsymbol{u}_{mtt}, \boldsymbol{u}_{mtt}\right) = \left(\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{f}(\boldsymbol{u}_{mt}), \boldsymbol{u}_{mtt}\right),$$

而由条件(1.5)知

$$(\boldsymbol{u}_{mttt}, \boldsymbol{u}_{mtt})\left((-1)^M \Delta^M \boldsymbol{u}_{mt}, \boldsymbol{u}_{mtt}\right) + \left((-1)^M \Delta^M \boldsymbol{u}_{mtt}, \boldsymbol{u}_{mtt}\right) = \left(\frac{\partial \boldsymbol{f}(\boldsymbol{u}_{mt})}{\partial \boldsymbol{u}_{mt}} \boldsymbol{u}_{mtt}, \boldsymbol{u}_{mtt}\right) \le k_0 (\boldsymbol{u}_{mtt}, \boldsymbol{u}_{mtt}),$$

从而有

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}_{mtt}, \boldsymbol{u}_{mtt}) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\nabla^M \boldsymbol{u}_{mt}, \nabla^M \boldsymbol{u}_{mt}\right) + \left(\nabla^M \boldsymbol{u}_{mtt}, \nabla^M \boldsymbol{u}_{mtt}\right) \le k_0 (\boldsymbol{u}_{mtt}, \boldsymbol{u}_{mtt}),$$

两边从 0 到 t 积分得

$$\begin{aligned}&\frac{1}{2}(\boldsymbol{u}_{mtt}, \boldsymbol{u}_{mtt}) - \frac{1}{2}\left(\boldsymbol{u}_{mtt}(x,0), \boldsymbol{u}_{mtt}(x,0)\right) + \frac{1}{2}\left(\nabla^M \boldsymbol{u}_{mt}, \nabla^M \boldsymbol{u}_{mt}\right)\\ &-\frac{1}{2}\left(\nabla^M \boldsymbol{\psi}_m, \nabla^M \boldsymbol{\psi}_m\right) + \left[\nabla^M \boldsymbol{u}_{mtt}, \nabla^M \boldsymbol{u}_{mtt}\right] \le k_0 [\boldsymbol{u}_{mtt}, \boldsymbol{u}_{mtt}],\end{aligned}$$

下证 $\left(\boldsymbol{u}_{mtt}(x,0), \boldsymbol{u}_{mtt}(x,0)\right)$ 有界。

方程组(2.4)两边同乘以 $g''_{mik}(t)$ 得

$$(u_{mitt}, g''_{mik}(t)\omega_k) + \left((-1)^M \Delta^M u_{mi}, g''_{mik}(t)\omega_k\right) + \left((-1)^M \Delta^M u_{mit}, g''_{mik}(t)\omega_k\right) = \left(f_i(\boldsymbol{u}_{mt}), g''_{mik}(t)\omega_k\right),$$

关于 k 从 1 到 m 作和得

$$\left(u_{mitt},u_{mitt}\right)+\left(\left(-1\right)^{M}\Delta^{M}u_{mi},u_{mitt}\right)+\left(\left(-1\right)^{M}\Delta^{M}u_{mit},u_{mitt}\right)=\left(f_i\left(\boldsymbol{u}_{mt}\right),u_{mitt}\right),$$

关于 i 从 1 到 N 作和得

$$\left(\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt}\right)+\left(\left(-1\right)^{M}\Delta^{M}\boldsymbol{u}_{m},\boldsymbol{u}_{mtt}\right)+\left(\left(-1\right)^{M}\Delta^{M}\boldsymbol{u}_{mt},\boldsymbol{u}_{mtt}\right)=\left(f\left(\boldsymbol{u}_{mt}\right),\boldsymbol{u}_{mtt}\right),$$

令 $t=0$ 得

$$\begin{aligned}&\left(\boldsymbol{u}_{mtt}\left(x,0\right),\boldsymbol{u}_{mtt}\left(x,0\right)\right)+\left(\left(-1\right)^{M}\Delta^{M}\boldsymbol{u}_{m}\left(x,0\right),\boldsymbol{u}_{mtt}\left(x,0\right)\right)\\&+\left(\left(-1\right)^{M}\Delta^{M}\boldsymbol{u}_{mt}\left(x,0\right),\boldsymbol{u}_{mtt}\left(x,0\right)\right)=\left(\boldsymbol{f}\left(\boldsymbol{\psi}_m\right),\boldsymbol{u}_{mtt}\left(x,0\right)\right),\end{aligned}$$

$$\left(\boldsymbol{f}\left(\boldsymbol{\psi}_m\right),\boldsymbol{u}_{mtt}\left(x,0\right)\right)\le\frac{1}{2\varepsilon_2}\left(\boldsymbol{f}\left(\boldsymbol{\psi}_m\right),\boldsymbol{f}\left(\boldsymbol{\psi}_m\right)\right)+\frac{\varepsilon_2}{2}\left(\boldsymbol{u}_{mtt}\left(x,0\right),\boldsymbol{u}_{mtt}\left(x,0\right)\right),$$

(其中 $\boldsymbol{\psi}_m=\left(\psi_{m1}\left(x\right),\cdots,\psi_{mN}\left(x\right)\right)^{\mathrm{T}}$)。

下面证明 $\left(\boldsymbol{f}\left(\boldsymbol{\psi}_m\right),\boldsymbol{f}\left(\boldsymbol{\psi}_m\right)\right)$ 有界。

$$\left(\boldsymbol{f}\left(\boldsymbol{\psi}_m\right),\boldsymbol{f}\left(\boldsymbol{\psi}_m\right)\right)\le\left(a_1+b_1\left|\boldsymbol{\psi}_m\right|^{\frac{p}{2}},a_1+b_1\left|\boldsymbol{\psi}_m\right|^{\frac{p}{2}}\right)=a_1^2\left|\Omega\right|+2a_1b_1\int_\Omega\left|\boldsymbol{\psi}_m\right|^{\frac{p}{2}}\mathrm{d}x+b_1^2\int_\Omega\left|\boldsymbol{\psi}_m\right|^{p}\mathrm{d}x,$$

由于当 $m\to+\infty$ 时， $\boldsymbol{\psi}_m\to\boldsymbol{\psi}$ 在 $H^{2M}\left(\Omega\right)\cap H_0^M\left(\Omega\right)$ 中强收敛，由 Sobolev 嵌入定理可知 $\left|\boldsymbol{\psi}_m\right|_{L^p(\Omega)}\le\mathrm{const}$ ，其中当 $2M<n$ 时， $2\le p<\dfrac{2n}{n-2M}$ ；当 $2M\ge n$ 时， $2\le p<+\infty$ 。
所以

$$\int_\Omega\left|\boldsymbol{\psi}_m\right|^{p}\mathrm{d}x=\left|\boldsymbol{\psi}_m\right|_{L^p(\Omega)}^{p}\le\mathrm{const},$$

从而有

$$\int_\Omega\left|\boldsymbol{\psi}_m\right|^{\frac{p}{2}}\mathrm{d}x\le\left(\int_\Omega\left|\boldsymbol{\psi}_m\right|^{\frac{p}{2}\times2}\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega1^2\mathrm{d}x\right)^{\frac{1}{2}}=\left|\Omega\right|^{\frac{1}{2}}\left|\boldsymbol{\psi}_m\right|_{L^p(\Omega)}^{\frac{p}{2}}\le\mathrm{const},$$

