

The Initial Boundary Value Problems for a Class of n -Dimensional Nonlinear Evolution Systems of Higher Order

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Abstract

In this paper, we study a class of n -dimensional nonlinear wave system of high order by using the Galerkin method, Sobolev embedding theorem and prior estimate of solution, and prove the existence and uniqueness of global strong solution to the initial boundary value problem.

Keywords

Nonlinear Wave System, Prior Estimate, Galerkin Method, Sobolev Embedding Theorem, The Global Strong Solution

一类高阶 n 维非线性发展方程组的初边值问题

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摘 要

本文研究了一类高阶 n 维非线性波动方程组, 通过解的先验估计, 利用Galerkin方法和Sobolev嵌入定理,

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证明了初边值问题整体强解的存在性和唯一性。

关键词

非线性波动方程组, 先验估计, Galerkin方法, Sobolev嵌入定理, 整体强解

1. 引言

在本文中, $\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^N u_i v_i$, $(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \mathbf{u} \cdot \mathbf{v} dx$, $\|\mathbf{u}\|_{L^2(\Omega)}^2 = (\mathbf{u}, \mathbf{u})$,

$$[\mathbf{u}, \mathbf{v}] = \int_0^t (\mathbf{u}, \mathbf{v}) dt, \quad \|\mathbf{u}\|_{L^2(\Omega)}^2 = [\mathbf{u}, \mathbf{u}], \quad |\mathbf{u}| = \left(\sum_{i=1}^N u_i^2 \right)^{\frac{1}{2}},$$

$$\mathbf{u} = \mathbf{u}(x, t) = (u_1, u_2, \dots, u_N)^T, \quad \mathbf{v} = \mathbf{v}(x, t) = (v_1, v_2, \dots, v_N)^T,$$

$$\mathbf{f}(\mathbf{u}_t) = (f_1(\mathbf{u}_t), f_2(\mathbf{u}_t), \dots, f_N(\mathbf{u}_t))^T, \quad \boldsymbol{\varphi}(x) = (\varphi_1(x), \varphi_2(x), \dots, \varphi_N(x))^T,$$

$$\boldsymbol{\psi}(x) = (\psi_1(x), \psi_2(x), \dots, \psi_N(x))^T$$

其中 $\boldsymbol{\varphi}(x) \in H^{2M}(\Omega) \cap H_0^M(\Omega)$, $\boldsymbol{\psi}(x) \in H^{2M}(\Omega) \cap H_0^M(\Omega)$, “T” 表示转置。

有三种因素影响弹性杆内波的传播: 非线性、弥散以及耗散。文[1]研究了在上述三种因素影响下的细长弹性杆中纵向应变波的传播问题。

文[2]在此基础上研究了一类四阶非线性耗散色散波动方程的初边值问题, 非线性耗散色散波动方程组关于空间变量导数为 2 阶的情形在文[3]中有所研究, 本文主要在此基础上研究了高阶的情形, 关于高阶 n 维非线性耗散色散波动方程组的研究在已有文献中还未见到。

本文主要研究了下述在粘弹性力学中具有实际背景的一类高阶 n 维非线性波动方程组:

$$\begin{cases} \mathbf{u}_{tt} + (-1)^M \Delta^M \mathbf{u} + (-1)^M \Delta^M \mathbf{u}_t + (-1)^M \Delta^M \mathbf{u}_{tt} = \mathbf{f}(\mathbf{u}_t), & (x, t) \in \Omega \times [0, T], & (1.1) \\ \mathbf{u}|_{t=0} = \boldsymbol{\varphi}(x), \quad \mathbf{u}_t|_{t=0} = \boldsymbol{\psi}(x), & & (1.2) \\ D^\gamma \mathbf{u}|_{\partial\Omega \times [0, T]} = 0, \quad 0 \leq |\gamma| \leq M-1 & & (1.3) \end{cases}$$

其中 Ω 为 R^n 中具有光滑边界的有界区域, $T > 0$, M 为正整数, $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$ 为多重指标, $\gamma_i (i=1, 2, \dots, n)$ 为非负整数, 且 $|\gamma| = \gamma_1 + \gamma_2 + \dots + \gamma_n$. $\boldsymbol{\varphi}(x) \in H^{2M}(\Omega) \cap H_0^M(\Omega)$, $\boldsymbol{\psi}(x) \in H^{2M}(\Omega) \cap H_0^M(\Omega)$, 假设 \mathbf{f} 满足:

$$\mathbf{f}(\mathbf{0}) = \mathbf{0}, \quad \mathbf{f} \in C^1 \quad (1.4)$$

且 Jacobian 矩阵 $\frac{\partial \mathbf{f}}{\partial \mathbf{u}}$ 半有界, 即 $\exists k_0 > 0, \forall \xi \in R^n$ 满足:

$$\left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}} \xi, \xi \right) \leq k_0 (\xi, \xi), \quad (1.5)$$

$$|\mathbf{f}(\mathbf{u})| \leq a_1 + b_1 |\mathbf{u}|^p \quad (a_1, b_1 \text{ 为正常数}), \quad (1.6)$$

当 $2M < n$ 时, $2 \leq p < \frac{2n}{n-2M}$; 当 $2M \geq n$ 时, $2 \leq p < +\infty$ 。

2. 主要结论

定义: $\mathbf{u} = \mathbf{u}(x, t)$ ($(x, t) \in \Omega \times [0, T], \Omega \subset \mathbb{R}^n$) 称为问题(1.1)~(1.3)在 $\Omega \times [0, T]$ 上的整体强解, 若

$$\begin{aligned} \mathbf{u} &\in L^\infty(0, T; H^{2M}(\Omega) \cap H_0^M(\Omega)), \\ \mathbf{u}_t &\in L^\infty(0, T; H^{2M}(\Omega) \cap H_0^M(\Omega)), \\ \mathbf{u}_{tt} &\in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^M(\Omega)), \end{aligned}$$

对一切 $\mathbf{d}(x, t) = (d_1(x, t), d_2(x, t), \dots, d_N(x, t)) \in C([0, T]; L^2(\Omega))$ 成立

$$\int_0^T (\mathbf{u}_{tt} + (-1)^M \Delta^M \mathbf{u} + (-1)^M \Delta^M \mathbf{u}_t + (-1)^M \Delta^M \mathbf{u}_{tt} - \mathbf{f}(\mathbf{u}_t), \mathbf{d}(x, t)) dt = 0$$

且

$$\mathbf{u}|_{t=0} = \boldsymbol{\varphi}(x), \mathbf{u}_t|_{t=0} = \boldsymbol{\psi}(x)$$

定理 1: 若 $\boldsymbol{\varphi}(x) \in H^{2M}(\Omega) \cap H_0^M(\Omega), \boldsymbol{\psi}(x) \in H^{2M}(\Omega) \cap H_0^M(\Omega)$, 条件(1.5)~(1.6)成立, 则问题(1.1)~(1.3)存在上述意义下的整体强解 $\mathbf{u} = \mathbf{u}(x, t)$ 。

定理 2: 若 $\boldsymbol{\varphi}(x) \in H^{2M}(\Omega) \cap H_0^M(\Omega), \boldsymbol{\psi}(x) \in H^{2M}(\Omega) \cap H_0^M(\Omega)$, 条件(1.5)~(1.6)成立, 则非线性问题(1.1)~(1.3)的整体强解唯一。

3. 主要结论的证明

首先介绍定理 1 的证明思想, 即第一步构造问题(1.1)~(1.3)的近似解, 然后对近似解作出先验估计, 在作出先验估计基础上, 由列紧性原理可证得问题(1.1)~(1.3)的整体强解。下面进行定理 1 的证明:

设 $\{w_j(x) | j=1, 2, \dots\}$ 为问题 $\begin{cases} (-1)^M \Delta^M w_j = \lambda_j w_j \\ \mathcal{D}^\gamma w_j(x)|_{\partial\Omega} = 0 \quad 0 \leq |\gamma| \leq M-1 \end{cases}$ 的特征函数系, 则由文[4]中的引理 1.1

有结论: $\{w_j(x) | j=1, 2, \dots\}$ 分别构成 $L^2(\Omega), H_0^M(\Omega), H^{2M} \cap H_0^M(\Omega)$ 的正交基底。

构造问题(1.1)~(1.3)的近似解

$$u_{mi} = u_{mi}(x, t) = \sum_{j=1}^m a_{mij}(t) w_j(x), \quad (2.1)$$

并满足初值条件

$$u_{mi}(x, 0) = \varphi_{mi}(x) = \sum_{j=1}^m a_{mij} w_j(x) \quad (a_{mij} \text{将在下面选取}), \quad (2.2)$$

$$u_{mit}(x, 0) = \psi_{mi}(x) = \sum_{j=1}^m b_{mij} w_j(x) \quad (b_{mij} \text{将在下面选取}). \quad (2.3)$$

由 Galerkin 方法, 该近似解应满足如下非线性常微分方程组的初值问题

$$(*) \begin{cases} (u_{mit}, w_k) + ((-1)^M \Delta^M u_{mi}, w_k) + ((-1)^M \Delta^M u_{mit}, w_k) + ((-1)^M \Delta^M u_{mit}, w_k) = (f_i(\mathbf{u}_m), w_k), 0 \leq t \leq T, (2.4) \\ u_{mi}(x, 0) = \varphi_{mi}(x), \\ u_{mit}(x, 0) = \psi_{mi}(x), \\ i = 1, 2, \dots, N; k = 1, 2, \dots, m. \end{cases}$$

由于 $\boldsymbol{\varphi}(x) = (\varphi_1(x), \varphi_2(x), \dots, \varphi_N(x))^T \in H^{2M}(\Omega) \cap H_0^M(\Omega)$,

$\boldsymbol{\psi}(x) = (\psi_1(x), \psi_2(x), \dots, \psi_N(x))^T \in H^{2M}(\Omega) \cap H_0^M(\Omega)$, $\{w_j(x) | j=1, 2, \dots\}$ 构成

$H^{2M}(\Omega) \cap H_0^M(\Omega)$ 的一组正交基, 可选取适当的 a_{mij}, b_{mij} ($i=1, 2, \dots, N; j=1, 2, 3, \dots$) 使得当 $m \rightarrow +\infty$ 时, 在空间 $H^{2M}(\Omega) \cap H_0^M(\Omega)$ 中成立强收敛, $\varphi_{mi}(x) \rightarrow \varphi_i(x)$, $\psi_{mi}(x) \rightarrow \psi_i(x)$, ($i=1, 2, \dots, N$). 因假设 $f \in C^1$, 由非线性常微分方程组理论中 Peano 存在性定理知问题 (*) 存在局部解 $u_{mi}(x, t)$ ($i=1, 2, 3, \dots, N$), 为得到问题 (*) 在 $[0, T]$ 上的整体解, 需证明问题 (*) 中方程的右端 $(f_i(\mathbf{u}_{mt}), w_k)$ 的绝对值能用一与 $\mathbf{u}_{mt}(x, t)$ 无关的正常数控制住, 为此作出近似解 $u_{mi}(x, t)$ ($i=1, 2, 3, \dots, N$) 的先验估计如下:

