Research on the Implementation of the Optimal Implementation of the Multi-Asset Option

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Abstract

In this paper, we study the problem of determining the optimal implementation boundary of multi-asset option, and establish a mathematical model of multidimensional Black-Scholes equation with singular inner boundary function vector \( s = s(t) = (s_1(t), \ldots, s_m(t)), 0 < t < T \). In multi-dimension region \( \Omega \equiv \{(s, t) | s \in \mathbb{R}^m, t \in (0, T)\} \), the option price function is an unknown function. The exact solution \( u(s, t) \) of the mathematical model is obtained by using the matrix theory and the generalized characteristic function method. And the exponential function vector expression of the singular inner boundary is obtained \( \left(s_1(t), \ldots, s_m(t)\right) = \left(\partial_t e^{\omega(t)} \ldots, \partial_t e^{\omega(t)}\right) \). It is demonstrated that: when any \( t \in (0, T) \), the maximum value \( \max_{s \in \mathbb{R}^m} u(s, t) \) of the solution \( u(s, t) \) of the region \( R^n : 0 < s_j < \infty, j = 1, \ldots, m \) is obtained on the singular boundary, namely \( u(s(t), t) = \max_{s \in \mathbb{R}^m} u(s, t) \). The free boundary problem A and free boundary problem B of Black-Scholes equation are solved. The free boundary of problem A and B is expressed by the function vector \( R^n : 0 < s_j < \infty, j = 1, \ldots, m \).

The free boundary of the problem A and problem B coincides with the singular inner boundary. So the vector expression of the exponential function is the best implementation of the boundary. The exponential function vector \( \left(s_1(t), \ldots, s_m(t)\right) = \left(\partial_t e^{\omega(t)} \ldots, \partial_t e^{\omega(t)}\right) \) satisfies the condition \( \omega = -\frac{1}{s_k(t)} \frac{ds_k(t)}{dt} \), \( k = 1, \ldots, m \); and \( \omega_k \) is calculated by \( \omega_k = \sum_{j=1}^{k} b_{kj} \left[1 + \sum_{n=0}^{k-1} \frac{1}{2} \sum_{n=0}^{k-1} a_{mn} + q_n - r \right] \); the formula shows that \( \omega_k \) is only determined by all the parameters appearing in the multidimensional Black-Scholes equation.

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多资产期权确定最佳实施边界问题的研究

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摘要

本文研究多资产期权确定最佳实施边界的问题，建立了多维Black-Scholes方程在多维区域
\[ \Omega = \{(s,t) | s \in \mathbb{R}^m, t \in (0,T)\} \] 具有奇异内边界函数向量 \( s = (s_1(t), \ldots, s_m(t)) \), \( 0 < t < T \) 的数学模型，期权价格函数为未知函数。应用矩阵理论和广义特征函数法获得了期权价格函数的精确解 \( u(s,t) \)。并获得了奇异内边界的指数函数向量表达式
\[
\left( e_{11}\omega_j, \ldots, e_{m1}\omega_{j1} \right)
\]
同时获得了 Black-Scholes方程的自由边界问题A和自由边界问题B的精确解和其自由边界的指数函数向量表达式
\[
\left( e_{11}\omega_j, \ldots, e_{m1}\omega_{j1} \right)
\]
为最佳实施边界。指数函数向量
\[
\left( e_{11}\omega_j, \ldots, e_{m1}\omega_{j1} \right)
\]
满足条件 \( \omega_k = -\frac{1}{s_i(t)} \frac{d s_i(t)}{d t} \), \( k = 1, \ldots, m \)；且有 \( \omega_k \) 的计算公式
\[
\omega_k = \sum_{j=1}^n b_j \left[ \frac{1}{2} + \sum_{a_{m+1}}^n \left( a_{m+1} + q_j - r \right) c_{aj} \right]
\]
公式表明 \( \omega_k, k = 1, \ldots, m \) 由多维Black-Scholes方程中出现的所有参数
\( a_{ij}, q_j, r \) 唯一确定。