因此， $\left(\boldsymbol{f}\left(\boldsymbol{\psi}_m\right),\boldsymbol{f}\left(\boldsymbol{\psi}_m\right)\right)$ 有界。

将 $\left(\left(-1\right)^{M}\Delta^{M}\boldsymbol{u}_{m}\left(x,0\right),\boldsymbol{u}_{mtt}\left(x,0\right)\right)$ 移至等式右边得

$$-\left(\left(-1\right)^{M}\Delta^{M}\boldsymbol{u}_{m}\left(x,0\right),\boldsymbol{u}_{mtt}\left(x,0\right)\right)=\left(\left(-1\right)^{M+1}\Delta^{M}\boldsymbol{u}_{m}\left(x,0\right),\boldsymbol{u}_{mtt}\left(x,0\right)\right),$$

因为 $\Delta^{M}\boldsymbol{u}_{m}\left(x,0\right)=\Delta^{M}\boldsymbol{\varphi}_m$ ，所以

$$\left(\left(-1\right)^{M+1}\Delta^{M}\boldsymbol{u}_{m}\left(x,0\right),\boldsymbol{u}_{mtt}\left(x,0\right)\right)=\left(\left(-1\right)^{M+1}\Delta^{M}\boldsymbol{\varphi}_m,\boldsymbol{u}_{mtt}\left(x,0\right)\right),$$

经计算得

$$\left(\left(-1\right)^{M+1}\Delta^{M}\boldsymbol{u}_{m}\left(x,0\right),\boldsymbol{u}_{mtt}\left(x,0\right)\right)\le\frac{1}{2\varepsilon_1}\left(\left(-1\right)^{M+1}\Delta^{M}\boldsymbol{\varphi}_m,\left(-1\right)^{M+1}\Delta^{M}\boldsymbol{\varphi}_m\right)+\frac{\varepsilon_1}{2}\left(\boldsymbol{u}_{mtt}\left(x,0\right),\boldsymbol{u}_{mtt}\left(x,0\right)\right),$$

将 $\left(\left(-1\right)^{M}\Delta^{M}\boldsymbol{u}_{m}\left(x,0\right),\boldsymbol{u}_{mtt}\left(x,0\right)\right)$ 移至等式右边得

$$-\left(\left(-1\right)^{M}\Delta^{M}\boldsymbol{u}_{mt}\left(x,0\right),\boldsymbol{u}_{mtt}\left(x,0\right)\right)=\left(\left(-1\right)^{M+1}\Delta^{M}\boldsymbol{u}_{mt}\left(x,0\right),\boldsymbol{u}_{mtt}\left(x,0\right)\right),$$

因为 $\Delta^{M}\boldsymbol{u}_{mt}\left(x,0\right)=\Delta^{M}\boldsymbol{\psi}_m$ ，所以

$$\left(\left(-1\right)^{M+1}\Delta^{M}\boldsymbol{u}_{mt}\left(x,0\right),\boldsymbol{u}_{mtt}\left(x,0\right)\right)=\left(\left(-1\right)^{M+1}\Delta^{M}\boldsymbol{\psi}_m,\boldsymbol{u}_{mtt}\left(x,0\right)\right),$$

经计算得

$$\left(\left(-1\right)^{M+1}\Delta^M \boldsymbol{u}_{mt}\left(x,0\right),\boldsymbol{u}_{mtt}\left(x,0\right)\right)\le\frac{1}{2\varepsilon_3}\left(\left(-1\right)^{M+1}\Delta^M\boldsymbol{\psi}_m,\left(-1\right)^{M+1}\Delta^M\boldsymbol{\psi}_m\right)+\frac{\varepsilon_3}{2}\left(\boldsymbol{u}_{mtt}\left(x,0\right),\boldsymbol{u}_{mtt}\left(x,0\right)\right),$$

所以

$$\begin{aligned}\left(\boldsymbol{u}_{mtt}\left(x,0\right),\boldsymbol{u}_{mtt}\left(x,0\right)\right)&=-\left(\left(-1\right)^M\Delta^M\boldsymbol{\varphi}_m,\boldsymbol{u}_{mtt}\left(x,0\right)\right)-\left(\left(-1\right)^M\Delta^M\boldsymbol{\psi}_m,\boldsymbol{u}_{mtt}\left(x,0\right)\right)\;+\left(\boldsymbol{f}\left(\boldsymbol{\psi}_m\right),\boldsymbol{u}_{mtt}\left(x,0\right)\right)\\&\le\frac{\varepsilon_1}{2}\left(\boldsymbol{u}_{mtt}\left(x,0\right),\boldsymbol{u}_{mtt}\left(x,0\right)\right)+\frac{1}{2\varepsilon_1}\left(\Delta^M\boldsymbol{\varphi}_m,\Delta^M\boldsymbol{\varphi}_m\right)+\frac{\varepsilon_3}{2}\left(\boldsymbol{u}_{mtt}\left(x,0\right),\boldsymbol{u}_{mtt}\left(x,0\right)\right)+\frac{1}{2\varepsilon_3}\left(\Delta^M\boldsymbol{\psi}_m,\Delta^M\boldsymbol{\psi}_m\right)\\&\quad+\frac{1}{2\varepsilon_2}\left(\boldsymbol{f}\left(\boldsymbol{\psi}_m\right),\boldsymbol{f}\left(\boldsymbol{\psi}_m\right)\right)+\frac{\varepsilon_2}{2}\left(\boldsymbol{u}_{mtt}\left(x,0\right),\boldsymbol{u}_{mtt}\left(x,0\right)\right),\end{aligned}$$

因此

$$\left(1-\frac{\varepsilon_1}{2}-\frac{\varepsilon_2}{2}-\frac{\varepsilon_3}{2}\right)\left(\boldsymbol{u}_{mtt}\left(x,0\right),\boldsymbol{u}_{mtt}\left(x,0\right)\right)\le\frac{1}{2\varepsilon_1}\left(\Delta^M\boldsymbol{\varphi}_m,\Delta^M\boldsymbol{\varphi}_m\right)+\frac{1}{2\varepsilon_3}\left(\Delta^M\boldsymbol{\psi}_m,\Delta^M\boldsymbol{\psi}_m\right)+\frac{1}{2\varepsilon_2}\left(\boldsymbol{f}\left(\boldsymbol{\psi}_m\right),\boldsymbol{f}\left(\boldsymbol{\psi}_m\right)\right).$$

取 $\varepsilon_1,\varepsilon_2,\varepsilon_3$ 充分小，使得 $\left(1-\frac{\varepsilon_1}{2}-\frac{\varepsilon_2}{2}-\frac{\varepsilon_3}{2}\right)\ge\frac{1}{2}$，