引理 3.1: 设 $\boldsymbol{\varphi}(x) \in H_0^M(\Omega), \boldsymbol{\psi}(x) \in H_0^M(\Omega)$, 选取 a_{mij}, b_{mij} , 使当 $m \rightarrow +\infty$ 时, $\mathbf{u}_m(x, 0) \rightarrow \boldsymbol{\varphi}(x)$ 及 $\mathbf{u}_{mt}(x, 0) \rightarrow \boldsymbol{\psi}(x)$ 在 $H_0^M(\Omega)$ 中强收敛, 若条件(1.5), (1.6)式成立, 则有估计:

$$|\mathbf{u}_m|_{L^2(\Omega)}^2 + |\mathbf{u}_{mt}|_{L^2(\Omega)}^2 + |\nabla^M \mathbf{u}_m|_{L^2(\Omega)}^2 + |\nabla^M \mathbf{u}_{mt}|_{L^2(\Omega)}^2 + \|\nabla^M \mathbf{u}_{mt}\|_{L^2(\Omega)}^2 \leq C_1 \quad (C_1 \text{ 为与 } \mathbf{u}_m \text{ 无关的正常数}).$$

引理 3.2: 设 $\boldsymbol{\varphi}(x) \in H^{2M}(\Omega) \cap H_0^M(\Omega), \boldsymbol{\psi}(x) \in H^{2M}(\Omega) \cap H_0^M(\Omega)$, 选取 a_{mij}, b_{mij} , 使当 $m \rightarrow +\infty$ 时, $\mathbf{u}_m(x, 0) \rightarrow \boldsymbol{\varphi}(x)$ 及 $\mathbf{u}_{mt}(x, 0) \rightarrow \boldsymbol{\psi}(x)$ 在 $H^{2M}(\Omega) \cap H_0^M(\Omega)$ 中强收敛, 若条件(1.5), (1.6)式成立, 则有估计:

$$|\mathbf{u}_{mt}|_{L^2(\Omega)}^2 + |\nabla^M \mathbf{u}_{mt}|_{L^2(\Omega)}^2 + |\Delta^M \mathbf{u}_{mt}|_{L^2(\Omega)}^2 + \|\Delta^M \mathbf{u}_{mt}\|_{L^2(\Omega)}^2 \leq C_2 \quad (C_2 \text{ 为与 } \mathbf{u}_m \text{ 无关的正常数}).$$

引理 3.3: 设 $\boldsymbol{\varphi}(x) \in H^{2M}(\Omega) \cap H_0^M(\Omega), \boldsymbol{\psi}(x) \in H^{2M}(\Omega) \cap H_0^M(\Omega)$, 选取 a_{mij}, b_{mij} 使当 $m \rightarrow +\infty$ 时, $\mathbf{u}_m(x, 0) \rightarrow \boldsymbol{\varphi}(x)$ 及 $\mathbf{u}_{mt}(x, 0) \rightarrow \boldsymbol{\psi}(x)$ 在 $H^{2M}(\Omega) \cap H_0^M(\Omega)$ 中强收敛, 若条件(1.5), (1.6)式成立, 则有估计:

$$|\nabla^M \mathbf{u}_{mt}|_{L^2(\Omega)}^2 + |\Delta^M \mathbf{u}_m|_{L^2(\Omega)}^2 + |\Delta^M \mathbf{u}_{mt}|_{L^2(\Omega)}^2 + \|\Delta^M \mathbf{u}_{mt}\|_{L^2(\Omega)}^2 \leq C_3 \quad (C_3 \text{ 为与 } \mathbf{u}_m \text{ 无关的正常数}).$$

引理 3.4: 在引理 3.3 条件下成立不等式

$$|\Delta^M \mathbf{u}_{mt}|_{L^2(\Omega)}^2 \leq C_4 \quad (C_4 \text{ 为与 } \mathbf{u}_m \text{ 无关的正常数}).$$

引理 3.1 的证明:

方程(2.4)两边同乘 $a'_{mik}(t)$

$$\begin{aligned} & (u_{mit}, a'_{mik}(t)w_k) + \left((-1)^M \Delta^M u_{mi}, a'_{mik}(t)w_k\right) + \left((-1)^M \Delta^M u_{mit}, a'_{mik}(t)w_k\right) + \left((-1)^M \Delta^M u_{mit}, a'_{mik}(t)w_k\right) \\ & = (f_i(\mathbf{u}_{mt}), a'_{mik}(t)w_k), \end{aligned}$$

关于 k 从 1 到 m 作和得

$$(u_{mit}, u_{mit}) + \left((-1)^M \Delta^M u_{mi}, u_{mit}\right) + \left((-1)^M \Delta^M u_{mit}, u_{mit}\right) + \left((-1)^M \Delta^M u_{mit}, u_{mit}\right) = (f_i(\mathbf{u}_{mt}), u_{mit}),$$

关于 i 从 1 到 N 作和得

$$(\mathbf{u}_{mt}, \mathbf{u}_{mt}) + \left((-1)^M \Delta^M \mathbf{u}_m, \mathbf{u}_{mt}\right) + \left((-1)^M \Delta^M \mathbf{u}_{mt}, \mathbf{u}_{mt}\right) + \left((-1)^M \Delta^M \mathbf{u}_{mt}, \mathbf{u}_{mt}\right) = (\mathbf{f}(\mathbf{u}_{mt}), \mathbf{u}_{mt}),$$

而由条件(1.5)知

$$(\mathbf{f}(\mathbf{u}_{mt}), \mathbf{u}_{mt}) = (\mathbf{f}(\mathbf{u}_{mt}) - \mathbf{f}(\mathbf{0}), \mathbf{u}_{mt}) = \left(\frac{\partial \mathbf{f}(\mathbf{u}_{mt})}{\partial \mathbf{u}_{mt}} \Big|_{\theta \mathbf{u}_{mt}} \mathbf{u}_{mt}, \mathbf{u}_{mt} \right) \leq k_0 (\mathbf{u}_{mt}, \mathbf{u}_{mt}) \quad (0 < \theta < 1),$$

首先利用格林公式, 然后两端从 0 到 t 积分 ($0 \leq t \leq T$), 得

$$\begin{aligned} & \frac{1}{2}(\mathbf{u}_{mt}, \mathbf{u}_{mt}) - \frac{1}{2}(\boldsymbol{\psi}_m, \boldsymbol{\psi}_m) + \frac{1}{2}(\nabla^M \mathbf{u}_m, \nabla^M \mathbf{u}_m) - \frac{1}{2}(\nabla^M \boldsymbol{\varphi}_m, \nabla^M \boldsymbol{\varphi}_m) \\ & + [\nabla^M \mathbf{u}_{mt}, \nabla^M \mathbf{u}_{mt}] + \frac{1}{2}(\nabla^M \mathbf{u}_{mt}, \nabla^M \mathbf{u}_{mt}) - \frac{1}{2}(\nabla^M \boldsymbol{\psi}_m, \nabla^M \boldsymbol{\psi}_m) \leq k_0 [\mathbf{u}_{mt}, \mathbf{u}_{mt}], \end{aligned}$$

这里, $\boldsymbol{\varphi}_m(x) = (\varphi_{m1}(x), \varphi_{m2}(x), \dots, \varphi_{mN}(x))^T$, $\boldsymbol{\psi}_m(x) = (\psi_{m1}(x), \psi_{m2}(x), \dots, \psi_{mN}(x))^T$, 从而有不等式

$$\begin{aligned} & \frac{1}{2}(\mathbf{u}_{mt}, \mathbf{u}_{mt}) + \frac{1}{2}(\nabla^M \mathbf{u}_m, \nabla^M \mathbf{u}_m) + [\nabla^M \mathbf{u}_{mt}, \nabla^M \mathbf{u}_{mt}] + \frac{1}{2}(\nabla^M \mathbf{u}_{mt}, \nabla^M \mathbf{u}_{mt}) \\ & \leq k_0 [\mathbf{u}_{mt}, \mathbf{u}_{mt}] + \frac{1}{2}(\boldsymbol{\psi}_m, \boldsymbol{\psi}_m) + \frac{1}{2}(\nabla^M \boldsymbol{\varphi}_m, \nabla^M \boldsymbol{\varphi}_m) + \frac{1}{2}(\nabla^M \boldsymbol{\psi}_m, \nabla^M \boldsymbol{\psi}_m), \end{aligned}$$

两边加上 $[\mathbf{u}_m, \mathbf{u}_m]$,

左边将它估计为 $\frac{1}{2}(\mathbf{u}_m, \mathbf{u}_m) - \frac{1}{2}(\boldsymbol{\varphi}_m, \boldsymbol{\varphi}_m)$,

右边将它化为 $[\mathbf{u}_m, \mathbf{u}_{mt}] \leq [\mathbf{u}_m, \mathbf{u}_m] + [\mathbf{u}_{mt}, \mathbf{u}_{mt}]$,

从而有

$$\begin{aligned} & \frac{1}{2}(\mathbf{u}_m, \mathbf{u}_m) + \frac{1}{2}(\mathbf{u}_{mt}, \mathbf{u}_{mt}) + \frac{1}{2}(\nabla^M \mathbf{u}_m, \nabla^M \mathbf{u}_m) + [\nabla^M \mathbf{u}_{mt}, \nabla^M \mathbf{u}_{mt}] + \frac{1}{2}(\nabla^M \mathbf{u}_{mt}, \nabla^M \mathbf{u}_{mt}) \\ & \leq \frac{1}{2}(\boldsymbol{\varphi}_m, \boldsymbol{\varphi}_m) + \frac{1}{2}(\boldsymbol{\psi}_m, \boldsymbol{\psi}_m) + \frac{1}{2}(\nabla^M \boldsymbol{\varphi}_m, \nabla^M \boldsymbol{\varphi}_m) + \frac{1}{2}(\nabla^M \boldsymbol{\psi}_m, \nabla^M \boldsymbol{\psi}_m) + (k_0 + 1)[\mathbf{u}_{mt}, \mathbf{u}_{mt}] + [\mathbf{u}_m, \mathbf{u}_m] \end{aligned}$$

由于当 $m \rightarrow +\infty$ 时, $\boldsymbol{\varphi}_m \rightarrow \boldsymbol{\varphi}$ 及 $\boldsymbol{\psi}_m \rightarrow \boldsymbol{\psi}$ 在 $H_0^M(\Omega)$ 中强收敛, 所以当 $m \rightarrow +\infty$ 时, $(\boldsymbol{\varphi}_m, \boldsymbol{\varphi}_m) \rightarrow (\boldsymbol{\varphi}, \boldsymbol{\varphi})$, $(\boldsymbol{\psi}_m, \boldsymbol{\psi}_m) \rightarrow (\boldsymbol{\psi}, \boldsymbol{\psi})$, $(\nabla^M \boldsymbol{\varphi}_m, \nabla^M \boldsymbol{\varphi}_m) \rightarrow (\nabla^M \boldsymbol{\varphi}, \nabla^M \boldsymbol{\varphi})$ 及 $(\nabla^M \boldsymbol{\psi}_m, \nabla^M \boldsymbol{\psi}_m) \rightarrow (\nabla^M \boldsymbol{\psi}, \nabla^M \boldsymbol{\psi})$. 以上不等式右边前四项能用一与 m 无关的正常数来控制. 由 Gronwall 不等式得

$$|\mathbf{u}_m|_{L^2(\Omega)}^2 + |\mathbf{u}_{mt}|_{L^2(\Omega)}^2 + |\nabla^M \mathbf{u}_m|_{L^2(\Omega)}^2 + |\nabla^M \mathbf{u}_{mt}|_{L^2(\Omega)}^2 + \|\nabla^M \mathbf{u}_{mt}\|_{L^2(\Omega)}^2 \leq C_1$$

引理 3.1 证毕!