关键词

多资产期权，最佳实施边界，自由边界问题，多维Black-Scholes方程

1. 引言

期权是风险管理的核心工具，姜礼尚[1]对期权定价理论作了系统深入的阐述，利用偏微分方程理论和方法对期权理论作深入的定性和定量分析，特别对美式期权展开了深入的讨论。美式期权合约中具有提前实施的条款，因此最佳实施边界的确定对于美式期权具有特殊意义。在美式期权定价研究中，姜礼尚[1]建立了 Black-Scholes 方程的自由边界问题，对最佳实施边界 \( s = s(t), 0 < t < T \) 作了很多深入的研究，
得到很多重要的结论。其中包括 $s(t)$ 的位置，$s(t)$ 的单调性，$s(t)$ 的上下界以及 $s(t)$ 的凸性等，并给出了 $s(t)$ 在 $t = T$ 附近的渐近表达式。这些结果增加了对最佳实施边界的认知，对美式期权定价的数值计祘产生了重要的影响。期权定价问题历来是金融经济学中的重要研究课题之一[1]-[8]，多年来，众多经济学者与研究人员对这一问题进行不断深入的研究，但是这些研究大多是围绕具有单个资产的期权进行的。多资产期权在现代金融市场中占有重要的地位，研究多资产(或单个资产)期权定价模型大多是围绕数值解法进行的[9]-[21]。美礼尚[1]建立了关于期权价格函数 $V = V(s, t) = V(s_1, \ldots, s_m, t)$ 的多维 Black-Scholes 方程

$$
\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{k,j=1}^{m} a_{kj} s_k s_j \frac{\partial^2 V}{\partial s_k \partial s_j} + \sum_{k=1}^{m} (r - q_k) s_k \frac{\partial V}{\partial s_k} - rV = 0, (s_1, \ldots, s_m) \in \mathbb{R}_+^m, 0 < t < T \tag{01}
$$

其中矩阵 $A = (a_{kj})_{m \times m}$ 为实对称非负矩阵。研究关于方程(01)的多资产期权的数学模型。由于

$$
\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{k,j=1}^{m} a_{kj} s_k s_j \frac{\partial^2 V}{\partial s_k \partial s_j} + \sum_{k=1}^{m} (r - q_k) s_k \frac{\partial V}{\partial s_k} - rV = 0 \tag{02}
$$

故方程(01)可改记为

$$
\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{k,j=1}^{m} a_{kj} s_k s_j \frac{\partial^2 V}{\partial s_k \partial s_j} + \sum_{k=1}^{m} (r - q_k) s_k \frac{\partial V}{\partial s_k} - rV = 0 \tag{03}
$$

本文研究多资产期权确定最佳实施边界的问题，建立了多维 Black-Scholes 方程在多维区域 $\Omega = \{(s, t) \mid s \in \mathbb{R}_+^m, t \in (0, T]\}$ 具有奇异内边界函数向量 $s = s(t) = (s_1(t), \ldots, s_m(t)), 0 < t < T$ 的数学模型，期权价格函数 $u(s, t)$ 为未知函数。应用矩阵理论和广义特征函数法获得了数学模型的精确解 $u(s, t)$。并获得了奇异内边界的指数函数向量表达式 $u(s, t) = (s_1(t), \ldots, s_m(t))$. 证明了：当任意 $t \in (0, T)$，数学模型的解 $u(s, t)$ 在奇异内边界取于 $\mathbb{R}_+^m$ 区域 $s_j < \infty, j = 1, \ldots, m$ 中的最大值，即