当 $m\to+\infty$, $\boldsymbol{\psi}_m\left(x\right)$ 在 $H^{2M}\left(\Omega\right)\cap H_0^M\left(\Omega\right)$ 中强收敛于 $\boldsymbol{\psi}\left(x\right)$，故当 $m\to+\infty$，$\left(\Delta^M\boldsymbol{\psi}_m,\Delta^M\boldsymbol{\psi}_m\right)\to\left(\Delta^M\boldsymbol{\psi},\Delta^M\boldsymbol{\psi}\right)$，$\left(\Delta^M\boldsymbol{\psi}_m,\Delta^M\boldsymbol{\psi}_m\right)$ 能用一与 m 无关的正常数控制住，同样道理，$\left(\Delta^M\boldsymbol{\varphi}_m,\Delta^M\boldsymbol{\varphi}_m\right)$ 能用一与 m 无关的正常数控制住，类似于前面的推导，$\left(\boldsymbol{f}\left(\boldsymbol{\psi}_m\right),\boldsymbol{f}\left(\boldsymbol{\psi}_m\right)\right)$ 也能用一与 m 无关的正常数控制住，类似前面推导知 $\frac{1}{2}\left(\Delta^M\boldsymbol{\psi}_m,\Delta^M\boldsymbol{\psi}_m\right)$ 能用一与 m 无关的正常数控制住，所以 $\left|\boldsymbol{u}_{mtt}\left(x,0\right)\right|_{L^2(\Omega)}^2\le\mathrm{const}$。

又由已证的不等式

$$\frac{1}{2}\left(\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt}\right)+\frac{1}{2}\left(\nabla^M\boldsymbol{u}_{mt},\nabla^M\boldsymbol{u}_{mt}\right)+\left[\nabla^M\boldsymbol{u}_{mtt},\nabla^M\boldsymbol{u}_{mtt}\right]\le\frac{1}{2}\left(\boldsymbol{u}_{mtt}\left(x,0\right),\boldsymbol{u}_{mtt}\left(x,0\right)\right)+\frac{1}{2}\left(\Delta^M\boldsymbol{\psi}_m,\Delta^M\boldsymbol{\psi}_m\right)+k_0\left[\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt}\right],$$

因为 $\left|\boldsymbol{u}_{mtt}\left(x,0\right)\right|_{L^2(\Omega)}^2$ 有界，类似于前面的推导知 $\frac{1}{2}\left(\Delta^M\boldsymbol{\psi}_m,\Delta^M\boldsymbol{\psi}_m\right)$ 也能用一与 m 无关的正常数控制住，所以

$$\frac{1}{2}\left(\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt}\right)+\frac{1}{2}\left(\nabla^M\boldsymbol{u}_{mt},\nabla^M\boldsymbol{u}_{mt}\right)+\left[\nabla^M\boldsymbol{u}_{mtt},\nabla^M\boldsymbol{u}_{mtt}\right]\le k_0\left[\boldsymbol{u}_{mtt},\boldsymbol{u}_{mtt}\right]+C,$$

其中 C 为正常数，再根据 Gronwall 不等式得

$$\left|\boldsymbol{u}_{mtt}\right|_{L^2(\Omega)}^2+\left\|\nabla^M\boldsymbol{u}_{mtt}\right\|_{L^2(\Omega)}^2+\left|\nabla^M\boldsymbol{u}_{mt}\right|_{L^2(\Omega)}^2\le\mathrm{const}\quad\left(0\le t\le T\right),$$

引理 2.2 证毕！

引理 2.3：若条件(1.5)，(1.6)成立，则有估计式

$$\left|\nabla^M\boldsymbol{u}_{mt}\right|_{L^2(\Omega)}^2+\left|\Delta^M\boldsymbol{u}_m\right|_{L^2(\Omega)}^2+\left\|\Delta^M\boldsymbol{u}_{mt}\right\|_{L^2(\Omega)}^2\le\mathrm{const}.$$

证明：方程(2.4)两边同乘以 $\lambda_k g'_{mik}\left(t\right)$ 得

$$\left(u_{mitt},\lambda_k g'_{mik}\left(t\right)\omega_k\right)+\left(\left(-1\right)^M\Delta^M u_{mi},\lambda_k g'_{mik}\left(t\right)\omega_k\right)+\left(\left(-1\right)^M\Delta^M u_{mit},\lambda_k g'_{mik}\left(t\right)\omega_k\right)=\left(f_i\left(\boldsymbol{u}_{mt}\right),\lambda_k g'_{mik}\left(t\right)\omega_k\right),$$

关于 k 从 1 到 m 作和得

$$\left(u_{mitt},\left(-1\right)^M\Delta^M u_{mit}\right)+\left(\left(-1\right)^M\Delta^M u_{mi},\left(-1\right)^M\Delta^M u_{mit}\right)+\left(\left(-1\right)^M\Delta^M u_{mit},\left(-1\right)^M\Delta^M u_{mit}\right)=\left(f_i\left(\boldsymbol{u}_{mt}\right),\left(-1\right)^M\Delta^M u_{mit}\right),$$

关于 i 从 1 到 N 作和得

$$\left(\boldsymbol{u}_{mtt},(-1)^M\Delta^M\boldsymbol{u}_{mt}\right)+\left((-1)^M\Delta^M\boldsymbol{u}_m,(-1)^M\Delta^M\boldsymbol{u}_{mt}\right)+\left((-1)^M\Delta^M\boldsymbol{u}_{mt},(-1)^M\Delta^M\boldsymbol{u}_{mt}\right)=\left(\boldsymbol{f}(\boldsymbol{u}_{mt}),(-1)^M\Delta^M\boldsymbol{u}_{mt}\right),$$

对于右边项，

$$\left(\boldsymbol{f}(\boldsymbol{u}_{mt}),(-1)^M\Delta^M\boldsymbol{u}_{mt}\right)\le\frac{1}{2\varepsilon_1}\left(\boldsymbol{f}(\boldsymbol{u}_{mt}),\boldsymbol{f}(\boldsymbol{u}_{mt})\right)+\frac{\varepsilon_1}{2}\left((-1)^M\Delta^M\boldsymbol{u}_{mt},(-1)^M\Delta^M\boldsymbol{u}_{mt}\right),$$

取 ε_1 充分小，使得 $1-\frac{\varepsilon_1}{2}\ge\frac{1}{2}$，所以

$$\left(\boldsymbol{u}_{mtt},(-1)^M\Delta^M\boldsymbol{u}_{mt}\right)+\left((-1)^M\Delta^M\boldsymbol{u}_m,(-1)^M\Delta^M\boldsymbol{u}_{mt}\right)+\frac{1}{2}\left(\Delta^M\boldsymbol{u}_{mt},\Delta^M\boldsymbol{u}_{mt}\right)\le\frac{1}{2\varepsilon_1}\left(\boldsymbol{f}(\boldsymbol{u}_{mt}),\boldsymbol{f}(\boldsymbol{u}_{mt})\right),$$

由 *Green* 公式得

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\Delta^M\boldsymbol{u}_{mt},\Delta^M\boldsymbol{u}_{mt}\right)+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\Delta^M\boldsymbol{u}_m,\Delta^M\boldsymbol{u}_m\right)+\frac{1}{2}\left(\Delta^M\boldsymbol{u}_{mt},\Delta^M\boldsymbol{u}_{mt}\right)\le\frac{1}{2\varepsilon_1}\left(\boldsymbol{f}(\boldsymbol{u}_{mt}),\boldsymbol{f}(\boldsymbol{u}_{mt})\right),$$

两边从 0 到 t 积分得

$$\begin{aligned}&\frac{1}{2}\left(\Delta^M\boldsymbol{u}_{mt},\Delta^M\boldsymbol{u}_{mt}\right)+\frac{1}{2}\left(\Delta^M\boldsymbol{u}_m,\Delta^M\boldsymbol{u}_m\right)+\frac{1}{2}\left[\Delta^M\boldsymbol{u}_{mt},\Delta^M\boldsymbol{u}_{mt}\right]\\&\le\frac{1}{2}\left(\nabla^M\boldsymbol{\psi}_m,\nabla^M\boldsymbol{\psi}_m\right)+\frac{1}{2}\left(\Delta^M\boldsymbol{\varphi}_m,\Delta^M\boldsymbol{\varphi}_m\right)\ +\frac{1}{2\varepsilon_1}\left[\boldsymbol{f}(\boldsymbol{u}_{mt}),\boldsymbol{f}(\boldsymbol{u}_{mt})\right],\end{aligned}$$

