Advances in Applied Mathematics 应用数学进展, 2014, 3, 134-139
Published Online August 2014 in Hans. http://www.hanspub.org/journal/aam
http://dx.doi.org/10.12677/aam.2014.33020

Uncertainty Principle for a Kind of Quaternionic Linear Canonical Transform

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Received: May 20th, 2014; revised: Jun. 18th, 2014; accepted: Jun. 27th, 2014

Abstract

In this paper, based on the properties of the left-sided quaternionic linear canonical transform (QLCT), an uncertainty principle is established for the left-sided QLCT. It states that the product of the variances of quaternion-valued signals in the spatial and frequency domains has a lower bound and only a 2D Gaussian signal minimizes the uncertainty principle.

Keywords

Quaternion, Left-Sided Quaternionic Linear Canonical Transform, Uncertainty Principle

一类四元数线性正则变换的不确定性原理

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收稿日期：2014年5月20日；修回日期：2014年6月18日；录用日期：2014年6月27日

摘　要

基于左边四元数线性正则变换的相关性态，本文建立左边四元数线性正则变换的不确定性原理。其表明

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四元数值的信号在时域和频域中方差的乘积具有下界，仅有二维的高斯信号能满足不确定性原理的等式。

关键词

四元数，左边四元数线性正则变换，不确定性原理

1. 引言

Hamilton 在 1843 年提出四元数的概念[1] [2]，作为最简单的超复数，人们不断地对其进行探讨，结合经典的傅立叶变换，学者们提出了四元数傅立叶变换[3]-[7]。由于四元数乘法的不可交换性，人们可以定义三种不同类型的四元数傅立叶变换，即左边四元数傅立叶变换，右边四元数傅立叶变换和双边四元数傅立叶变换。目前，众多文献在四元数域内运用右边四元数傅立叶变换取得了大量的研究成果。基于右边四元数傅立叶变换的性质，文献[7]解决了右边四元数傅立叶变换的不确定性原理。

经典的线性正则变换是由 Moshinsky 和 Collins 在 20 世纪 70 年代首先提出来的[8] [9]，作为傅立叶变换，分数阶傅立叶变换和菲涅尔变换的推广，其运用更加灵活且计算的复杂性和傅立叶变换的计算量大致相当[10]。与分数阶傅立叶变换一样，线性正则变换最早是被用于微分方程求解和光学系统分析，随着 20 世纪 90 年代分数阶傅立叶变换的发展，线性正则变换也开始逐渐在信号处理领域受到重视[11]-[18]。最近，结合四元数和线性正则变换，文献[19]提出了四元数线性正则变换，运用线性正则变换的相关性质推出了右边四元数线性正则变换的不确定性原理。其表明四元数值的信号在时域和频域的方差的乘积具有下界，仅有二维的高斯信号能最小化不确定性原理的不等式。

然而，右边四元数线性正则变换的不确定性原理不能直接推广到左边四元数线性正则变换和双边四元数线性正则变换。本文着重探讨左边四元数线性正则变换的不确定性原理。其主要难点在于文献[19]中定义的内积意义下的 Plancherel 定理不再成立。为此，为了克服四元数乘法的不可交换性，在 Plancherel 定理的证明过程中，借助四元数循环相乘的对称性质，我们运用标量内积的形式来解决此问题。结合已有的右边四元数线性正则变换不确定性原理的证明[19]，本文证明了左边四元数线性正则变换的不确定性原理。

2. 预备知识

四元数是一种将复数推广到四维空间的代数，即

$$H=\left\{q \mid q=q_r+q_i i+q_j j+q_k k, q_r, q_j, q_k \in R\right\},$$

其中 i，j，k 满足 $ij=-ji=k$， $jk=-kj=i$， $ik=-ki=-j$， $i^2=j^2=k^2=ijk=-1$。四元数 q 可以表示成标量 q_0 和非标量 $\underline{q}$ (通常也叫纯四元数)和的形式，即 $q=q_0+\underline{q}=q_0+q_1 i+q_2 j+q_3 k$。通过改变纯四元数的符号我们得到 q 的共轭 $\bar{q}=q_0-\underline{q}=q_0-q_1 i-q_2 j-q_3 k$，同时 q 的模 $|q|=\sqrt{q\bar{q}}=\sqrt{q_0^2+q_1^2+q_2^2+q_3^2}$。四元数值的函数 $f: R^2 \rightarrow H$ 可表示为 $f(X)=f_0(X)+f_1(X) i+f_2(X) j+f_3(X) k$， $f_k: R^2 \rightarrow R$，$k=0,1,2,3$，$X=(x_1, x_2)$。同时，对定义在 R^2 上的四元数值的函数 f，g 引入对称的标量内积如下