$$
u(s, t, t) = \max_{s_j \in \mathbb{R}_+^m} u(s, t, t) \quad \text{同时获得了 Black-Scholes 方程的自由边界问题 A 和自由边界问题 B 的精确解和其自由边界的指数函数向量表达式} (s_1(t), \ldots, s_m(t)) = \left( g_1 e^{\alpha(t)}, \ldots, g_m e^{\alpha(t)} \right), \quad \text{问题 A 和问题 B 的自由边界与奇异内边界重合。从而指数函数向量表达式} s(t) = (s_1(t), \ldots, s_m(t)) = \left( g_1 e^{\alpha(t)}, \ldots, g_m e^{\alpha(t)} \right) \quad \text{为最佳实施边界，指数函数向量} \quad \text{为最佳实施边界。指数函数向量} s(t) = \left( g_1 e^{\alpha(t)}, \ldots, g_m e^{\alpha(t)} \right), \quad \text{满足条件} \omega_k = -\frac{1}{s_j(t)} \frac{ds_j(t)}{dt}, \quad \text{且有} \omega_k \quad \text{的计算公式} \omega_k = \sum_{j=1}^{m} b_{kj} \left[ \frac{1}{2} \sum_{n=1}^{m} \left(\frac{1}{2} a_{nn} + q_n - r \right) c_{nj} \right]; \quad \text{公式表明} \omega_k, k = 1, \ldots, m \quad \text{由多维 Black-Scholes 方程中出现的所有参数} a_{kj}, \quad q_j, \quad r \quad \text{唯一确定。}

2. 主要结果

2.1. 多资产期权的数学模型 I 的研究

引入记号

$$
\Omega = \{(s, t) \mid s \in \mathbb{R}_+^m, t \in (0, T]\}, \quad \Omega_+ = \{(s, t) \mid s \in E_+(t), t \in (0, T]\}, \quad \Omega_- = \{(s, t) \mid s \in E_-(t), t \in (0, T]\}
$$

$E_+(t) \colon \nu < s_j < s_j(t), j = 1, \ldots, m; E_-(t) \colon s_j(t) < s_j < \infty, j = 1, \ldots, m; E_0(t) \colon 0 \leq s_j \leq s_j(t), j = 1, \ldots, m; E_1(t) \colon s_j(t) \leq s_j < \infty, j = 1, \ldots, m.$
数学模型 I (多维 Black-Scholes 方程具有奇异内边界的终值问题):

\[
\frac{\partial u}{\partial t} + \frac{1}{2} \sum_{k,j=1}^{m} a_{kj} \left( s_k \frac{\partial}{\partial s_k} \right) \left( s_j \frac{\partial}{\partial s_j} \right) u + \sum_{k=1}^{m} \left( r - q_k - \frac{1}{2} a_{kk} \right) s_k \frac{\partial u}{\partial s_k} - ru = -f(s,t), s = (s_1, \ldots, s_n) \in \mathbb{R}_+^n, 0 < t < T
\]

\[u(s,T) = \phi(s)\]

\[
\lim_{s \to 0^+} |u| < \infty, \lim_{s \to +\infty} |u| < \infty
\]

数学模型 I 是关于多资产期权的数学模型，它是多维 Black-Scholes 方程在区域 \(\Omega\) 具有奇异内边界 \(s(t) \equiv (s_1(t), \ldots, s_k(t))\), \(0 < t < T\) 的终值问题，未知函数 \(u(s,t)\) 为期权价格函数。

其中：方程的自由项为

\[f(s,t) = \prod_{k=1}^{n} \gamma_k(t) \gamma_k(t) \delta(s_k - s_k(t))\]

\(\delta(s_k - s_k(t))\) 为狄拉克 \(\delta\)-函数；\(\delta(s - s(t))\) 为 \(m\) 维狄拉克 \(\delta\)-函数；\(s(t) \equiv (s_1(t), \ldots, s_k(t))\), \(A = (a_{ij})_{m \times m}\) 为实对称非负矩阵。

数学模型 I.1 (多维 Black-Scholes 方程的终值问题):

\[
\frac{\partial u}{\partial t} + \frac{1}{2} \sum_{k,j=1}^{m} a_{kj} \left( s_k \frac{\partial}{\partial s_k} \right) \left( s_j \frac{\partial}{\partial s_j} \right) u + \sum_{k=1}^{m} \left( r - q_k - \frac{1}{2} a_{kk} \right) s_k \frac{\partial u}{\partial s_k} - ru = 0, s = (s_1, \ldots, s_n) \in \mathbb{R}_+^n, 0 < t < T
\]

\[u(s,T) = \phi(s)\]

\[
\lim_{s \to 0^+} |u| < \infty, \lim_{s \to +\infty} |u| < \infty
\]