由引理 2.1，引理 2.2 知，$\left|\boldsymbol{u}_{mt}\right|^2_{H^M(\Omega)}\le\mathrm{const}\ \left(0\le t\le T\right)$，

$$\left(\boldsymbol{f}(\boldsymbol{u}_{mt}),\boldsymbol{f}(\boldsymbol{u}_{mt})\right)\le\left(a_1+b_1\left|\boldsymbol{u}_{mt}\right|^{\frac{p}{2}},a_1+b_1\left|\boldsymbol{u}_{mt}\right|^{\frac{p}{2}}\right)=a_1^2\left|\Omega\right|+2a_1b_1\int_\Omega\left|\boldsymbol{u}_{mt}\right|^{\frac{p}{2}}\mathrm{d}x+b_1^2\int_\Omega\left|\boldsymbol{u}_{mt}\right|^p\mathrm{d}x,$$

由 Sobolev 嵌入定理得

(注意到当 $2M<n$ 时，$2\le p<\frac{2n}{n-2M}$；当 $2M\ge n$ 时，$2\le p<+\infty$)，

$$\int_\Omega\left|\boldsymbol{u}_{mt}\right|^p\mathrm{d}x=\left|\boldsymbol{u}_{mt}\right|^p_{L^p(\Omega)}\le\mathrm{const}\,,\quad\int_\Omega\left|\boldsymbol{u}_{mt}\right|^{\frac{p}{2}}\mathrm{d}x\le\left(\int_\Omega\left|\boldsymbol{u}_{mt}\right|^{\frac{p}{2}\times2}\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega1^2\mathrm{d}x\right)^{\frac{1}{2}}\le C\left|\boldsymbol{u}_{mt}\right|^{\frac{p}{2}}_{L^p(\Omega)}\le\mathrm{const}.$$

所以 $\left(\boldsymbol{f}(\boldsymbol{u}_{mt}),\boldsymbol{f}(\boldsymbol{u}_{mt})\right)\le\mathrm{const}$ 。在下面不等式

$$\begin{aligned}&\frac{1}{2}\left(\Delta^M\boldsymbol{u}_{mt},\Delta^M\boldsymbol{u}_{mt}\right)+\frac{1}{2}\left(\Delta^M\boldsymbol{u}_m,\Delta^M\boldsymbol{u}_m\right)+\frac{1}{2}\left[\Delta^M\boldsymbol{u}_{mt},\Delta^M\boldsymbol{u}_{mt}\right]\\&\le\frac{1}{2}\left(\nabla^M\boldsymbol{\psi}_m,\nabla^M\boldsymbol{\psi}_m\right)+\frac{1}{2}\left(\Delta^M\boldsymbol{\varphi}_m,\Delta^M\boldsymbol{\varphi}_m\right)\ +\frac{1}{2\varepsilon_1}\left[\boldsymbol{f}(\boldsymbol{u}_{mt}),\boldsymbol{f}(\boldsymbol{u}_{mt})\right],\end{aligned}$$

中，由于当 $m\to+\infty$ 时，$\boldsymbol{\varphi}_m(x)$ 在 $H^{2M}(\Omega)\cap H_0^M(\Omega)$ 中强收敛于 $\boldsymbol{\varphi}(x)$，$\boldsymbol{\psi}_m(x)$ 在 $H^{2M}(\Omega)\cap H_0^M(\Omega)$ 强收敛于 $\boldsymbol{\psi}(x)$，因此当 $m\to+\infty$，$\frac{1}{2}\left(\nabla^M\boldsymbol{\psi}_m,\nabla^M\boldsymbol{\psi}_m\right)+\frac{1}{2}\left(\Delta^M\boldsymbol{\varphi}_m,\Delta^M\boldsymbol{\varphi}_m\right)$ 收敛于 $\frac{1}{2}\left(\nabla^M\boldsymbol{\psi},\nabla^M\boldsymbol{\psi}\right)+\frac{1}{2}\left(\Delta^M\boldsymbol{\varphi},\Delta^M\boldsymbol{\varphi}\right)$，

$\frac{1}{2}\left(\nabla^M\boldsymbol{\psi}_m,\nabla^M\boldsymbol{\psi}_m\right)+\frac{1}{2}\left(\Delta^M\boldsymbol{\varphi}_m,\Delta^M\boldsymbol{\varphi}_m\right)$

能用一与 m 无关的正常数控制住，因此由以上不等式得

$$\frac{1}{2}\left(\Delta^M\boldsymbol{u}_m,\Delta^M\boldsymbol{u}_m\right)+\frac{1}{2}\left(\Delta^M\boldsymbol{u}_{mt},\Delta^M\boldsymbol{u}_{mt}\right)+\frac{1}{2}\left[\Delta^M\boldsymbol{u}_{mt},\Delta^M\boldsymbol{u}_{mt}\right]\le\mathrm{const},$$

即

$$\left|\nabla^M \boldsymbol{u}_{mt}\right|^2_{L^2(\Omega)} + \left|\Delta^M \boldsymbol{u}_m\right|^2_{L^2(\Omega)} + \left\|\Delta^M \boldsymbol{u}_{mt}\right\|^2_{L^2(\Omega)} \le \text{const}.$$

引理 2.3 证毕！

引理 2.4：在引理 2.3 条件下成立不等式 $\left|\Delta^M \boldsymbol{u}_{mt}\right|^2_{L^2(\Omega)} \le \text{const}\ \ (0 \le t \le T)$。

证明：方程(2.4)两边同乘以 $\lambda_k g'_{mik}(t)$，

$$\left(u_{mitt}, \lambda_k g'_{mik}(t)\omega_k\right) + \left((-1)^M \Delta^M u_{mi}, \lambda_k g'_{mik}(t)\omega_k\right) + \left((-1)^M \Delta^M u_{mit}, \lambda_k g'_{mik}(t)\omega_k\right) = \left(f_i(\boldsymbol{u}_{mt}), \lambda_k g'_{mik}(t)\omega_k\right),$$

关于 k 从 1 到 m 作和

$$\left(u_{mitt}, (-1)^M \Delta^M u_{mit}\right) + \left((-1)^M \Delta^M u_{mi}, (-1)^M \Delta^M u_{mit}\right) + \left((-1)^M \Delta^M u_{mit}, (-1)^M \Delta^M u_{mit}\right) = \left(f_i(\boldsymbol{u}_{mt}), (-1)^M \Delta^M u_{mit}\right)$$

关于 i 从 1 到 N 作和

$$\left(\boldsymbol{u}_{mtt}, (-1)^M \Delta^M \boldsymbol{u}_{mt}\right) + \left((-1)^M \Delta^M \boldsymbol{u}_m, (-1)^M \Delta^M \boldsymbol{u}_{mt}\right) + \left((-1)^M \Delta^M \boldsymbol{u}_{mt}, (-1)^M \Delta^M \boldsymbol{u}_{mt}\right) = \left(\boldsymbol{f}(\boldsymbol{u}_{mt}), (-1)^M \Delta^M \boldsymbol{u}_{mt}\right)$$