$$\langle f, g\rangle=Sc[(f, g)]=\int_{R^2} Sc\left[f(X) \overline{g(X)}\right] \mathrm{d}_X^2 \tag{1}$$

其中 $\mathrm{d}_X^2=\mathrm{d}_{x_1} \mathrm{d}_{x_2}$，$(f, g)=\int f(X) \overline{g(X)} \mathrm{d}_X^2$ 为向量值内积，$\overline{g(X)}$ 为 $g(X)$ 的共轭函数。在(1)中取 $f=g$ 可得 f 的模为 $\|f\|^2=(f, f)=\langle f, f\rangle=\int_{R^2}|f(X)|^2 \mathrm{d}_X^2$，空间 $L^2(R^2 ; H)=\left\{f \mid f: R^2 \rightarrow H,\|f\|<\infty\right\}$，对 $\forall f, g \in L^2(R^2 ; H)$，下面的 Cauchy-Schwarz 不等式成立

$$\left|\left\langle f,g\right\rangle\right| \le \left|\left(f,g\right)\right| \le \| f\|\|g\| \tag{2}$$

3. 主要结论

本节中首先给出左边四元数线性正则变换的定义，然后证明 Parseval 等式与左边线性正则变换的偏导性质。

当 $i=1,2$ 时，$\boldsymbol{A}_i=\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}\in R^{2\times 2}$ 是一个系数矩阵且满足 $\left|\boldsymbol{A}_i\right|=1$，$b_i\neq 0$，则对 $f\in L^2\left(R^2;H\right)$，左边四元数线性正则变换定义为

$$L_l^{i,j}\left(f\right)\left(U\right)=\int_{R^2} K_{A_1}^i\left(x_1,u_1\right)K_{A_2}^j\left(x_2,u_2\right)f\left(X\right)\mathrm{d}_X^2$$

其中 $K_{A_1}^i\left(x_1,u_1\right)=\dfrac{1}{\sqrt{2\pi i b_1}}\mathrm{e}^{i\left(\frac{a_1}{2b_1}x_1^2-\frac{1}{b_1}x_1u_1+\frac{d_1}{2b_1}u_1^2\right)}$，$K_{A_2}^j\left(x_2,u_2\right)=\dfrac{1}{\sqrt{2\pi j b_2}}\mathrm{e}^{j\left(\frac{a_2}{2b_2}x_2^2-\frac{1}{b_2}x_2u_2+\frac{d_2}{2b_2}u_2^2\right)}$。

定理 1 当 $i=1,2$ 时，若 $f_i\in L^2\left(R^2;H\right)$，则如下关系成立

$$\left\langle f_1,f_2\right\rangle=\left\langle L_l^{i,j}\left(f_1\right),L_l^{i,j}\left(f_2\right)\right\rangle$$

特别地，当 $f_1=f_2=f$ 时，可以得到 Parseval 等式，即 $\|f\|=\left\|L_l^{i,j}\left(f\right)\right\|$。

证明 运用对称的标量内积(1)，得到

$$\left\langle L_l^{i,j}\left(f_1\right),L_l^{i,j}\left(f_2\right)\right\rangle=Sc\left[\left(L_l^{i,j}\left(f_1\right),L_l^{i,j}\left(f_2\right)\right)\right]=Sc\left[\int_{R^2} L_l^{i,j}\left(f_1\right)\overline{L_l^{i,j}\left(f_2\right)}\mathrm{d}_U^2\right]$$