数学模型 I.2 (多维 Black-Scholes 方程具有奇异内边界和齐次终值条件的终值问题):

\[
\frac{\partial u}{\partial t} + \frac{1}{2} \sum_{k,j=1}^{m} a_{kj} \left( s_k \frac{\partial}{\partial s_k} \right) \left( s_j \frac{\partial}{\partial s_j} \right) u + \sum_{k=1}^{m} \left( r - q_k - \frac{1}{2} a_{kk} \right) s_k \frac{\partial u}{\partial s_k} - ru = -f(s,t), s = (s_1, \ldots, s_n) \in \mathbb{R}_+^n, 0 < t < T
\]

\[u(s,T) = 0\]

\[
\lim_{s \to 0^+} |u| < \infty, \lim_{s \to +\infty} |u| < \infty
\]

2.1.1. Black-Scholes 方程数学模型 I 的求解

记偏微分算子

\[L = \frac{1}{2} \sum_{k,j=1}^{m} a_{kj} \left( s_k \frac{\partial}{\partial s_k} \right) \left( s_j \frac{\partial}{\partial s_j} \right) + \sum_{k=1}^{m} \left( r - q_k - \frac{1}{2} a_{kk} \right) s_k \frac{\partial}{\partial s_k} - r\]

先考虑 \(m\) 维 Euler 方程在半无界区域 \(\mathbb{R}_+^m\) 的特征值问题 I

\[
LE(s) = \frac{1}{2} \sum_{k,j=1}^{m} a_{kj} \left( s_k \frac{\partial}{\partial s_k} \right) \left( s_j \frac{\partial}{\partial s_j} \right) + \sum_{k=1}^{m} \left( r - q_k - \frac{1}{2} a_{kk} \right) s_k \frac{\partial}{\partial s_k} - sE(s) - \lambda E(s), s = (s_1, \ldots, s_n) \in \mathbb{R}_+^m
\]

\[
\lim_{s \to 0^+} |\chi E(s)| < \infty, \lim_{s \to +\infty} |\chi E(s)| < \infty
\]

为求解特征值问题 I 我们建立了引理 1.1~引理 1.6。
引理 1.1: 设 \( A = (a_{ij}) \in R^{m \times n} \) 为正定矩阵，则存在正线下三角矩阵 \( B = (b_{ij}) \in R^{m \times n} \) 满足 \( A = BB^T \) 且分解是唯一的：且有

1. 正线下三角矩阵 \( B \) 的行列式 \( |B| = \prod_{j=1}^{n} b_{jj} > 0 \)，且 \( A = BB^T \)；
2. 由 \( (B^T)^{-1} \) 唯一确定 \( (B^T)^{-1} \)；
3. 记 \( (B^T)^{-1} = C = (c_{ij}) \)，则 \( C \) 为正线上三角矩阵，
4. 记 \( I(k; X) \approx \sum_{j=1}^{n} c_{kj} x_j \)，\( x = (x_1, \ldots, x_n) \in R^n \)；

从而有 \( I(k; X) > 0, k = 1, \ldots, m \)。

证明：由矩阵理论 (22) 即知存在正线下三角矩阵 \( B = (b_{ij}) \in R^{m \times n} \) 满足 \( A = BB^T \) 且分解是唯一的。由 \( A = BB^T \) 有 \( |A| = |B|^T \)。

下证 4) 由于 \( C \) 为正线上三角矩阵，即有 \( \sum_{n=1}^{m} c_{kj} \sum_{j=1}^{n} c_{jk} x_j = \sum_{n=1}^{m} c_{kj} \sum_{j=1}^{n} c_{jk} x_j \)。从而有

\[
I(k; X) \approx \sum_{n=1}^{m} c_{kj} \sum_{j=1}^{n} c_{jk} x_j = \sum_{n=1}^{m} c_{kj} \sum_{j=1}^{n} c_{jk} x_j = \sum_{n=1}^{m} c_{kj} \sum_{j=1}^{n} c_{jk} x_j \]