从而有

$$\left((-1)^M \Delta^M \boldsymbol{u}_{mt}, (-1)^M \Delta^M \boldsymbol{u}_{mt}\right) = -\left(\boldsymbol{u}_{mtt}, (-1)^M \Delta^M \boldsymbol{u}_{mt}\right) - \left((-1)^M \Delta^M \boldsymbol{u}_m, (-1)^M \Delta^M \boldsymbol{u}_{mt}\right) + \left(\boldsymbol{f}(\boldsymbol{u}_{mt}), (-1)^M \Delta^M \boldsymbol{u}_{mt}\right)$$

而

$$-\left(\boldsymbol{u}_{mtt}, (-1)^M \Delta^M \boldsymbol{u}_{mt}\right) \le \frac{\varepsilon_1}{2}\left(\Delta^M \boldsymbol{u}_{mt}, \Delta^M \boldsymbol{u}_{mt}\right) + \frac{1}{2\varepsilon_1}\left(\boldsymbol{u}_{mtt}, \boldsymbol{u}_{mtt}\right),$$

$$-\left((-1)^M \Delta^M \boldsymbol{u}_m, (-1)^M \Delta^M \boldsymbol{u}_{mt}\right) \le \frac{\varepsilon_2}{2}\left(\Delta^M \boldsymbol{u}_{mt}, \Delta^M \boldsymbol{u}_{mt}\right) + \frac{1}{2\varepsilon_2}\left(\Delta^M \boldsymbol{u}_m, \Delta^M \boldsymbol{u}_m\right),$$

$$\left(\boldsymbol{f}(\boldsymbol{u}_{mt}), (-1)^M \Delta^M \boldsymbol{u}_{mt}\right) \le \frac{\varepsilon_3}{2}\left((-1)^M \Delta^M \boldsymbol{u}_{mt}, (-1)^M \Delta^M \boldsymbol{u}_{mt}\right) + \frac{1}{2\varepsilon_3}\left(\boldsymbol{f}(\boldsymbol{u}_{mt}), \boldsymbol{f}(\boldsymbol{u}_{mt})\right),$$

$$\begin{aligned}\left(\Delta^M \boldsymbol{u}_{mt}, \Delta^M \boldsymbol{u}_{mt}\right) &\le \frac{\varepsilon_1}{2}\left((-1)^M \Delta^M \boldsymbol{u}_{mt}, (-1)^M \Delta^M \boldsymbol{u}_{mt}\right) + \frac{1}{2\varepsilon_1}\left(\boldsymbol{u}_{mtt}, \boldsymbol{u}_{mtt}\right) \\ &+ \frac{\varepsilon_2}{2}\left((-1)^M \Delta^M \boldsymbol{u}_{mt}, (-1)^M \Delta^M \boldsymbol{u}_{mt}\right) + \frac{1}{2\varepsilon_2}\left(\Delta^M \boldsymbol{u}_m, \Delta^M \boldsymbol{u}_m\right) \\ &+ \frac{\varepsilon_3}{2}\left(\Delta^M \boldsymbol{u}_{mt}, \Delta^M \boldsymbol{u}_{mt}\right) + \frac{1}{2\varepsilon_3}\left(\boldsymbol{f}(\boldsymbol{u}_m), \boldsymbol{f}(\boldsymbol{u}_m)\right),\end{aligned}$$

所以

$$\left(1 - \frac{\varepsilon_1}{2} - \frac{\varepsilon_2}{2} - \frac{\varepsilon_3}{2}\right)\left(\Delta^M \boldsymbol{u}_{mt}, \Delta^M \boldsymbol{u}_{mt}\right) \le \frac{1}{2\varepsilon_1}\left(\boldsymbol{u}_{mtt}, \boldsymbol{u}_{mtt}\right) + \frac{1}{2\varepsilon_2}\left(\Delta^M \boldsymbol{u}_m, \Delta^M \boldsymbol{u}_m\right) + \frac{1}{2\varepsilon_3}\left(\boldsymbol{f}(\boldsymbol{u}_{mt}), \boldsymbol{f}(\boldsymbol{u}_{mt})\right).$$

由引理 2.3 的结论和引理 2.3 的证明过程及引理 2.2 知 $\left(\Delta^M \boldsymbol{u}_m, \Delta^M \boldsymbol{u}_m\right) \le \text{const}$，$\left(\boldsymbol{f}(\boldsymbol{u}_{mt}), \boldsymbol{f}(\boldsymbol{u}_{mt})\right) \le \text{const}$，$\left(\boldsymbol{u}_{mtt}, \boldsymbol{u}_{mtt}\right) \le \text{const}$, 取 ε_1，ε_2，ε_3 充分小, 使 $1 - \frac{\varepsilon_1}{2} - \frac{\varepsilon_2}{2} - \frac{\varepsilon_3}{2} \ge \frac{1}{2}$，所以 $\left(\Delta^M \boldsymbol{u}_{mt}, \Delta^M \boldsymbol{u}_{mt}\right) \le \text{const}$，即 $\left|\Delta^M \boldsymbol{u}_{mt}\right|^2_{L^2(\Omega)} \le \text{const}.$

引理 2.5 证毕！

定义：$u = u(x,t)$ 称为问题(1.1)~(1.4)于 $\Omega \times [0,T)$ 上的整体强解，若

$$\begin{aligned} &u \in L^\infty\left(0,T; H^{2M}(\Omega) \cap H_0^M(\Omega)\right), \\ &u_t \in L^\infty\left(0,T; H_0^M(\Omega) \cap H^{2M}(\Omega)\right), \\ &u_{tt} \in L^\infty\left(0,T; L^2(\Omega)\right) \cap L^2\left(0,T; H_0^M(\Omega)\right). \end{aligned}$$

对一切 $\boldsymbol{u}(x,t)\in C([0,T];L^2(\Omega))$ 成立

$$\int_0^T\left(\boldsymbol{u}_{tt}+(-1)^M\Delta^M\boldsymbol{u}+(-1)^M\Delta^M\boldsymbol{u}_t-\boldsymbol{f}(\boldsymbol{u}_t),\boldsymbol{u}(x,t)\right)\mathrm{d}t=0,\text{且 }\boldsymbol{u}\big|_{t=0}=\boldsymbol{\varphi}(x),\boldsymbol{u}_t\big|_{t=0}=\boldsymbol{\psi}(x).$$

定理 1：若 $\boldsymbol{\varphi}(x)\in H^{2M}(\Omega)\cap H_0^M(\Omega)$，
$\boldsymbol{\psi}(x)\in H^{2M}(\Omega)\cap H_0^M(\Omega)$，条件(1.5)，(1.6)成立，则问题(I)存在上述意义下的整体强解 $\boldsymbol{u}=\boldsymbol{u}(x,t)$。
证明：把(2.1)~(2.3)代入(II)得

$$\begin{cases}\sum_{j=1}^m g''_{mij}(t)(\omega_j,\omega_k)+\sum_{j=1}^m\left((-1)^M\Delta^M\omega_j,\omega_k\right)g'_{mij}(t)\\+\sum_{j=1}^m\left((-1)^M\Delta^M\omega_j,\omega_k\right)g_{mij}(t)=\left(f_i(\boldsymbol{u}_{mt}),\omega_k\right),k=1,2,\cdots,m,\\g_{mij}(0)=\xi_{mij},\\g'_{mij}(0)=\eta_{mij}.\end{cases}$$