其中

$$\begin{aligned}\int_{R^2} L_l^{i,j}\left(f_1\right)\overline{L_l^{i,j}\left(f_2\right)}\mathrm{d}_U^2&=\int_{R^2}\left[\int_{R^2} K_{A_1}^i\left(x_1,u_1\right)K_{A_2}^j\left(x_2,u_2\right)f_1\left(X\right)\mathrm{d}_X^2\right]\times\overline{\left[\int_{R^2} K_{A_1}^i\left(y_1,u_1\right)K_{A_2}^j\left(y_2,u_2\right)f_2\left(Y\right)\mathrm{d}_Y^2\right]}\mathrm{d}_U^2\\&=\int_{R^6} K_{A_1}^i\left(x_1,u_1\right)K_{A_2}^j\left(x_2,u_2\right)f_1\left(X\right)\overline{f_2\left(Y\right)}K_{A_2}^{-j}\left(y_2,u_2\right)K_{A_1}^{-i}\left(y_1,u_1\right)\mathrm{d}_X^2\mathrm{d}_Y^2\mathrm{d}_U^2\end{aligned}$$

应用四元数循环相乘的对称性质，即

$$Sc\left[qrs\right]=Sc\left[rsq\right]\quad \forall q,r,s\in H$$

可得

$$\begin{aligned}&\left\langle L_l^{i,j}\left(f_1\right),L_l^{i,j}\left(f_2\right)\right\rangle\\&=\int_{R^6} Sc\left[K_{A_2}^j\left(x_2,u_2\right)f_1\left(X\right)\overline{f_2\left(Y\right)}K_{A_2}^{-j}\left(y_2,u_2\right)\left(K_{A_1}^{-i}\left(y_1,u_1\right)K_{A_1}^i\left(x_1,u_1\right)\right)\right]\mathrm{d}_X^2\mathrm{d}_Y^2\mathrm{d}_U^2\\&=\int_{R^5} Sc\left[K_{A_2}^j\left(x_2,u_2\right)f_1\left(X\right)\overline{f_2\left(Y\right)}K_{A_2}^{-j}\left(y_2,u_2\right)\left(\int_R\frac{1}{2\pi b_1}\mathrm{e}^{i\frac{a_1}{2b_1}\left(x_1^2-y_1^2\right)}\mathrm{e}^{i\frac{y_1-x_1}{b_1}u_1}\mathrm{d}_{u_1}\right)\right]\mathrm{d}_X^2\mathrm{d}_Y^2\mathrm{d}_{u_2}\\&=\int_{R^5} Sc\left[K_{A_2}^j\left(x_2,u_2\right)f_1\left(X\right)\overline{f_2\left(Y\right)}K_{A_2}^{-j}\left(y_2,u_2\right)\mathrm{e}^{i\frac{a_1}{2b_1}\left(x_1^2-y_1^2\right)}\delta\left(y_1-x_1\right)\right]\mathrm{d}_X^2\mathrm{d}_Y^2\mathrm{d}_{u_2}\\&=\int_{R^4} Sc\left[K_{A_2}^j\left(x_2,u_2\right)f_1\left(X\right)\overline{f_2\left(x_1,y_2\right)}K_{A_2}^{-j}\left(y_2,u_2\right)\right]\mathrm{d}_X^2\mathrm{d}_{y_2}\mathrm{d}_{u_2}\\&=\int_{R^4} Sc\left[f_1\left(X\right)\overline{f_2\left(x_1,y_2\right)}\left(K_{A_2}^{-j}\left(y_2,u_2\right)K_{A_2}^j\left(x_2,u_2\right)\right)\right]\mathrm{d}_X^2\mathrm{d}_{y_2}\mathrm{d}_{u_2}\end{aligned}$$

$$= \int_{R^2} Sc\left[f_1(X)\overline{f_2(x_1,y_2)}\left(\int_{R^2} \frac{1}{2\pi b_2} e^{j\frac{a_2}{2b_2}\left(x_2^2-y_2^2\right)} e^{j\frac{y_2-x_2}{b_2}u_2} \mathrm{d}_{u_2}\mathrm{d}_{y_2} \right)\right]\mathrm{d}_X^2$$

$$= \int_{R^2} Sc\left[f_1(X)\overline{f_2(x_1,y_2)}\left(\int_{R} e^{j\frac{a_2}{2b_2}\left(x_2^2-y_2^2\right)} \left(\int_{R} \frac{1}{2\pi b_2} e^{j\frac{y_2-x_2}{b_2}u_2} \mathrm{d}_{u_2} \right)\mathrm{d}_{y_2} \right)\right]\mathrm{d}_X^2$$

$$= \int_{R^2} Sc\left[f_1(X)\overline{f_2(x_1,y_2)}\left(\int_{R} e^{j\frac{a_2}{2b_2}\left(x_2^2-y_2^2\right)} \delta(y_2-x_2)\mathrm{d}_{y_2} \right)\right]\mathrm{d}_X^2$$

$$= \int_{R^2} Sc\left[f_1(X)\overline{f_2(X)}\right]\mathrm{d}_X^2 = \langle f_1, f_2\rangle,$$