引理 3.2 的证明:

对方程组(2.4)两边关于 t 求导得

$$(\mathbf{u}_{mitt}, w_k) + \left((-1)^M \Delta^M \mathbf{u}_{mit}, w_k \right) + \left((-1)^M \Delta^M \mathbf{u}_{mitt}, w_k \right) + \left((-1)^M \Delta^M \mathbf{u}_{mitt}, w_k \right) = \left(\frac{d}{dt} f_i(\mathbf{u}_{mt}), w_k \right)$$

两边同乘以 $a_{mik}''(t)$ 得

$$\begin{aligned} & (\mathbf{u}_{mitt}, a_{mik}''(t) w_k) + \left((-1)^M \Delta^M \mathbf{u}_{mit}, a_{mik}''(t) w_k \right) + \left((-1)^M \Delta^M \mathbf{u}_{mitt}, a_{mik}''(t) w_k \right) \\ & + \left((-1)^M \Delta^M \mathbf{u}_{mitt}, a_{mik}''(t) w_k \right) = \left(\frac{d}{dt} f_i(\mathbf{u}_{mt}), a_{mik}''(t) w_k \right) \end{aligned}$$

关于 k 从 1 到 m 作和得

$$(\mathbf{u}_{mitt}, \mathbf{u}_{mitt}) + \left((-1)^M \Delta^M \mathbf{u}_{mit}, \mathbf{u}_{mitt} \right) + \left((-1)^M \Delta^M \mathbf{u}_{mitt}, \mathbf{u}_{mitt} \right) + \left((-1)^M \Delta^M \mathbf{u}_{mitt}, \mathbf{u}_{mitt} \right) = \left(\frac{d}{dt} f_i(\mathbf{u}_{mt}), \mathbf{u}_{mitt} \right)$$

关于 i 从 1 到 N 作和得

$$(\mathbf{u}_{mitt}, \mathbf{u}_{mitt}) + \left((-1)^M \Delta^M \mathbf{u}_{mt}, \mathbf{u}_{mitt} \right) + \left((-1)^M \Delta^M \mathbf{u}_{mitt}, \mathbf{u}_{mitt} \right) + \left((-1)^M \Delta^M \mathbf{u}_{mitt}, \mathbf{u}_{mitt} \right) = \left(\frac{d}{dt} f(\mathbf{u}_{mt}), \mathbf{u}_{mitt} \right)$$

而由条件(1.5)知 $\left(\frac{d}{dt} \mathbf{f}(\mathbf{u}_{mt}), \mathbf{u}_{mt}\right) = \left(\frac{\partial \mathbf{f}(\mathbf{u}_{mt})}{\partial \mathbf{u}_{mt}} \cdot \mathbf{u}_{mt}, \mathbf{u}_{mt}\right) \leq k_0(\mathbf{u}_{mt}, \mathbf{u}_{mt})$,

从而有

$$\frac{1}{2} \frac{d}{dt}(\mathbf{u}_{mt}, \mathbf{u}_{mt}) + \frac{1}{2} \frac{d}{dt}(\nabla^M \mathbf{u}_{mt}, \nabla^M \mathbf{u}_{mt}) + \frac{1}{2} \frac{d}{dt}(\nabla^M \mathbf{u}_{mt}, \nabla^M \mathbf{u}_{mt}) + (\nabla^M \mathbf{u}_{mt}, \nabla^M \mathbf{u}_{mt}) \leq k_0(\mathbf{u}_{mt}, \mathbf{u}_{mt})$$

两边从 0 到 t ($0 \leq t \leq T$) 积分得

$$\begin{aligned} & \frac{1}{2}(\mathbf{u}_{mt}, \mathbf{u}_{mt}) - \frac{1}{2}(\mathbf{u}_{mt}(x, 0), \mathbf{u}_{mt}(x, 0)) + \frac{1}{2}(\nabla^M \mathbf{u}_{mt}, \nabla^M \mathbf{u}_{mt}) - \frac{1}{2}(\nabla^M \boldsymbol{\psi}_m, \nabla^M \boldsymbol{\psi}_m) \\ & + \frac{1}{2}(\nabla^M \mathbf{u}_{mt}, \nabla^M \mathbf{u}_{mt}) - \frac{1}{2}(\nabla^M \mathbf{u}_{mt}(x, 0), \nabla^M \mathbf{u}_{mt}(x, 0)) + [\nabla^M \mathbf{u}_{mt}, \nabla^M \mathbf{u}_{mt}] \leq k_0[\mathbf{u}_{mt}, \mathbf{u}_{mt}] \end{aligned}$$

下证 $(\mathbf{u}_{mt}(x, 0), \mathbf{u}_{mt}(x, 0)) + (\nabla^M \mathbf{u}_{mt}(x, 0), \nabla^M \mathbf{u}_{mt}(x, 0))$ 有界(能用一与 m 无关正常数控制住)。

方程组(2.4)两边同乘以 $a_{mik}''(t)$ 得

$$\begin{aligned} & (\mathbf{u}_{mit}, a_{mik}''(t)w_k) + ((-1)^M \Delta^M \mathbf{u}_{mi}, a_{mik}''(t)w_k) + ((-1)^M \Delta^M \mathbf{u}_{mit}, a_{mik}''(t)w_k) + ((-1)^M \Delta^M \mathbf{u}_{mit}, a_{mik}''(t)w_k) \\ & = (f_i(\mathbf{u}_{mt}), a_{mik}''(t)w_k), \end{aligned}$$

关于 k 从 1 到 m 作和得

$$(\mathbf{u}_{mit}, \mathbf{u}_{mit}) + ((-1)^M \Delta^M \mathbf{u}_{mi}, \mathbf{u}_{mit}) + ((-1)^M \Delta^M \mathbf{u}_{mit}, \mathbf{u}_{mit}) + ((-1)^M \Delta^M \mathbf{u}_{mit}, \mathbf{u}_{mit}) = (f_i(\mathbf{u}_{mt}), \mathbf{u}_{mit})$$

关于 i 从 1 到 N 作和得

$$(\mathbf{u}_{mt}, \mathbf{u}_{mt}) + ((-1)^M \Delta^M \mathbf{u}_m, \mathbf{u}_{mt}) + ((-1)^M \Delta^M \mathbf{u}_{mt}, \mathbf{u}_{mt}) + ((-1)^M \Delta^M \mathbf{u}_{mt}, \mathbf{u}_{mt}) = (\mathbf{f}(\mathbf{u}_{mt}), \mathbf{u}_{mt})$$

令 $t = 0$ 得

$$\begin{aligned} & (\mathbf{u}_{mt}(x, 0), \mathbf{u}_{mt}(x, 0)) + ((-1)^M \Delta^M \mathbf{u}_m(x, 0), \mathbf{u}_{mt}(x, 0)) + ((-1)^M \Delta^M \mathbf{u}_{mt}(x, 0), \mathbf{u}_{mt}(x, 0)) \\ & + ((-1)^M \Delta^M \mathbf{u}_{mt}(x, 0), \mathbf{u}_{mt}(x, 0)) = (\mathbf{f}(\boldsymbol{\psi}_m), \mathbf{u}_{mt}(x, 0)) \end{aligned} \quad (**)$$

$$(\mathbf{f}(\boldsymbol{\psi}_m), \mathbf{u}_{mt}(x, 0)) \leq \frac{1}{2\varepsilon_1}(\mathbf{f}(\boldsymbol{\psi}_m), \mathbf{f}(\boldsymbol{\psi}_m)) + \frac{\varepsilon_1}{2}(\mathbf{u}_{mt}(x, 0), \mathbf{u}_{mt}(x, 0))$$

(其中 $\boldsymbol{\psi}_m = (\psi_{m1}(x), \dots, \psi_{mN}(x))^T$),

下面证明 $(\mathbf{f}(\boldsymbol{\psi}_m), \mathbf{f}(\boldsymbol{\psi}_m))$ 有界。

由于

$$(\mathbf{f}(\boldsymbol{\psi}_m), \mathbf{f}(\boldsymbol{\psi}_m)) \leq \left(a_1 + b_1 |\boldsymbol{\psi}_m|^{\frac{p}{2}}, a_1 + b_1 |\boldsymbol{\psi}_m|^{\frac{p}{2}} \right) = a_1^2 |\Omega| + 2a_1 b_1 \int_{\Omega} |\boldsymbol{\psi}_m|^{\frac{p}{2}} dx + b_1^2 \int_{\Omega} |\boldsymbol{\psi}_m|^p dx.$$

当 $m \rightarrow +\infty$ 时, $\boldsymbol{\psi}_m \rightarrow \boldsymbol{\psi}$ 在 $H^{2M}(\Omega) \cap H_0^M(\Omega)$ 中强收敛, 由 Sobolev 嵌入定理[5]可知 $|\boldsymbol{\psi}_m|_{L^p(\Omega)} \leq \text{const}$,

其中当 $2M < n$ 时, $2 \leq p < \frac{2n}{n-2M}$; 当 $2M \geq n$ 时, $2 \leq p < +\infty$ 。

所以 $\int_{\Omega} |\boldsymbol{\psi}_m|^p dx = |\boldsymbol{\psi}_m|_{L^p(\Omega)}^p \leq \text{const}$,