由引理 2.1~引理 2.4 知 $\left|\left(f_i(\boldsymbol{u}_{mt}),\omega_k\right)\right|$ 可用一与 $\boldsymbol{u}_{mt}$ 无关的正常数控制住，因此由常微分方程理论知上面方程组有整体解 $g_{mij}(t)$，所以(II)有整体解 $u_{mi}(x,t)$。由引理 2.1~引理 2.4 知

$$|\boldsymbol{u}_m|_{L^2(\Omega)}^2\le\text{const}t,\quad|\boldsymbol{u}_{mtt}|_{L^2(\Omega)}^2\le\text{const}t,$$

$$|\boldsymbol{u}_{mt}|_{L^2(\Omega)}^2\le\text{const}t,\quad\left|\Delta^M\boldsymbol{u}_{mt}\right|_{L^2(\Omega)}^2\le\text{const}t,$$

$$\left|\nabla^M\boldsymbol{u}_{mt}\right|_{L^2(\Omega)}^2\le\text{const}t,\quad\left|\Delta^M\boldsymbol{u}_m\right|_{L^2(\Omega)}^2\le\text{const}t.$$

由弱紧性可得，存在 $\{\boldsymbol{u}_m\}$ 的一个子序列，不妨设为 $\{\boldsymbol{u}_v\}$，当 $v\to\infty$ 时，

$$\boldsymbol{u}_v\text{ 弱*收敛到 }\boldsymbol{u}\tag{2.5}$$

于 $L^\infty([0,T],H^{2M}(\Omega)\cap H_0^M(\Omega))$ 中，

$$\boldsymbol{u}_{vt}\text{ 弱*收敛到 }\boldsymbol{u}_t\tag{2.6}$$

于 $L^\infty([0,T],H^{2M}(\Omega)\cap H_0^M(\Omega))\subset L^\infty([0,T],L^2(\Omega))$ 中，

$$\boldsymbol{u}_{vtt}\text{ 弱*收敛到 }\boldsymbol{u}_{tt}\tag{2.7}$$

于 $L^2([0,T],H_0^M(\Omega))\cap L^\infty([0,T],L^2(\Omega))$ 中，由引理 2.1~引理 2.4 知，$\boldsymbol{u}_{vt},\boldsymbol{u}_{vtt},\nabla\boldsymbol{u}_{vt}$ (由内插不等式知 $|\nabla\boldsymbol{u}_{vt}|_{L^2(\Omega)}$ 能用 $|\boldsymbol{u}_{vt}|_{L^2(\Omega)}$ 与 $\left|\nabla^M\boldsymbol{u}_{vt}\right|_{L^2(\Omega)}$ 控制住，而 $|\boldsymbol{u}_{vt}|_{L^2(\Omega)}$ 与 $\left|\nabla^M\boldsymbol{u}_{vt}\right|_{L^2(\Omega)}$ 是有界的，故 $|\nabla\boldsymbol{u}_{vt}|_{L^2(\Omega)}$ 有界)。
于 $L^\infty(0,T;L^2(\Omega))\subset L^2(0,T;L^2(\Omega))=L^2(Q_T)$ $(Q_T=\Omega\times(0,T))$ 中有界，因此 $\boldsymbol{u}_{vt}$ 于 $H^1(Q_T)$ 中有界，由 $H^1(Q_T)$ 紧嵌入到 $L^2(Q_T)$ 中知，$\boldsymbol{u}_{vt}$ 可选出一子列(仍记为 $\boldsymbol{u}_{vt}$)使 $\boldsymbol{u}_{vt}$ 在 $L^2(Q_T)$ 中强收敛且几乎处处收敛到 $\boldsymbol{u}_t$，

$$\begin{aligned}\left(\boldsymbol{f}(\boldsymbol{u}_{vt}),\boldsymbol{f}(\boldsymbol{u}_{vt})\right)&\le\left(a_1+b_1|\boldsymbol{u}_{vt}|^{\frac{p}{2}},a_1+b_1|\boldsymbol{u}_{vt}|^{\frac{p}{2}}\right)\\&=a_1{}^2|\Omega|+2a_1b_1\int_\Omega|\boldsymbol{u}_{vt}|^{\frac{p}{2}}\mathrm{d}x+b_1{}^2\int_\Omega|\boldsymbol{u}_{vt}|^p\mathrm{d}x,\end{aligned}$$

由引理 2.1 及引理 2.4 知 $|\boldsymbol{u}_{vt}|_{H^M(\Omega)}^2\le\text{const}t$，由 Sobolev 嵌入定理得 $|\boldsymbol{u}_{vt}|_{L^p(\Omega)}^2\le C|\boldsymbol{u}_{vt}|_{H^M(\Omega)}^2\le\text{const}t$ (当 $2M<n$ 时，$2\le p<\dfrac{2n}{n-2M}$；当 $2M\ge n$ 时，$2\le p<+\infty$)。

因为 $\int_\Omega|\boldsymbol{u}_{vt}|^p\mathrm{d}x=|\boldsymbol{u}_{vt}|_{L^p(\Omega)}^p\le\text{const}t$，$\int_\Omega|\boldsymbol{u}_{vt}|^{\frac{p}{2}}\mathrm{d}x\le\left(\int_\Omega|\boldsymbol{u}_{vt}|^p\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega1^2\mathrm{d}x\right)^{\frac{1}{2}}=|\boldsymbol{u}_{vt}|_{L^p(\Omega)}^{\frac{p}{2}}\le\text{const}t$，

所以

$$\left(\boldsymbol{f}\left(\boldsymbol{u}_{vt}\right),\boldsymbol{f}\left(\boldsymbol{u}_{vt}\right)\right)=\left|\boldsymbol{f}\left(\boldsymbol{u}_{vt}\right)\right|_{L^2(\Omega)}^2\le \text{const}.$$

由于当 $v\to+\infty$ 时，$\boldsymbol{u}_{vt}\to\boldsymbol{u}_t$ 在 $L^2\left(Q_T\right)$ 中强收敛且几乎处处收敛及 $\left(\boldsymbol{f}\left(\boldsymbol{u}_{vt}\right),\boldsymbol{f}\left(\boldsymbol{u}_{vt}\right)\right)=\left|\boldsymbol{f}\left(\boldsymbol{u}_{vt}\right)\right|_{L^2(\Omega)}^2\le \text{const}$，由[7](p. 11 引理 1.3)可知 $\boldsymbol{f}\left(\boldsymbol{u}_{vt}\right)\to\boldsymbol{f}\left(\boldsymbol{u}_t\right)$ 在 $L^2\left(Q_T\right)$ 中弱收敛。

在 $\left(u_{mitt},\omega_k\right)+\left(\left(-1\right)^M\Delta^M u_{mi},\omega_k\right)+\left(\left(-1\right)^M\Delta^M u_{mit},\omega_k\right)=\left(f_i\left(\boldsymbol{u}_{mt}\right),\omega_k\right)$ 中取 $m=v$ 得

$$\left(u_{vitt},\omega_k\right)+\left(\left(-1\right)^M\Delta^M u_{vi},\omega_k\right)+\left(\left(-1\right)^M\Delta^M u_{vit},\omega_k\right)=\left(f_i\left(\boldsymbol{u}_{vt}\right),\omega_k\right),$$