从而有

\[
I(k; X) = [c_{k1}, c_{k2}, \ldots, c_{km}] C^T X, k = 1, \ldots, m
\]
为正定矩阵，则 $A^T A^{-1}$ 为正定矩阵。由 (12) 式，有

$$\sum_{j=1}^{m} X^T A^{-1} X_j > 0, X \neq 0$$

由 $X$ 的任意性，分别令 $X = y_p e_p, y_p \neq 0, p = 1, \cdots, m$，其中 $e_1 = (1, 0, 0, \cdots, 0)^T, e_2 = (0, 1, 0, \cdots, 0)^T, \cdots, e_m = (0, 0, \cdots, 0, 1)^T$。

由 (13) 式即有

$$y_p \sum_{n=1}^{m} c_{kn} c_{pn} y_p > 0, p = 1, \cdots, m$$

从而

$$\sum_{p=1}^{m} \sum_{n=1}^{m} c_{kn} c_{pn} y_p^2 > 0, y_p \neq 0, p = 1, \cdots, m$$

再记 $y_p^2 \equiv x_p, p = 1, \cdots, m$：有 $x_p > 0, p = 1, \cdots, m$ 和

$$\sum_{p=1}^{m} \sum_{n=1}^{m} c_{kn} c_{pn} x_p^2 = \sum_{p=1}^{m} \sum_{n=1}^{m} c_{kn} c_{pn} x_p = I(k; X), k = 1, \cdots, m$$

由 (14)，(15) 两式即有：当 $X = (x_1, \cdots, x_m)^T, x_j > 0, j = 1, \cdots, m$，有 $I(k; X) > 0, k = 1, \cdots, m$；显然也有：当 $X = (x_1, \cdots, x_m)^T, x_j < 0, j = 1, \cdots, m$，有 $I(k; X) < 0, k = 1, \cdots, m$。引理证毕。

记

$$\begin{bmatrix} \ln s_1 \\ \vdots \\ \ln s_m \end{bmatrix} \equiv \ln s = \alpha \in \mathbb{R}^m, \text{ 作 } \mathbb{R}^m \text{ 到 } \mathbb{R}^m \text{ 的线性变换}$$

$$\Phi(\alpha) = B^{-1} \alpha \equiv \ln \eta = \begin{bmatrix} \ln \eta_1 \\ \vdots \\ \ln \eta_m \end{bmatrix} \in \mathbb{R}^m$$

记 $E(s) = E^r(\ln s) = E^r(B \ln \eta) \equiv Y(\eta)$

引理 1.2：若 $\ln s = B \ln \eta$，则有

$$\nabla_\eta = B^T \nabla_s$$

其中 $\nabla_\eta \equiv \begin{bmatrix} \eta_1 \frac{\partial}{\partial \eta_1} \\ \vdots \\ \eta_m \frac{\partial}{\partial \eta_m} \end{bmatrix}, \nabla_s \equiv \begin{bmatrix} s_1 \frac{\partial}{\partial s_1} \\ \vdots \\ s_m \frac{\partial}{\partial s_m} \end{bmatrix}$ 为向量变系数偏微分算子。

证明：(16) 式即


\[
\begin{bmatrix}
\ln s_1 \\
\vdots \\
\ln s_m
\end{bmatrix} = \begin{bmatrix}
\sum_{p=1}^{m} b_{pp} \ln \eta_p \\
\vdots \\
\sum_{p=1}^{m} b_{mp} \ln \eta_p
\end{bmatrix}
\]

由(17)式即有

\[
E(s) = E^*(\ln s_1, \cdots, \ln s_m) = E^*\left(\sum_{p=1}^{m} b_{pp} \ln \eta_p, \cdots, \sum_{p=1}^{m} b_{mp} \ln \eta_p\right) = Y(\eta)
\]

由复合函数的求导法则

\[
\eta_k \frac{\partial Y(\eta)}{\partial \eta_k} = \frac{\partial Y(\eta)}{\partial \ln \eta_k} = b_{kk} \frac{\partial E(s)}{\partial \ln s_k} + \cdots + b_{mk} \frac{\partial E(s)}{\partial \ln s_m} = \begin{bmatrix} s_1 \frac{\partial}{\partial \eta_1} \\ \vdots \\ s_m \frac{\partial}{\partial \eta_m} \end{bmatrix} E(s)
\]