其中 $\delta(y_1-x_1)=\int_R \frac{1}{2\pi b_1} e^{j\frac{y_1-x_1}{b_1}u_1}\mathrm{d}_{u_1}$ 表示 y_1-x_1 的狄拉克 δ 函数。

定理 1 说明四元数值函数在时域中能量和在四元数线性正则变换域中能量是守恒的。接下来，我们探讨左边线性正则变换的偏导性质。

引理 2 当 $i=1,2$ 时，对 $f\in L^2\left(R^2;H\right)$，有如下等式成立

$$\left\| u_i L_l^{i,j}(f)(U)\right\|^2 = b_i^2 \left\| \frac{\partial}{\partial x_i} f(X)\right\|^2$$

证明 因为

$$\left\| u_i L_l^{i,j}(f)(U)\right\|^2 = Sc\left[\int_{R^2} u_i^2 \left| L_l^{i,j}(f)(U)\right|^2 \mathrm{d}_U^2 \right]$$

当 $i=1$ 时，

$$\int_{R^2} u_1^2 \left| L_l^{i,j}(f)(U)\right|^2 d_U^2 = \int_{R^6} u_1^2 K_{A_1}^i(x_1,u_1) K_{A_2}^j(x_2,u_2) f(X)\overline{f(Y)} K_{A_2}^{-j}(y_2,u_2) K_{A_1}^{-i}(y_1,u_1)\mathrm{d}_X^2\mathrm{d}_Y^2\mathrm{d}_U^2$$

则运用应用四元数循环相乘的对称性质可得

$$\left\| u_1 L_l^{i,j}(f)(U)\right\|^2$$

$$= Sc[\int_{R^2} u_1^2 \left| L_l^{i,j}(f)(U)\right|^2 \mathrm{d}_U^2]$$

$$= Sc\left[\int_{R^6} \left(u_1^2 K_{A_1}^{-i}(y_1,u_1) K_{A_1}^{i}(x_1,u_1)\right) K_{A_2}^j(x_2,u_2) f(X)\overline{f(Y)} K_{A_2}^{-j}(y_2,u_2)\mathrm{d}_X^2\mathrm{d}_Y^2\mathrm{d}_U^2 \right]$$

$$= Sc\left[\int_{R^5} \left(\int_{R} \frac{1}{2\pi b_1} u_1^2 e^{i\frac{a_1}{2b_1}\left(x_1^2-y_1^2\right)} e^{i\frac{y_1-x_1}{b_1}u_1} du_1 \right) K_{A_2}^j(x_2,u_2) f(X)\overline{f(Y)} K_{A_2}^{-j}(y_2,u_2)\mathrm{d}_X^2\mathrm{d}_Y^2\mathrm{d}_{u_2} \right]$$

$$= -b_1^2 Sc\left[\int_{R^5} \left(e^{i\frac{a_1}{2b_1}\left(x_1^2-y_1^2\right)} \frac{\partial^2}{\partial x_1^2}\delta(y_1-x_1)\right) K_{A_2}^j(x_2,u_2) f(X)\overline{f(Y)} K_{A_2}^{-j}(y_2,u_2)\mathrm{d}_X^2\mathrm{d}_Y^2\mathrm{d}_{u_2} \right]$$

$$= -b_1^2 Sc\left[\int_{R^5} e^{i\frac{a_1}{2b_1}\left(x_1^2-y_1^2\right)} K_{A_2}^j(x_2,u_2) \frac{\partial^2}{\partial x_1^2}\delta(y_1-x_1) f(X)\overline{f(Y)} K_{A_2}^{-j}(y_2,u_2)\mathrm{d}_X^2\mathrm{d}_Y^2\mathrm{d}_{u_2} \right]$$