从而有

$$\int_{\Omega} |\boldsymbol{\psi}_m|^{\frac{p}{2}} dx \leq \left(\int_{\Omega} |\boldsymbol{\psi}_m|^{\frac{p}{2} \times 2} dx \right)^{\frac{1}{2}} \left(\int_{\Omega} 1^2 dx \right)^{\frac{1}{2}} = |\Omega|^{\frac{1}{2}} |\boldsymbol{\psi}_m|_{L^p(\Omega)}^{\frac{p}{2}} \leq \text{const},$$

因此, $(f(\boldsymbol{\psi}_m), f(\boldsymbol{\psi}_m))$ 有界。

将 (**) 中的项 $((-1)^M \Delta^M \mathbf{u}_m(x, 0), \mathbf{u}_{mtt}(x, 0))$ 移至等式右边得

$$-((-1)^M \Delta^M \mathbf{u}_m(x, 0), \mathbf{u}_{mtt}(x, 0)) = ((-1)^{M+1} \Delta^M \mathbf{u}_m(x, 0), \mathbf{u}_{mtt}(x, 0)),$$

应用带 ε 的柯西不等式得

$$((-1)^{M+1} \Delta^M \mathbf{u}_m(x, 0), \mathbf{u}_{mtt}(x, 0)) \leq \frac{1}{2\varepsilon_2} ((-1)^{M+1} \Delta^M \mathbf{u}_m(x, 0), (-1)^{M+1} \Delta^M \mathbf{u}_m(x, 0)) + \frac{\varepsilon_2}{2} (\mathbf{u}_{mtt}(x, 0), \mathbf{u}_{mtt}(x, 0))$$

故 (***) 可变为

$$\begin{aligned} & (\mathbf{u}_{mtt}(x, 0), \mathbf{u}_{mtt}(x, 0)) + ((-1)^M \Delta^M \mathbf{u}_{mt}(x, 0), \mathbf{u}_{mtt}(x, 0)) + ((-1)^M \Delta^M \mathbf{u}_{mtt}(x, 0), \mathbf{u}_{mtt}(x, 0)) \\ & \leq (f(\boldsymbol{\psi}_m), \mathbf{u}_{mtt}(x, 0)) + \frac{1}{2\varepsilon_2} ((-1)^{M+1} \Delta^M \mathbf{u}_m(x, 0), (-1)^{M+1} \Delta^M \mathbf{u}_m(x, 0)) + \frac{\varepsilon_2}{2} (\mathbf{u}_{mtt}(x, 0), \mathbf{u}_{mtt}(x, 0)), \end{aligned}$$

再将项 $((-1)^M \Delta^M \mathbf{u}_{mt}(x, 0), \mathbf{u}_{mtt}(x, 0))$ 移至等式右边得

$$-((-1)^M \Delta^M \mathbf{u}_{mt}(x, 0), \mathbf{u}_{mtt}(x, 0)) = ((-1)^{M+1} \Delta^M \mathbf{u}_{mt}(x, 0), \mathbf{u}_{mtt}(x, 0)),$$

应用带 ε 的柯西不等式得

$$((-1)^{M+1} \Delta^M \mathbf{u}_{mt}(x, 0), \mathbf{u}_{mtt}(x, 0)) \leq \frac{1}{2\varepsilon_3} ((-1)^{M+1} \Delta^M \mathbf{u}_{mt}(x, 0), (-1)^{M+1} \Delta^M \mathbf{u}_{mt}(x, 0)) + \frac{\varepsilon_3}{2} (\mathbf{u}_{mtt}(x, 0), \mathbf{u}_{mtt}(x, 0))$$

故 (***) 可变为

$$\begin{aligned} & (\mathbf{u}_{mtt}(x, 0), \mathbf{u}_{mtt}(x, 0)) + ((-1)^M \Delta^M \mathbf{u}_{mtt}(x, 0), \mathbf{u}_{mtt}(x, 0)) \\ & \leq \frac{1}{2\varepsilon_1} (f(\boldsymbol{\psi}_m), f(\boldsymbol{\psi}_m)) + \frac{\varepsilon_1}{2} (\mathbf{u}_{mtt}(x, 0), \mathbf{u}_{mtt}(x, 0)) \\ & \quad + \frac{1}{2\varepsilon_2} ((-1)^{M+1} \Delta^M \mathbf{u}_m(x, 0), (-1)^{M+1} \Delta^M \mathbf{u}_m(x, 0)) + \frac{\varepsilon_2}{2} (\mathbf{u}_{mtt}(x, 0), \mathbf{u}_{mtt}(x, 0)) \\ & \quad + \frac{1}{2\varepsilon_3} ((-1)^{M+1} \Delta^M \mathbf{u}_{mt}(x, 0), (-1)^{M+1} \Delta^M \mathbf{u}_{mt}(x, 0)) + \frac{\varepsilon_3}{2} (\mathbf{u}_{mtt}(x, 0), \mathbf{u}_{mtt}(x, 0)), \end{aligned}$$

因为 $\Delta^M \mathbf{u}_m(x, 0) = \Delta^M \boldsymbol{\varphi}_m$, $\Delta^M \mathbf{u}_{mt}(x, 0) = \Delta^M \boldsymbol{\psi}_m$ 所以在上式中

$$\begin{aligned} & ((-1)^{M+1} \Delta^M \mathbf{u}_m(x, 0), (-1)^{M+1} \Delta^M \mathbf{u}_m(x, 0)) = ((-1)^{M+1} \Delta^M \boldsymbol{\varphi}_m, (-1)^{M+1} \Delta^M \boldsymbol{\varphi}_m), \\ & ((-1)^{M+1} \Delta^M \mathbf{u}_{mt}(x, 0), (-1)^{M+1} \Delta^M \mathbf{u}_{mt}(x, 0)) = ((-1)^{M+1} \Delta^M \boldsymbol{\psi}_m, (-1)^{M+1} \Delta^M \boldsymbol{\psi}_m), \end{aligned}$$

从而有

$$\begin{aligned} & (\mathbf{u}_{mtt}(x, 0), \mathbf{u}_{mtt}(x, 0)) + ((-1)^M \Delta^M \mathbf{u}_{mtt}(x, 0), \mathbf{u}_{mtt}(x, 0)) \\ & \leq \frac{1}{2\varepsilon_1} (f(\boldsymbol{\psi}_m), f(\boldsymbol{\psi}_m)) + \frac{\varepsilon_1}{2} (\mathbf{u}_{mtt}(x, 0), \mathbf{u}_{mtt}(x, 0)) \\ & \quad + \frac{1}{2\varepsilon_2} ((-1)^{M+1} \Delta^M \boldsymbol{\varphi}_m(x, 0), (-1)^{M+1} \Delta^M \boldsymbol{\varphi}_m(x, 0)) + \frac{\varepsilon_2}{2} (\mathbf{u}_{mtt}(x, 0), \mathbf{u}_{mtt}(x, 0)) \\ & \quad + \frac{1}{2\varepsilon_3} ((-1)^{M+1} \Delta^M \boldsymbol{\psi}_m(x, 0), (-1)^{M+1} \Delta^M \boldsymbol{\psi}_m(x, 0)) + \frac{\varepsilon_3}{2} (\mathbf{u}_{mtt}(x, 0), \mathbf{u}_{mtt}(x, 0)), \end{aligned}$$

移项整理得

$$\begin{aligned} & \left(1 - \frac{\varepsilon_1}{2} - \frac{\varepsilon_2}{2} - \frac{\varepsilon_3}{2}\right) (\mathbf{u}_{mtt}(x,0), \mathbf{u}_{mtt}(x,0)) + (\nabla^M \mathbf{u}_{mtt}(x,0), \nabla^M \mathbf{u}_{mtt}(x,0)) \\ & \leq \frac{1}{2\varepsilon_2} (\Delta^M \boldsymbol{\varphi}_m, \Delta^M \boldsymbol{\varphi}_m) + \frac{1}{2\varepsilon_3} (\Delta^M \boldsymbol{\psi}_m, \Delta^M \boldsymbol{\psi}_m) + \frac{1}{2\varepsilon_1} (f(\boldsymbol{\psi}_m), f(\boldsymbol{\psi}_m)), \end{aligned}$$

取 $\varepsilon_1, \varepsilon_2, \varepsilon_3$ 充分小, 使得 $\left(1 - \frac{\varepsilon_1}{2} - \frac{\varepsilon_2}{2} - \frac{\varepsilon_3}{2}\right) \geq \frac{1}{2}$ 。

由于当 $m \rightarrow +\infty$ 时, $\boldsymbol{\psi}_m \rightarrow \boldsymbol{\psi}$ 在 $H^{2M}(\Omega) \cap H_0^M(\Omega)$ 中强收敛, 故当 $m \rightarrow +\infty$ 时, $(\Delta^M \boldsymbol{\psi}_m, \Delta^M \boldsymbol{\psi}_m) \rightarrow (\Delta^M \boldsymbol{\psi}, \Delta^M \boldsymbol{\psi})$, 故 $(\Delta^M \boldsymbol{\psi}_m, \Delta^M \boldsymbol{\psi}_m)$ 能用一与 m 无关的正常数控制住, 同样道理 $(\Delta^M \boldsymbol{\varphi}_m, \Delta^M \boldsymbol{\varphi}_m)$ 也能用一与 m 无关的正常数控制住, 所以 $(\mathbf{u}_{mtt}(x,0), \mathbf{u}_{mtt}(x,0)) + (\nabla^M \mathbf{u}_{mtt}(x,0), \nabla^M \mathbf{u}_{mtt}(x,0))$ 有界。

又由已证不等式

$$\begin{aligned} & \frac{1}{2} (\mathbf{u}_{mtt}, \mathbf{u}_{mtt}) + \frac{1}{2} (\nabla^M \mathbf{u}_{mtt}, \nabla^M \mathbf{u}_{mtt}) + \frac{1}{2} (\nabla^M \mathbf{u}_{mtt}, \nabla^M \mathbf{u}_{mtt}) + [\nabla^M \mathbf{u}_{mtt}, \nabla^M \mathbf{u}_{mtt}] \\ & \leq \frac{1}{2} (\mathbf{u}_{mtt}(x,0), \mathbf{u}_{mtt}(x,0)) + \frac{1}{2} (\nabla^M \boldsymbol{\psi}_m, \nabla^M \boldsymbol{\psi}_m) + \frac{1}{2} (\nabla^M \mathbf{u}_{mtt}(x,0), \nabla^M \mathbf{u}_{mtt}(x,0)) + k_0 [\mathbf{u}_{mtt}, \mathbf{u}_{mtt}], \end{aligned}$$

因为 $(\mathbf{u}_{mtt}(x,0), \mathbf{u}_{mtt}(x,0)) + (\nabla^M \mathbf{u}_{mtt}(x,0), \nabla^M \mathbf{u}_{mtt}(x,0))$ 有界, $\frac{1}{2} (\nabla^M \boldsymbol{\psi}_m, \nabla^M \boldsymbol{\psi}_m)$ 又能用一与 m 无关的正常数控制住, 所以

$$\frac{1}{2} (\mathbf{u}_{mtt}, \mathbf{u}_{mtt}) + \frac{1}{2} (\nabla^M \mathbf{u}_{mtt}, \nabla^M \mathbf{u}_{mtt}) + \frac{1}{2} (\nabla^M \mathbf{u}_{mtt}, \nabla^M \mathbf{u}_{mtt}) + [\nabla^M \mathbf{u}_{mtt}, \nabla^M \mathbf{u}_{mtt}] \leq k_0 [\mathbf{u}_{mtt}, \mathbf{u}_{mtt}] + C,$$

其中 C 为与 m 无关的正常数, 再根据 Gronwall 不等式得

$$|\mathbf{u}_{mtt}|_{L^2(\Omega)}^2 + |\nabla^M \mathbf{u}_{mtt}|_{L^2(\Omega)}^2 + |\nabla^M \mathbf{u}_{mtt}|_{L^2(\Omega)}^2 + \|\nabla^M \mathbf{u}_{mtt}\|_{L^2(\Omega)}^2 \leq C_2.$$

引理 3.2 证毕!