两边同乘 $d_{ki}\left(t\right)\in C\left[0,T\right]\left(k=1,2,\cdots;i=1,2,\cdots,N\right)$ 得

$$\left(u_{vitt},d_{ki}\left(t\right)\omega_k\right)+\left(\left(-1\right)^M\Delta^M u_{vi},d_{ki}\left(t\right)\omega_k\right)+\left(\left(-1\right)^M\Delta^M u_{vit},d_{ki}\left(t\right)\omega_k\right)=\left(f_i\left(\boldsymbol{u}_{vt}\right),d_{ki}\left(t\right)\omega_k\right),$$

关于 $k=1,2,\cdots,v'\left(v'\le v\right)$ 求和得

$$\left(u_{vitt},\sum_{k=1}^{v'}d_{ki}\left(t\right)\omega_k\left(x\right)\right)+\left(\left(-1\right)^M\Delta^M u_{vi},\sum_{k=1}^{v'}d_{ki}\left(t\right)\omega_k\left(x\right)\right)+\left(\left(-1\right)^M\Delta^M u_{vit},\sum_{k=1}^{v'}d_{ki}\left(t\right)\omega_k\left(x\right)\right)=\left(f_i\left(\boldsymbol{u}_{vt}\right),\sum_{k=1}^{v'}d_{ki}\left(t\right)\omega_k\left(x\right)\right)$$

关于 t 从 0 到 T 积分得

$$\begin{aligned}&\int_0^T\left(u_{vitt},\sum_{k=1}^{v'}d_{ki}\left(t\right)\omega_k\left(x\right)\right)\mathrm{d}t+\int_0^T\left(\left(-1\right)^M\Delta^M u_{vi},\sum_{k=1}^{v'}d_{ki}\left(t\right)\omega_k\left(x\right)\right)\mathrm{d}t\\&+\int_0^T\left(\left(-1\right)^M\Delta^M u_{vit},\sum_{k=1}^{v'}d_{ki}\left(t\right)\omega_k\left(x\right)\right)\mathrm{d}t=\int_0^T\left(f_i\left(\boldsymbol{u}_{vt}\right),\sum_{k=1}^{v'}d_{ki}\left(t\right)\omega_k\left(x\right)\right)\mathrm{d}t\end{aligned}\quad\left(i=1,2,\cdots,N\right),$$

由(2.5)~(2.7)及 $\left\{\boldsymbol{f}\left(\boldsymbol{u}_{vt}\right)\right\}$ 于 $L^2\left(Q_T\right)$ 中弱收敛于 $\boldsymbol{f}\left(\boldsymbol{u}_t\right)$ (对应有 $f_i\left(\boldsymbol{u}_{vt}\right)$ 于 $L^2\left(Q_T\right)$ 中弱收敛于 $f_i\left(\boldsymbol{u}_t\right)\left(i=1,2,\cdots,N\right)$)，在上式中令 $v\to+\infty$ 得

$$\begin{aligned}&\int_0^T\left(u_{itt},\sum_{k=1}^{v'}d_{ki}\left(t\right)\omega_k\left(x\right)\right)\mathrm{d}t+\int_0^T\left(\left(-1\right)^M\Delta^M u_{i},\sum_{k=1}^{v'}d_{ki}\left(t\right)\omega_k\left(x\right)\right)\mathrm{d}t\\&+\int_0^T\left(\left(-1\right)^M\Delta^M u_{it},\sum_{k=1}^{v'}d_{ki}\left(t\right)\omega_k\left(x\right)\right)\mathrm{d}t=\int_0^T\left(f_i\left(\boldsymbol{u}_{t}\right),\sum_{k=1}^{v'}d_{ki}\left(t\right)\omega_k\left(x\right)\right)\mathrm{d}t,\end{aligned}\quad\left(i=1,2,\cdots,N\right),$$

由于 $\left\{\omega_k\left(x\right)\right\}_{k=1}^{+\infty}$ 构成 $L^2\left(\Omega\right)$ 的一组标准正交基，而 $d_{ki}\left(t\right)\in C\left[0,T\right]$ $\left(k=1,2,\cdots;i=1,2,\cdots,N\right)$，$\left\{\sum_{k=1}^{v'}d_{ki}\left(t\right)\omega_k\left(x\right)\middle|v'=1,2,\cdots;i=1,2,\cdots,N\right\}$ 在空间 $C\left(\left[0,T\right];L^2\left(\Omega\right)\right)$ 中稠密，因此对任意 $\mu_i\left(x,t\right)\in C\left(\left[0,T\right];L^2\left(\Omega\right)\right)\left(i=1,2,\cdots,N\right)$ 成立

$$\begin{aligned}&\int_0^T\left(u_{itt}\left(x,t\right),\mu_i\left(x,t\right)\right)\mathrm{d}t+\int_0^T\left(\left(-1\right)^M\Delta^M u_i\left(x,t\right),\mu_i\left(x,t\right)\right)\mathrm{d}t\\&+\int_0^T\left(\left(-1\right)^M\Delta^M u_{it}\left(x,t\right),\mu_i\left(x,t\right)\right)\mathrm{d}t=\int_0^T\left(f_i\left(\boldsymbol{u}_t\right),\mu_i\left(x,t\right)\right)\mathrm{d}t,\end{aligned}\quad\left(i=1,2,\cdots,N\right),$$

上式关于 $i=1,2,\cdots,N$ 求和得对任意 $\boldsymbol{u}\left(x,t\right)=\left(\mu_1\left(x,t\right),\mu_2\left(x,t\right),\cdots,\mu_N\left(x,t\right)\right)^T\in C\left(\left[0,T\right];L^2\left(\Omega\right)\right)$ 成立

$$\begin{aligned}&\int_0^T\left(\boldsymbol{u}_{tt}\left(x,t\right),\boldsymbol{u}\left(x,t\right)\right)\mathrm{d}t+\int_0^T\left(\left(-1\right)^M\Delta^M\boldsymbol{u}\left(x,t\right),\boldsymbol{u}\left(x,t\right)\right)\mathrm{d}t\\&+\int_0^T\left(\left(-1\right)^M\Delta^M\boldsymbol{u}_t\left(x,t\right),\boldsymbol{u}\left(x,t\right)\right)\mathrm{d}t=\int_0^T\left(\boldsymbol{f}\left(\boldsymbol{u}_t\right),\boldsymbol{u}\left(x,t\right)\right)\mathrm{d}t.\end{aligned}$$

最后证明 $\boldsymbol{u}$ 满足初始条件 $\boldsymbol{u}(x,0)=\boldsymbol{\varphi}(x)$，由(2.5)，(2.6)知，$u_{vi}(x,t),u_i(x,t)\in C\left([0,T],H^{2M}(\Omega)\cap H_0^M(\Omega)\right)$，故当 $v\to\infty$ 时，由(2.5)知 $u_{vi}(x,0)$ 弱收敛到 $u_i(x,0)$ 于 $H^{2M}(\Omega)\cap H_0^M(\Omega)$ 中，又已知 $v\to\infty$ 时，$u_{vi}(x,0)\to\varphi_i(x)$ $(i=1,2,\cdots,N)$ 在 $H^{2M}(\Omega)\cap H_0^M(\Omega)$ 中强收敛，就得到了 $u_i(x,0)=\varphi_i(x)(i=1,2,\cdots,N)$。所以 $\boldsymbol{u}(x,0)=\boldsymbol{\varphi}(x)$。