$$= -b_1^2 Sc\left[\int_{R^5} \frac{\partial^2}{\partial x_1^2}\delta(y_1-x_1) f(X)\overline{f(Y)} \left(K_{A_2}^{-j}(y_2,u_2) K_{A_2}^j(x_2,u_2)\right) e^{i\frac{a_1}{2b_1}\left(x_1^2-y_1^2\right)} \mathrm{d}_X^2\mathrm{d}_Y^2\mathrm{d}_{u_2} \right]$$

$$= -b_1^2 Sc\left[\int_{R^3} \frac{\partial^2}{\partial x_1^2}\delta(y_1 - x_1) f(X)\overline{f(y_1, x_2)} e^{i\frac{a_1}{2b_1}(x_1^2 - y_1^2)} d_{y_1} d_X^2\right]$$

$$= -b_1^2 Sc\left[\int_{R^2}\left(\frac{\partial^2}{\partial x_1^2} f(X)\right)\overline{f(X)} d_X^2\right] = b_1^2 Sc\left[\int_{R^2}\left|\frac{\partial}{\partial x_1} f(X)\right|^2 d_X^2\right] = b_1^2\left\|\frac{\partial}{\partial x_1} f(X)\right\|^2,$$

当 $i = 2$ 时，同理可得相应结论。证毕。

若当 $k = 1,2$ 时，若 f，$x_k f \in L^2(R^2; H)$ 且 $L_l^{i,j}(f)$，$u_k L_l^{i,j}(f) \in L^2(R^2; H)$，那么 f 的空间宽度 Δx_k 定义为 $\Delta x_k := \sqrt{Var_k(f)}$，其中

$$Var_k(f) = \frac{\|x_k f\|^2}{\|f\|^2} = \frac{\int_{R^2} x_k^2 |f(X)|^2 d_X^2}{\int_{R^2} |f(X)|^2 d_X^2}$$

类似地，在四元数域定义谱宽度为 $\Delta u_k := \sqrt{Var_k(L_l^{i,j}(f))}$，其中

$$Var_k(L_l^{i,j}(f)) = \frac{\|u_k L_l^{i,j}(f)\|^2}{\|L_l^{i,j}(f)\|^2} = \frac{\int_{R^2} u_k^2 |L_l^{i,j}(f)|^2 d_U^2}{\int_{R^2} |L_l^{i,j}(f)|^2 d_U^2}$$

定理 3 当 $k = 1,2$ 时，若 f，$x_k f \in L^2(R^2; H)$ 且 $L_l^{i,j}(f)$，$u_k L_l^{i,j}(f) \in L^2(R^2; H)$，则如下不确定性原理成立

$$\Delta x_1 \Delta u_1 \ge \frac{b_1}{2},\quad \Delta x_2 \Delta u_2 \ge \frac{b_2}{2}$$

进一步可得

$$\Delta x_1 \Delta x_2 \Delta u_1 \Delta u_2 \ge \frac{b_1 b_2}{4}$$

等式成立当且仅当 f 是二维高斯函数，即 $f(X) = \gamma e^{-\frac{c_1 x_1^2 + c_2 x_2^2}{2}}$，其中 c_1，c_2 是正实数，$\gamma = \sqrt[4]{\frac{c_1 c_2}{\pi^2}}\|f\|$。

证明 应用引理 2 和 Cauchy-Schwarz 不等式(2)，可得

$$\|x_k f\|^2 \|u_k L_l^{i,j}(f)\|^2 = Sc\left[\int_{R^2} x_k^2 |f(X)|^2 d_X^2\right] \cdot Sc\left[\int_{R^2} u_k^2 |L_l^{i,j}(f)|^2 d_U^2\right]$$

$$= Sc\left[\int_{R^2} x_k^2 |f(X)|^2 d_X^2\right] \cdot b_k^2 Sc\left[\int_{R^2}\left|\frac{\partial}{\partial x_k} f(X)\right|^2 d_X^2\right]$$

$$\ge b_k^2 \cdot Sc\left|\int_{R^2} x_k \overline{f(X)} \frac{\partial}{\partial x_k} f(X) d_X^2\right|^2,$$

剩下的证明同[19]的定理 5.1 证明类似，略去。

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