引理 3.3 的证明:

方程(2.4)两边同乘 $\lambda_k a'_{mik}(t)$ 得

$$\begin{aligned} & (\mathbf{u}_{mit}, \lambda_k a'_{mik}(t) w_k) + \left((-1)^M \Delta^M \mathbf{u}_{mi}, \lambda_k a'_{mik}(t) w_k\right) + \left((-1)^M \Delta^M \mathbf{u}_{mit}, \lambda_k a'_{mik}(t) w_k\right) \\ & + \left((-1)^M \Delta^M \mathbf{u}_{mit}, \lambda_k a'_{mik}(t) w_k\right) = (f_i(\mathbf{u}_{mi}), \lambda_k a'_{mik}(t) w_k), \end{aligned}$$

关于 k 从 1 到 m 作和得

$$\begin{aligned} & (\mathbf{u}_{mit}, (-1)^M \Delta^M \mathbf{u}_{mit}) + \left((-1)^M \Delta^M \mathbf{u}_{mi}, (-1)^M \Delta^M \mathbf{u}_{mit}\right) + \left((-1)^M \Delta^M \mathbf{u}_{mit}, (-1)^M \Delta^M \mathbf{u}_{mit}\right) \\ & + \left((-1)^M \Delta^M \mathbf{u}_{mit}, (-1)^M \Delta^M \mathbf{u}_{mit}\right) = (f_i(\mathbf{u}_{mi}), (-1)^M \Delta^M \mathbf{u}_{mit}), \end{aligned}$$

关于 i 从 1 到 N 作和得

$$\begin{aligned} & (\mathbf{u}_{mit}, (-1)^M \Delta^M \mathbf{u}_{mit}) + \left((-1)^M \Delta^M \mathbf{u}_m, (-1)^M \Delta^M \mathbf{u}_{mit}\right) + \left((-1)^M \Delta^M \mathbf{u}_{mi}, (-1)^M \Delta^M \mathbf{u}_{mit}\right) \\ & + \left((-1)^M \Delta^M \mathbf{u}_{mit}, (-1)^M \Delta^M \mathbf{u}_{mi}\right) = (f(\mathbf{u}_{mi}), (-1)^M \Delta^M \mathbf{u}_{mi}), \end{aligned}$$

对于右边项经计算得

$$\left(\mathbf{f}(\mathbf{u}_{mt}), (-1)^M \Delta^M \mathbf{u}_{mt} \right) \leq \frac{1}{2\varepsilon_1} \left(\mathbf{f}(\mathbf{u}_{mt}), \mathbf{f}(\mathbf{u}_{mt}) \right) + \frac{\varepsilon_1}{2} \left((-1)^M \Delta^M \mathbf{u}_{mt}, (-1)^M \Delta^M \mathbf{u}_{mt} \right),$$

取 ε_1 充分小, 使得 $1 - \frac{\varepsilon_1}{2} \geq \frac{1}{2}$ 故成立如下不等式

$$\begin{aligned} & \left(\mathbf{u}_{mt}, (-1)^M \Delta^M \mathbf{u}_{mt} \right) + \left((-1)^M \Delta^M \mathbf{u}_m, (-1)^M \Delta^M \mathbf{u}_{mt} \right) + \frac{1}{2} \left((-1)^M \Delta^M \mathbf{u}_{mt}, (-1)^M \Delta^M \mathbf{u}_{mt} \right) \\ & + \left((-1)^M \Delta^M \mathbf{u}_{mt}, (-1)^M \Delta^M \mathbf{u}_{mt} \right) \leq \frac{1}{2\varepsilon_1} \left(\mathbf{f}(\mathbf{u}_{mt}), \mathbf{f}(\mathbf{u}_{mt}) \right), \end{aligned}$$

由格林公式得

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\nabla^M \mathbf{u}_{mt}, \nabla^M \mathbf{u}_{mt} \right) + \frac{1}{2} \frac{d}{dt} \left(\Delta^M \mathbf{u}_m, \Delta^M \mathbf{u}_m \right) + \frac{1}{2} \frac{d}{dt} \left(\Delta^M \mathbf{u}_{mt}, \Delta^M \mathbf{u}_{mt} \right) \\ & + \frac{1}{2} \left(\Delta^M \mathbf{u}_{mt}, \Delta^M \mathbf{u}_{mt} \right) \leq \frac{1}{2\varepsilon_1} \left(\mathbf{f}(\mathbf{u}_{mt}), \mathbf{f}(\mathbf{u}_{mt}) \right), \end{aligned}$$

两边从 0 到 t ($0 \leq t \leq T$) 积分并移项得

$$\begin{aligned} & \frac{1}{2} \left(\nabla^M \mathbf{u}_{mt}, \nabla^M \mathbf{u}_{mt} \right) + \frac{1}{2} \left(\Delta^M \mathbf{u}_m, \Delta^M \mathbf{u}_m \right) + \frac{1}{2} \left[\Delta^M \mathbf{u}_{mt}, \Delta^M \mathbf{u}_{mt} \right] + \frac{1}{2} \left(\Delta^M \mathbf{u}_{mt}, \Delta^M \mathbf{u}_{mt} \right) \\ & \leq \frac{1}{2} \left(\nabla^M \boldsymbol{\psi}_m, \nabla^M \boldsymbol{\psi}_m \right) + \frac{1}{2} \left(\Delta^M \boldsymbol{\varphi}_m, \Delta^M \boldsymbol{\varphi}_m \right) + \frac{1}{2} \left(\Delta^M \boldsymbol{\psi}_m, \Delta^M \boldsymbol{\psi}_m \right) + \frac{1}{2\varepsilon_1} \left[\mathbf{f}(\mathbf{u}_{mt}), \mathbf{f}(\mathbf{u}_{mt}) \right], \end{aligned}$$

由引理 3.1 知 $\|\mathbf{u}_{mt}\|_{H^M(\Omega)}^2 \leq \text{const}$ ($0 \leq t \leq T$),

$$\left(\mathbf{f}(\mathbf{u}_{mt}), \mathbf{f}(\mathbf{u}_{mt}) \right) \leq \left(a_1 + b_1 \|\mathbf{u}_{mt}\|_{L^p(\Omega)}^p, a_1 + b_1 \|\mathbf{u}_{mt}\|_{L^p(\Omega)}^p \right) = a_1 |\Omega| + 2a_1 b_1 \int_{\Omega} |\mathbf{u}_{mt}|^{\frac{p}{2}} dx + b_1^2 \int_{\Omega} |\mathbf{u}_{mt}|^p dx,$$

由 Sobolev 嵌入定理[5]得

$$\int_{\Omega} |\mathbf{u}_{mt}|^p dx = \|\mathbf{u}_{mt}\|_{L^p(\Omega)}^p \leq \text{const}, \int_{\Omega} |\mathbf{u}_{mt}|^{\frac{p}{2}} dx \leq \left(\int_{\Omega} |\mathbf{u}_{mt}|^{\frac{p}{2} \times 2} dx \right)^{\frac{1}{2}} \left(\int_{\Omega} 1^2 dx \right)^{\frac{1}{2}} = |\Omega|^{\frac{1}{2}} \|\mathbf{u}_{mt}\|_{L^p(\Omega)}^{\frac{p}{2}} \leq \text{const}$$

其中当 $2M < n$ 时, $2 \leq p < \frac{2n}{n-2M}$; 当 $2M \geq n$ 时, $2 \leq p < +\infty$ 。

所以 $\left(\mathbf{f}(\mathbf{u}_{mt}), \mathbf{f}(\mathbf{u}_{mt}) \right) \leq \text{const}$ 。在下面的不等式中,

$$\begin{aligned} & \frac{1}{2} \left(\nabla^M \mathbf{u}_{mt}, \nabla^M \mathbf{u}_{mt} \right) + \frac{1}{2} \left(\Delta^M \mathbf{u}_m, \Delta^M \mathbf{u}_m \right) + \frac{1}{2} \left[\Delta^M \mathbf{u}_{mt}, \Delta^M \mathbf{u}_{mt} \right] + \frac{1}{2} \left(\Delta^M \mathbf{u}_{mt}, \Delta^M \mathbf{u}_{mt} \right) \\ & \leq \frac{1}{2} \left(\nabla^M \boldsymbol{\psi}_m, \nabla^M \boldsymbol{\psi}_m \right) + \frac{1}{2} \left(\Delta^M \boldsymbol{\varphi}_m, \Delta^M \boldsymbol{\varphi}_m \right) + \frac{1}{2} \left(\Delta^M \boldsymbol{\psi}_m, \Delta^M \boldsymbol{\psi}_m \right) + \frac{1}{2\varepsilon_1} \left[\mathbf{f}(\mathbf{u}_{mt}), \mathbf{f}(\mathbf{u}_{mt}) \right], \end{aligned}$$