再证 $\boldsymbol{u}(x,t)$ 满足初始条件 $u_{it}(x,0)=\psi_i(x)(i=1,2,\cdots,N)$，由(2.6)，(2.7)知 $u_{vit}(x,t),u_{it}(x,t)\in C\left([0,T],L^2(\Omega)\right)$，因此，当 $v\to\infty$ 时，由(2.6)知 $u_{vit}(x,0)$ 弱收敛到 $u_{it}(x,0)$ 于 $L^2(\Omega)(i=1,2,\cdots,N)$ 中。又因为当 $v\to+\infty$ 时，$u_{vit}(x,0)\to\psi_i(x)$ $(i=1,2,\cdots,N)$ 在 $L^2(\Omega)$ 中强收敛，从而得到 $u_{it}(x,0)=\psi_i(x)(i=1,2,\cdots,N)$，所以 $\boldsymbol{u}_t(x,0)=\boldsymbol{\psi}(x)$。

所以 $\boldsymbol{u}=\boldsymbol{u}(x,t)$ 是问题(I)的强解。

3. 初边值问题(1.1)~(1.4)强解的唯一性

定理 2：若条件(1.5)，(1.6)成立，则非线性问题(I)整体强解唯一。

证明：设 $\boldsymbol{u}^1$，$\boldsymbol{u}^2$ 为非线性问题(I)的两个整体强解。

令 $\boldsymbol{w}=\boldsymbol{u}^1-\boldsymbol{u}^2$，则 $\boldsymbol{w}$ 满足

$$\begin{cases}\boldsymbol{w}_{tt}+(-1)^M\Delta^M\boldsymbol{w}+(-1)^M\Delta^M\boldsymbol{w}_t=f\left(\boldsymbol{u}_t^1\right)-f\left(\boldsymbol{u}_t^2\right),\\ D^\gamma\boldsymbol{w}(x,t)\Big|_{\partial\Omega\times[0,T]}=\mathbf{0},\ 0\le|\gamma|\le M-1,\\ \boldsymbol{w}(x,0)=\mathbf{0},\\ \boldsymbol{w}_t(x,0)=\mathbf{0}.\end{cases}$$

两边用 $\boldsymbol{w}_t$ 做内积，得

$$(\boldsymbol{w}_{tt},\boldsymbol{w}_t)+\left((-1)^M\Delta^M\boldsymbol{w},\boldsymbol{w}_t\right)+\left((-1)^M\Delta^M\boldsymbol{w}_t,\boldsymbol{w}_t\right)=\left(\frac{\partial\boldsymbol{f}(\boldsymbol{u}_t)}{\partial\boldsymbol{u}_t}\Big|_{\boldsymbol{u}_t^2+\theta\boldsymbol{w}_t}\boldsymbol{w}_t,\boldsymbol{w}_t\right)\quad(0<\theta<1),$$

由条件(1.5)知

$$\left(\frac{\partial\boldsymbol{f}(\boldsymbol{u}_t)}{\partial\boldsymbol{u}_t}\Big|_{\boldsymbol{u}_t^2+\theta\boldsymbol{w}_t}\boldsymbol{w}_t,\boldsymbol{w}_t\right)\le k_0(\boldsymbol{w}_t,\boldsymbol{w}_t)\ (0<\theta<1),$$

因此

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{w}_t,\boldsymbol{w}_t)+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\nabla^M\boldsymbol{w},\nabla^M\boldsymbol{w}\right)+\left(\nabla^M\boldsymbol{w}_t,\nabla^M\boldsymbol{w}_t\right)\le k_0(\boldsymbol{w}_t,\boldsymbol{w}_t)$$

两边从 0 到 t 积分得

$$\frac{1}{2}(\boldsymbol{w}_t,\boldsymbol{w}_t)+\frac{1}{2}\left(\nabla^M\boldsymbol{w},\nabla^M\boldsymbol{w}\right)+\left[\nabla^M\boldsymbol{w}_t,\nabla^M\boldsymbol{w}_t\right]\le k_0[\boldsymbol{w}_t,\boldsymbol{w}_t],$$

两边同时加上 $[\boldsymbol{w},\boldsymbol{w}_t]+\left[\nabla^M\boldsymbol{w},\nabla^M\boldsymbol{w}_t\right]$，左边将它化为 $\frac{1}{2}(\boldsymbol{w},\boldsymbol{w})+\frac{1}{2}\left(\nabla^M\boldsymbol{w},\nabla^M\boldsymbol{w}\right)$，右边将它估计为

$$[\boldsymbol{w},\boldsymbol{w}_t]+\left[\nabla^M\boldsymbol{w},\nabla^M\boldsymbol{w}_t\right]\le[\boldsymbol{w},\boldsymbol{w}]+[\boldsymbol{w}_t,\boldsymbol{w}_t]+\frac{1}{2\varepsilon}\left[\nabla^M\boldsymbol{w},\nabla^M\boldsymbol{w}\right]+\frac{\varepsilon}{2}\left[\nabla^M\boldsymbol{w}_t,\nabla^M\boldsymbol{w}_t\right],$$

因此有

$$\begin{aligned}&\frac{1}{2}(\boldsymbol{w}_t,\boldsymbol{w}_t)+\frac{1}{2}(\boldsymbol{w},\boldsymbol{w})+\left(\nabla^M\boldsymbol{w},\nabla^M\boldsymbol{w}\right)+\left[\nabla^M\boldsymbol{w}_t,\nabla^M\boldsymbol{w}_t\right]\\&\le[\boldsymbol{w},\boldsymbol{w}]+[\boldsymbol{w}_t,\boldsymbol{w}_t]+k_0[\boldsymbol{w}_t,\boldsymbol{w}_t]+\frac{1}{2\varepsilon}\left[\nabla^M\boldsymbol{w},\nabla^M\boldsymbol{w}\right]+\frac{\varepsilon}{2}\left[\nabla^M\boldsymbol{w}_t,\nabla^M\boldsymbol{w}_t\right],\end{aligned}$$

取 ε 充分小使 $1-\frac{\varepsilon}{2}\geq\frac{1}{2}$，得

$$\begin{aligned}&\frac{1}{2}\left(\boldsymbol{w}_t,\boldsymbol{w}_t\right)+\frac{1}{2}\left(\boldsymbol{w},\boldsymbol{w}\right)+\left(\nabla^M\boldsymbol{w},\nabla^M\boldsymbol{w}\right)+\frac{1}{2}\left[\nabla^M\boldsymbol{w}_t,\nabla^M\boldsymbol{w}_t\right]\\&\leq\left[\boldsymbol{w},\boldsymbol{w}\right]+\left(k_0+1\right)\left[\boldsymbol{w}_t,\boldsymbol{w}_t\right]+\frac{1}{2\varepsilon}\left[\nabla^M\boldsymbol{w},\nabla^M\boldsymbol{w}\right],\end{aligned}$$

由 Gronwall 不等式得

$$\left|\boldsymbol{w}\right|_{L^2(\Omega)}^2+\left|\nabla^M\boldsymbol{w}\right|_{L^2(\Omega)}^2+\left|\boldsymbol{w}_t\right|_{L^2(\Omega)}^2=0.$$

所以 $w\equiv 0$ a.e $\boldsymbol{u}^1=\boldsymbol{u}^2$ 。

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