由于当 $m \rightarrow +\infty$ 时, $\boldsymbol{\varphi}_m \rightarrow \boldsymbol{\varphi}$, $\boldsymbol{\psi}_m \rightarrow \boldsymbol{\psi}$ 在 $H^{2M}(\Omega) \cap H_0^M(\Omega)$ 中强收敛, 所以当 $m \rightarrow +\infty$ 时,

$$\frac{1}{2} \left(\nabla^M \boldsymbol{\psi}_m, \nabla^M \boldsymbol{\psi}_m \right) + \frac{1}{2} \left(\Delta^M \boldsymbol{\varphi}_m, \Delta^M \boldsymbol{\varphi}_m \right) + \frac{1}{2} \left(\Delta^M \boldsymbol{\psi}_m, \Delta^M \boldsymbol{\psi}_m \right)$$

收敛于 $\frac{1}{2} \left(\nabla^M \boldsymbol{\psi}, \nabla^M \boldsymbol{\psi} \right) + \frac{1}{2} \left(\Delta^M \boldsymbol{\varphi}, \Delta^M \boldsymbol{\varphi} \right) + \frac{1}{2} \left(\Delta^M \boldsymbol{\psi}, \Delta^M \boldsymbol{\psi} \right)$,

$\frac{1}{2} \left(\nabla^M \boldsymbol{\psi}_m, \nabla^M \boldsymbol{\psi}_m \right) + \frac{1}{2} \left(\Delta^M \boldsymbol{\varphi}_m, \Delta^M \boldsymbol{\varphi}_m \right) + \frac{1}{2} \left(\Delta^M \boldsymbol{\psi}_m, \Delta^M \boldsymbol{\psi}_m \right)$ 能用一与 m 无关的正常数控制住, 因此由以上不等式得

$$\frac{1}{2}(\nabla^M \mathbf{u}_{mt}, \nabla^M \mathbf{u}_{mt}) + \frac{1}{2}(\Delta^M \mathbf{u}_m, \Delta^M \mathbf{u}_m) + \frac{1}{2}[\Delta^M \mathbf{u}_{mt}, \Delta^M \mathbf{u}_{mt}] + \frac{1}{2}(\Delta^M \mathbf{u}_{mt}, \Delta^M \mathbf{u}_{mt}) \leq \text{const},$$

即

$$|\nabla^M \mathbf{u}_{mt}|_{L^2(\Omega)}^2 + |\Delta^M \mathbf{u}_m|_{L^2(\Omega)}^2 + |\Delta^M \mathbf{u}_{mt}|_{L^2(\Omega)}^2 + \|\Delta^M \mathbf{u}_{mt}\|_{L^2(\Omega)}^2 \leq C_3 \quad (C_3 \text{ 为与 } \mathbf{u}_m \text{ 无关的正常数}).$$

引理 3.3 证毕!

引理 3.4 的证明:

方程(2.4)两边同乘以 $\lambda_k a_{mik}''(t)$, 得

$$\begin{aligned} & (\mathbf{u}_{mit}, \lambda_k a_{mik}''(t) w_k) + ((-1)^M \Delta^M \mathbf{u}_{mi}, \lambda_k a_{mik}''(t) w_k) + ((-1)^M \Delta^M \mathbf{u}_{mit}, \lambda_k a_{mik}''(t) w_k) \\ & + ((-1)^M \Delta^M \mathbf{u}_{mit}, \lambda_k a_{mik}''(t) w_k) = (f_i(\mathbf{u}_{mt}), \lambda_k a_{mik}''(t) w_k), \end{aligned}$$

关于 k 从 1 到 m 作和得

$$\begin{aligned} & (\mathbf{u}_{mit}, (-1)^M \Delta^M \mathbf{u}_{mit}) + ((-1)^M \Delta^M \mathbf{u}_{mi}, (-1)^M \Delta^M \mathbf{u}_{mit}) + ((-1)^M \Delta^M \mathbf{u}_{mit}, (-1)^M \Delta^M \mathbf{u}_{mit}) \\ & + ((-1)^M \Delta^M \mathbf{u}_{mit}, (-1)^M \Delta^M \mathbf{u}_{mit}) = (f_i(\mathbf{u}_{mt}), (-1)^M \Delta^M \mathbf{u}_{mit}), \end{aligned}$$

关于 i 从 1 到 N 作和得

$$\begin{aligned} & (\mathbf{u}_{mt}, (-1)^M \Delta^M \mathbf{u}_{mt}) + ((-1)^M \Delta^M \mathbf{u}_m, (-1)^M \Delta^M \mathbf{u}_{mt}) + ((-1)^M \Delta^M \mathbf{u}_{mt}, (-1)^M \Delta^M \mathbf{u}_{mt}) \\ & + ((-1)^M \Delta^M \mathbf{u}_{mt}, (-1)^M \Delta^M \mathbf{u}_{mt}) = (\mathbf{f}(\mathbf{u}_{mt}), (-1)^M \Delta^M \mathbf{u}_{mt}), \end{aligned}$$

移项得如下不等式

$$\begin{aligned} & ((-1)^M \Delta^M \mathbf{u}_{mt}, (-1)^M \Delta^M \mathbf{u}_{mt}) = -(\mathbf{u}_{mt}, (-1)^M \Delta^M \mathbf{u}_{mt}) - ((-1)^M \Delta^M \mathbf{u}_m, (-1)^M \Delta^M \mathbf{u}_{mt}) \\ & - ((-1)^M \Delta^M \mathbf{u}_{mt}, (-1)^M \Delta^M \mathbf{u}_{mt}) + (\mathbf{f}(\mathbf{u}_{mt}), (-1)^M \Delta^M \mathbf{u}_{mt}), \end{aligned}$$

而

$$\begin{aligned} & -(\mathbf{u}_{mt}, (-1)^M \Delta^M \mathbf{u}_{mt}) \leq \frac{\varepsilon_1}{2} ((-1)^M \Delta^M \mathbf{u}_{mt}, (-1)^M \Delta^M \mathbf{u}_{mt}) + \frac{1}{2\varepsilon_1} (\mathbf{u}_{mt}, \mathbf{u}_{mt}) \\ & = \frac{\varepsilon_1}{2} (\Delta^M \mathbf{u}_{mt}, \Delta^M \mathbf{u}_{mt}) + \frac{1}{2\varepsilon_1} (\mathbf{u}_{mt}, \mathbf{u}_{mt}), \\ & -((-1)^M \Delta^M \mathbf{u}_m, (-1)^M \Delta^M \mathbf{u}_{mt}) \\ & \leq \frac{\varepsilon_2}{2} ((-1)^M \Delta^M \mathbf{u}_{mt}, (-1)^M \Delta^M \mathbf{u}_{mt}) + \frac{1}{2\varepsilon_2} ((-1)^M \Delta^M \mathbf{u}_m, (-1)^M \Delta^M \mathbf{u}_m) \\ & = \frac{\varepsilon_2}{2} (\Delta^M \mathbf{u}_{mt}, \Delta^M \mathbf{u}_{mt}) + \frac{1}{2\varepsilon_2} (\Delta^M \mathbf{u}_m, \Delta^M \mathbf{u}_m), \\ & -((-1)^M \Delta^M \mathbf{u}_{mt}, (-1)^M \Delta^M \mathbf{u}_{mt}) \\ & \leq \frac{\varepsilon_3}{2} ((-1)^M \Delta^M \mathbf{u}_{mt}, (-1)^M \Delta^M \mathbf{u}_{mt}) + \frac{1}{2\varepsilon_3} ((-1)^M \Delta^M \mathbf{u}_{mt}, (-1)^M \Delta^M \mathbf{u}_{mt}) \\ & = \frac{\varepsilon_3}{2} (\Delta^M \mathbf{u}_{mt}, \Delta^M \mathbf{u}_{mt}) + \frac{1}{2\varepsilon_3} (\Delta^M \mathbf{u}_{mt}, \Delta^M \mathbf{u}_{mt}), \end{aligned}$$

$$\left(\mathbf{f}(\mathbf{u}_{mt}), (-1)^M \Delta^M \mathbf{u}_{mt} \right) \leq \frac{\varepsilon_4}{2} \left((-1)^M \Delta^M \mathbf{u}_{mt}, (-1)^M \Delta^M \mathbf{u}_{mt} \right) + \frac{1}{2\varepsilon_4} \left(\mathbf{f}(\mathbf{u}_{mt}), \mathbf{f}(\mathbf{u}_{mt}) \right),$$

所以

$$\begin{aligned} \left(\Delta^M \mathbf{u}_{mt}, \Delta^M \mathbf{u}_{mt} \right) &= \left((-1)^M \Delta^M \mathbf{u}_{mt}, (-1)^M \Delta^M \mathbf{u}_{mt} \right) \\ &\leq \frac{\varepsilon_1}{2} \left(\Delta^M \mathbf{u}_{mt}, \Delta^M \mathbf{u}_{mt} \right) + \frac{1}{2\varepsilon_1} \left(\mathbf{u}_{mt}, \mathbf{u}_{mt} \right) + \frac{\varepsilon_2}{2} \left(\Delta^M \mathbf{u}_{mt}, \Delta^M \mathbf{u}_{mt} \right) \\ &\quad + \frac{1}{2\varepsilon_2} \left(\Delta^M \mathbf{u}_m, \Delta^M \mathbf{u}_m \right) + \frac{\varepsilon_3}{2} \left(\Delta^M \mathbf{u}_{mt}, \Delta^M \mathbf{u}_{mt} \right) + \frac{1}{2\varepsilon_3} \left(\Delta^M \mathbf{u}_{mt}, \Delta^M \mathbf{u}_{mt} \right) \\ &\quad + \frac{\varepsilon_4}{2} \left(\Delta^M \mathbf{u}_{mt}, \Delta^M \mathbf{u}_{mt} \right) + \frac{1}{2\varepsilon_4} \left(\mathbf{f}(\mathbf{u}_{mt}), \mathbf{f}(\mathbf{u}_{mt}) \right), \end{aligned}$$

经计算得

$$\begin{aligned} &\left(1 - \frac{\varepsilon_1}{2} - \frac{\varepsilon_2}{2} - \frac{\varepsilon_3}{2} - \frac{\varepsilon_4}{2} \right) \left(\Delta^M \mathbf{u}_{mt}, \Delta^M \mathbf{u}_{mt} \right) \\ &\leq \frac{1}{2\varepsilon_1} \left(\mathbf{u}_{mt}, \mathbf{u}_{mt} \right) + \frac{1}{2\varepsilon_2} \left(\Delta^M \mathbf{u}_m, \Delta^M \mathbf{u}_m \right) + \frac{1}{2\varepsilon_3} \left(\Delta^M \mathbf{u}_{mt}, \Delta^M \mathbf{u}_{mt} \right) + \frac{1}{2\varepsilon_4} \left(\mathbf{f}(\mathbf{u}_{mt}), \mathbf{f}(\mathbf{u}_{mt}) \right) \end{aligned}$$

由引理 3.2 和引理 3.3 的结论知

$$\left(\mathbf{u}_{mt}, \mathbf{u}_{mt} \right) \leq \text{const}, \quad \left(\Delta^M \mathbf{u}_m, \Delta^M \mathbf{u}_m \right) \leq \text{const}, \quad \left(\Delta^M \mathbf{u}_{mt}, \Delta^M \mathbf{u}_{mt} \right) \leq \text{const}$$

类似于引理 3.2 的证明过程可证 $\left(\mathbf{f}(\mathbf{u}_{mt}), \mathbf{f}(\mathbf{u}_{mt}) \right) \leq \text{const}$,

取 $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ 充分小, 使得 $\left(1 - \frac{\varepsilon_1}{2} - \frac{\varepsilon_2}{2} - \frac{\varepsilon_3}{2} - \frac{\varepsilon_4}{2} \right) \geq \frac{1}{2}$, 所以

$$\left(\Delta^M \mathbf{u}_{mt}, \Delta^M \mathbf{u}_{mt} \right) \leq \text{const}, \text{ 即 } \left| \Delta^M \mathbf{u}_{mt} \right|_{L^2(\Omega)}^2 \leq C_4 \quad (C_4 \text{ 为与 } \mathbf{u}_m \text{ 无关的正常数}).$$

引理 3.4 证毕!

下面在引理 3.1~3.4 的基础上给出定理 1 的证明。

定理 1 的证明: 将(2.1)~(2.3)代入(2.4)及其初值条件得

$$\begin{cases} \sum_{j=1}^m a_{mij}''(t) (w_j, w_k) + \sum_{j=1}^m \left((-1)^M \Delta^M w_j, w_k \right) a_{mij}(t) + \sum_{j=1}^m \left((-1)^M \Delta^M w_j, w_k \right) a_{mij}'(t) \\ \quad + \sum_{j=1}^m \left((-1)^M \Delta^M w_j, w_k \right) a_{mij}''(t) = \left(f_i(\mathbf{u}_{mt}), w_k \right) \\ a_{mij}(0) = a_{mij} \\ a_{mij}'(0) = b_{mij} \\ k = 1, 2, \dots, m; i = 1, 2, \dots, N. \end{cases}$$

由引理 3.1~引理 3.4 的证明过程知 $\left| \left(f_i(\mathbf{u}_{mt}), w_k \right) \right|$ 有界, 因此由常微分方程理论知上述常微分方程组在 $[0, T]$ 上有整体解, 从而常微分方程组 (*) 在 $[0, T]$ 上有整体解 $\mathbf{u}_{mt}(x, t)$, 且有下列估计式成立:

$$\begin{aligned} \left| \mathbf{u}_m \right|_{L^2(\Omega)}^2 &\leq \text{const}, \quad \left| \mathbf{u}_{mt} \right|_{L^2(\Omega)}^2 \leq \text{const}, \quad \left| \mathbf{u}_{mt} \right|_{L^2(\Omega)}^2 \leq \text{const}, \quad \left\| \nabla^M \mathbf{u}_{mt} \right\|_{L^2(\Omega)}^2 \leq \text{const}, \\ \left| \nabla^M \mathbf{u}_{mt} \right|_{L^2(\Omega)}^2 &\leq \text{const}, \quad \left| \Delta^M \mathbf{u}_{mt} \right|_{L^2(\Omega)}^2 \leq \text{const}, \quad \left| \Delta^M \mathbf{u}_m \right|_{L^2(\Omega)}^2 \leq \text{const}. \end{aligned}$$

由列紧性原理知存在 $\{u_{mi}\}$ 的一个子序列, 不妨设为 $\{u_{vi}\}$, 使得当 $v \rightarrow \infty$ 时, u_{vi} 弱收敛到 u_i 于 $L^\infty([0, T], H^{2M}(\Omega) \cap H_0^M(\Omega))$, u_{vit} 弱收敛到 u_{it} 于

$$L^\infty([0, T], H^{2M}(\Omega) \cap H_0^M(\Omega)) \subset L^\infty([0, T], L^2(\Omega)), \quad u_{vit} \text{ 弱收敛到 } u_{it} \text{ 于}$$

$L^2([0, T], H_0^M(\Omega)) \cap L^\infty([0, T], L^2(\Omega))$, 另一方面, 由内插不等式知 $|\nabla u_{vt}|_{L^2(\Omega)}$ 可被 $|u_{vt}|_{L^2(\Omega)}$ 与 $|\nabla^M u_{vt}|_{L^2(\Omega)}$ 控制住, 而 $|u_{vt}|_{L^2(\Omega)}$ 与 $|\nabla^M u_{vt}|_{L^2(\Omega)}$ 是有界的, 所以 $|\nabla u_{vt}|_{L^2(\Omega)}$ 有界. 故由引理 3.1~引理 3.4 知 $u_{vt}, u_{vit}, \nabla u_{vt}$ 于 $L^\infty(0, T; L^2(\Omega)) \subset L^2(0, T; L^2(\Omega)) = L^2(Q_T)$ ($Q_T = \Omega \times (0, T)$) 中有界, 因此 u_{vt} 在 $H^1(Q_T)$ 中有界, 由 $H^1(Q_T)$ 紧致嵌入到 $L^2(Q_T)$ 知从 u_{vt} 可选取一子序列(仍记为 u_{vt}) 在 $L^2(Q_T)$ 中强收敛且几乎处处收敛到 u_t . 又由引理 3.3 的证明过程知 $(f(u_{vt}), f(u_{vt})) = |f(u_{vt})|_{L^2(\Omega)}^2 \leq \text{const}$, 由 [6] 中引理 1.3 知 $f(u_{vt}) \rightarrow f(u_t)$ 在 $L^2(Q_T)$ 中弱收敛. 在(2.4)中令 $m = v$ 将

$$(u_{mit}, w_k) + ((-1)^M \Delta^M u_{mi}, w_k) + ((-1)^M \Delta^M u_{mit}, w_k) + ((-1)^M \Delta^M u_{mit}, w_k) = (f_i(u_{mt}), w_k),$$

改为

$$(u_{vit}, w_k) + ((-1)^M \Delta^M u_{vi}, w_k) + ((-1)^M \Delta^M u_{vit}, w_k) + ((-1)^M \Delta^M u_{vit}, w_k) = (f_i(u_{vt}), w_k),$$

两边同乘 $d_{ki}(t)$, 其中 $d_{ki}(t) \in C[0, T], k = 1, 2, \dots; i = 1, 2, \dots, N$, 得

$$(u_{vit}, d_{ki}(t)w_k) + ((-1)^M \Delta^M u_{vi}, d_{ki}(t)w_k) + ((-1)^M \Delta^M u_{vit}, d_{ki}(t)w_k) + ((-1)^M \Delta^M u_{vit}, d_{ki}(t)w_k) \\ = (f_i(u_{vt}), d_{ki}(t)w_k),$$

对 $k = 1, 2, \dots, v'$ ($v' \leq v$) 求和得

$$\left(u_{vit}, \sum_{k=1}^{v'} d_{ki}(t)w_k(x) \right) + \left((-1)^M \Delta^M u_{vi}, \sum_{k=1}^{v'} d_{ki}(t)w_k(x) \right) + \left((-1)^M \Delta^M u_{vit}, \sum_{k=1}^{v'} d_{ki}(t)w_k(x) \right) \\ + \left((-1)^M \Delta^M u_{vit}, \sum_{k=1}^{v'} d_{ki}(t)w_k(x) \right) = \left(f_i(u_{vt}), \sum_{k=1}^{v'} d_{ki}(t)w_k(x) \right),$$

对 t 从 0 到 T 积分得

$$\int_0^T \left(u_{vit}, \sum_{k=1}^{v'} d_{ki}(t)w_k(x) \right) dt + \int_0^T \left((-1)^M \Delta^M u_{vi}, \sum_{k=1}^{v'} d_{ki}(t)w_k(x) \right) dt + \int_0^T \left((-1)^M \Delta^M u_{vit}, \sum_{k=1}^{v'} d_{ki}(t)w_k(x) \right) dt \\ + \int_0^T \left((-1)^M \Delta^M u_{vit}, \sum_{k=1}^{v'} d_{ki}(t)w_k(x) \right) dt = \int_0^T \left(f_i(u_{vt}), \sum_{k=1}^{v'} d_{ki}(t)w_k(x) \right) dt,$$

对应当 $v \rightarrow +\infty$ 时, $f(u_{vt}) \rightarrow f(u_t)$ 在 $L^2(Q_T)$ 中弱收敛, 有当 $v \rightarrow +\infty$ 时, $f_i(u_{vt}) \rightarrow f_i(u_t)$ 在 $L^2(Q_T)$ 中弱收敛 ($i = 1, 2, \dots, N$), 令 $v \rightarrow +\infty$ 得

$$\int_0^T \left(u_{it}, \sum_{k=1}^{v'} d_{ki}(t)w_k(x) \right) dt + \int_0^T \left((-1)^M \Delta^M u_i, \sum_{k=1}^{v'} d_{ki}(t)w_k(x) \right) dt + \int_0^T \left((-1)^M \Delta^M u_{it}, \sum_{k=1}^{v'} d_{ki}(t)w_k(x) \right) dt \\ + \int_0^T \left((-1)^M \Delta^M u_{it}, \sum_{k=1}^{v'} d_{ki}(t)w_k(x) \right) dt = \int_0^T \left(f_i(u_t), \sum_{k=1}^{v'} d_{ki}(t)w_k(x) \right) dt,$$

因为 $\{w_k(x)\}_{k=1}^{+\infty}$ 构成 $L^2(\Omega)$ 的一组标准正交基,

$$d_{ki}(t) \in C[0, T], k = 1, 2, \dots; i = 1, 2, \dots, N$$

$\left\{ \sum_{k=1}^{v'} d_{ki}(t) w_k(x) \mid v' = 1, 2, \dots; i = 1, 2, \dots, N \right\}$ 在空间 $C([0, T]; L^2(\Omega))$ 中稠密, 所以对于任意的 $d_i(x, t) \in C([0, T]; L^2(\Omega))$ ($i = 1, 2, \dots, N$) 成立

$$\begin{aligned}
 & \int_0^T (u_{iv}, d_i(x, t)) dt + \int_0^T \left((-1)^M \Delta^M u_i, d_i(x, t) \right) dt + \int_0^T \left((-1)^M \Delta^M u_{iv}, d_i(x, t) \right) dt \\
 & + \int_0^T \left((-1)^M \Delta^M u_{iv}, d_i(x, t) \right) dt = \int_0^T (f_i(u_i), d_i(x, t)) dt,
 \end{aligned}$$

上式关于 $i = 1, 2, \dots, N$ 求和得对任意

$$\begin{aligned}
 \mathbf{d}(x, t) &= (d_1(x, t), d_2(x, t), \dots, d_N(x, t))^T \in C([0, T]; L^2(\Omega)) \text{ 成立} \\
 & \int_0^T (\mathbf{u}_{iv}, \mathbf{d}(x, t)) dt + \int_0^T \left((-1)^M \Delta^M \mathbf{u}, \mathbf{d}(x, t) \right) dt + \int_0^T \left((-1)^M \Delta^M \mathbf{u}_{iv}, \mathbf{d}(x, t) \right) dt \\
 & + \int_0^T \left((-1)^M \Delta^M \mathbf{u}_{iv}, \mathbf{d}(x, t) \right) dt = \int_0^T (\mathbf{f}(\mathbf{u}_i), \mathbf{d}(x, t)) dt.
 \end{aligned}$$

下面验证初始条件 $\mathbf{u}(x, 0) = \boldsymbol{\varphi}(x)$ 。因为当 $v \rightarrow \infty$ 时, u_{vi} 弱收敛到 u_i 于 $L^\infty([0, T], H^{2M}(\Omega) \cap H_0^M(\Omega))$, u_{vii} 弱收敛到 u_{ii} 于

$$L^\infty([0, T], H^{2M}(\Omega) \cap H_0^M(\Omega)) \subset L^\infty([0, T], L^2(\Omega)),$$

因此 $u_{vi}(x, t), u_i(x, t) \in C([0, T], H^{2M}(\Omega) \cap H_0^M(\Omega))$,

所以, 当 $v \rightarrow +\infty$ 时 $u_{vi}(x, 0)$ 弱收敛到 $u_i(x, 0)$ 于 $H^{2M}(\Omega) \cap H_0^M(\Omega)$ 中, 而当 $v \rightarrow +\infty$ 时, $u_{vii}(x, 0) \rightarrow \varphi_i(x)$ ($i = 1, 2, \dots, N$) 在 $H^{2M}(\Omega) \cap H_0^M(\Omega)$ 中强收敛, 故

$$u_i(x, 0) = \varphi_i(x) (i = 1, 2, \dots, N), \text{ 即 } \mathbf{u}(x, 0) = \boldsymbol{\varphi}(x)。$$

再证初始条件 $\mathbf{u}_t(x, 0) = \boldsymbol{\psi}(x)$ 。因为当 $v \rightarrow \infty$ 时 u_{vii} 弱收敛到 u_{ii} 于

$L^\infty([0, T], H^{2M}(\Omega) \cap H_0^M(\Omega)) \subset L^\infty([0, T], L^2(\Omega))$, u_{viii} 弱收敛到 u_{iii} 于

$L^2([0, T], H_0^M(\Omega)) \cap L^\infty([0, T], L^2(\Omega))$, 因此 $u_{vii}(x, t), u_{ii}(x, t) \in C([0, T], L^2(\Omega))$, 所以, $u_{vii}(x, 0)$ 弱收敛到 $u_{ii}(x, 0)$ ($i = 1, 2, \dots, N$) 于 $L^2(\Omega)$ 中, 而当 $v \rightarrow +\infty$ 时, $u_{viii}(x, 0) \rightarrow \psi_i(x)$ ($i = 1, 2, \dots, N$) 在 $L^2(\Omega)$ 中强收敛, 故 $u_{ii}(x, 0) = \psi_i(x)$ ($i = 1, 2, \dots, N$), 即 $\mathbf{u}_t(x, 0) = \boldsymbol{\psi}(x)$ 。

综上所述由定义 $\mathbf{u} = \mathbf{u}(x, t)$ 为问题(1.1)~(1.3)的整体强解。

下面给出定理 2 的证明, 定理 2 是关于问题(1.1)~(1.3)强解唯一性的讨论。

定理 2 的证明: 设 $\mathbf{u}^1, \mathbf{u}^2$ 为非线性问题(1.1)~(1.3)的两个整体强解,

令 $\mathbf{w} = \mathbf{u}^1 - \mathbf{u}^2$, 则 \mathbf{w} 满足

$$\begin{cases} \mathbf{w}_{tt} + (-1)^M \Delta^M \mathbf{w} + (-1)^M \Delta^M \mathbf{w}_t + (-1)^M \Delta^M \mathbf{w}_{tt} = \mathbf{f}(\mathbf{u}_t^1) - \mathbf{f}(\mathbf{u}_t^2), \\ \mathbf{w}(x, 0) = 0, \mathbf{w}_t(x, 0) = 0, \\ \left. D^\gamma \mathbf{w}(x, t) \right|_{\partial\Omega \times [0, T]} = 0, \quad 0 \leq |\gamma| \leq M - 1. \end{cases}$$

两边用 \mathbf{w}_t 做内积, 得

$$\begin{aligned}
 & (\mathbf{w}_{tt}, \mathbf{w}_t) + \left((-1)^M \Delta^M \mathbf{w}, \mathbf{w}_t \right) + \left((-1)^M \Delta^M \mathbf{w}_t, \mathbf{w}_t \right) + \left((-1)^M \Delta^M \mathbf{w}_{tt}, \mathbf{w}_t \right) \\
 & = \left(\frac{\partial \mathbf{f}(\mathbf{u}_t)}{\partial \mathbf{u}_t} \Big|_{\mathbf{u}_t^2 + \theta \mathbf{w}_t}, \mathbf{w}_t, \mathbf{w}_t \right) \quad (0 < \theta < 1),
 \end{aligned}$$

由条件(1.5)知 $\left(\frac{\partial f(\mathbf{u}_t)}{\partial \mathbf{u}_t}\right)\Big|_{\mathbf{u}_t^2+\theta \mathbf{w}_t, \mathbf{w}_t, \mathbf{w}_t} \leq k_0(\mathbf{w}_t, \mathbf{w}_t)$ ($0 < \theta < 1$),

因此

$$\frac{1}{2} \frac{d}{dt}(\mathbf{w}_t, \mathbf{w}_t) + \frac{1}{2} \frac{d}{dt}(\nabla^M \mathbf{w}, \nabla^M \mathbf{w}) + (\nabla^M \mathbf{w}_t, \nabla^M \mathbf{w}_t) + \frac{1}{2} \frac{d}{dt}(\nabla^M \mathbf{w}_t, \nabla^M \mathbf{w}_t) \leq k_0(\mathbf{w}_t, \mathbf{w}_t),$$

两边从 0 到 t ($0 \leq t \leq T$) 积分得

$$\frac{1}{2}(\mathbf{w}_t, \mathbf{w}_t) + \frac{1}{2}(\nabla^M \mathbf{w}, \nabla^M \mathbf{w}) + [\nabla^M \mathbf{w}_t, \nabla^M \mathbf{w}_t] + \frac{1}{2}(\nabla^M \mathbf{w}_t, \nabla^M \mathbf{w}_t) \leq k_0[\mathbf{w}_t, \mathbf{w}_t],$$

两边同时加上 $[\mathbf{w}, \mathbf{w}_t] + [\nabla^M \mathbf{w}, \nabla^M \mathbf{w}_t]$, 左边将它化为 $\frac{1}{2}(\mathbf{w}, \mathbf{w}) + \frac{1}{2}(\nabla^M \mathbf{w}, \nabla^M \mathbf{w})$, 右边将它估计为

$$[\mathbf{w}, \mathbf{w}_t] + [\nabla^M \mathbf{w}, \nabla^M \mathbf{w}_t] \leq [\mathbf{w}, \mathbf{w}] + [\mathbf{w}_t, \mathbf{w}_t] + \frac{1}{2\varepsilon}[\nabla^M \mathbf{w}, \nabla^M \mathbf{w}] + \frac{\varepsilon}{2}[\nabla^M \mathbf{w}_t, \nabla^M \mathbf{w}_t],$$

因此有

$$\begin{aligned} & \frac{1}{2}(\mathbf{w}, \mathbf{w}) + \frac{1}{2}(\mathbf{w}_t, \mathbf{w}_t) + (\nabla^M \mathbf{w}, \nabla^M \mathbf{w}) + [\nabla^M \mathbf{w}_t, \nabla^M \mathbf{w}_t] + \frac{1}{2}(\nabla^M \mathbf{w}_t, \nabla^M \mathbf{w}_t) \\ & \leq [\mathbf{w}, \mathbf{w}] + (k_0 + 1)[\mathbf{w}_t, \mathbf{w}_t] + \frac{\varepsilon}{2}[\nabla^M \mathbf{w}_t, \nabla^M \mathbf{w}_t] + \frac{1}{2\varepsilon}[\nabla^M \mathbf{w}, \nabla^M \mathbf{w}], \end{aligned}$$

取 ε 充分小使得 $1 - \frac{\varepsilon}{2} \geq \frac{1}{2}$ 整理得

$$\begin{aligned} & \frac{1}{2}(\mathbf{w}, \mathbf{w}) + \frac{1}{2}(\mathbf{w}_t, \mathbf{w}_t) + (\nabla^M \mathbf{w}, \nabla^M \mathbf{w}) + \frac{1}{2}[\nabla^M \mathbf{w}_t, \nabla^M \mathbf{w}_t] + \frac{1}{2}(\nabla^M \mathbf{w}_t, \nabla^M \mathbf{w}_t) \\ & \leq [\mathbf{w}, \mathbf{w}] + (k_0 + 1)[\mathbf{w}_t, \mathbf{w}_t] + \frac{1}{2\varepsilon}[\nabla^M \mathbf{w}, \nabla^M \mathbf{w}], \end{aligned}$$

由 Gronwall 不等式得 $|\mathbf{w}|_{L^2(\Omega)}^2 + |\mathbf{w}_t}|_{L^2(\Omega)}^2 + |\nabla^M \mathbf{w}|_{L^2(\Omega)}^2 = 0$.

所以 $\mathbf{w} \equiv \mathbf{0}$ a.e $\mathbf{u}^1 = \mathbf{u}^2$ 。

唯一性证毕!

4. 结论

本文主要研究了在粘弹性力学中具有实际背景的一类高阶 n 维非线性波动方程组。要求方程的非线性项 f 一次连续可微且非线性项 f 满足条件: f 的 Jacobi 矩阵半有界。首先选取高阶调和算子的特征函数系作为一组基, 用 Galerkin 方法构造初边值问题的近似解, 用 4 个引理对近似解作出一系列的先验估计, 在先验估计的基础上, 由列紧性原理证明了初边值问题整体强解的存在性, 最后证明了初边值问题整体强解的唯一性。其中, 选取高阶调和算子的特征函数系作为一组基, 该特征函数系分别构成 $L^2(\Omega)$, $H_0^M(\Omega), H^{2M} \cap H_0^M(\Omega)$ 的正交基底, 这样对近似解作先验估计时会更方便。

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