从而

\[
\begin{bmatrix}
\eta_k \frac{\partial}{\partial \eta_k} \\
\vdots \\
\eta_m \frac{\partial}{\partial \eta_m}
\end{bmatrix} Y(\eta) = \begin{bmatrix}
b_{1k}, \cdots, b_{mk} \\
\vdots \\
b_{1m}, \cdots, b_{mm}
\end{bmatrix} \begin{bmatrix}
s_1 \frac{\partial}{\partial \eta_1} \\
\vdots \\
s_m \frac{\partial}{\partial \eta_m}
\end{bmatrix} E(s) = B^T E(s)
\]

即(18)式成立。引理证毕。

**引理1.3**：若 \( \ln s = B \ln \eta \)，则特征值问题 I 中方程(8)与特征值问题 II 中方程(22)等价。

**证明**：由 \( A = BB^T \)，有

\[
\sum_{k=1}^{m} a_k \begin{bmatrix}
s_k \frac{\partial}{\partial \eta_k} \\
\vdots \\
s_m \frac{\partial}{\partial \eta_m}
\end{bmatrix} E(s) = \begin{bmatrix}
s_1 \frac{\partial}{\partial \eta_1} \\
\vdots \\
s_m \frac{\partial}{\partial \eta_m}
\end{bmatrix} A E(s) = (B^T \nabla_s)^T (B^T \nabla_s) E(s)
\]
\[
\sum_{i,j=1}^{m} a_{ij} \left( s_i \frac{\partial}{\partial s_i} \right) \left( s_j \frac{\partial}{\partial s_j} \right) E(s) = (\nabla_q)^T \left( \nabla_q \right) Y(\eta) = \sum_{i=1}^{\infty} \eta_i \frac{\partial^2 Y(\eta)}{\partial \eta_i^2}
\]

(26)

记 \( C \equiv (B^T)^{-1} \)，由引理 1.1 枢阵 \( C \) 为正线上三角矩阵。

由(18)式有

\[
C \begin{bmatrix} \eta_1 \frac{\partial}{\partial \eta_1} \\ \vdots \\ \eta_m \frac{\partial}{\partial \eta_m} \end{bmatrix} = \begin{bmatrix} s_1 \frac{\partial}{\partial s_1} \\ \vdots \\ s_m \frac{\partial}{\partial s_m} \end{bmatrix}
\]

(27)

由矩阵乘法

\[
\begin{align*}
\sum_{p=1}^{m} c_{ip} \eta_p \frac{\partial}{\partial \eta_p} &= \begin{bmatrix} s_1 \frac{\partial}{\partial s_1} \\ \vdots \\ s_m \frac{\partial}{\partial s_m} \end{bmatrix} \\
\sum_{p=1}^{m} c_{pm} \eta_p \frac{\partial}{\partial \eta_p}
\end{align*}
\]

(28)

从而

\[
s_k \frac{\partial}{\partial s_k} = \sum_{p=1}^{m} c_{kp} \eta_p \frac{\partial}{\partial \eta_p}
\]

(29)

\[
\sum_{k=1}^{m} \left( r - q_k - \frac{1}{2} a_{ik} \right) s_k \frac{\partial E(s)}{\partial s_k} = \sum_{k=1}^{m} \left( r - q_k - \frac{1}{2} a_{ik} \right) \sum_{p=1}^{m} c_{kp} \eta_p \frac{\partial}{\partial \eta_p} Y(\eta)
\]

(30)

由于 \( C \) 为正线上三角矩阵有

\[
\sum_{i=1}^{m} c_{ip} \left( r - q_k - \frac{1}{2} a_{ik} \right) = \sum_{k=1}^{m} c_{ip} \left( r - q_k - \frac{1}{2} a_{ik} \right)
\]

记

\[
d_p = \sum_{k=1}^{m} \left( \frac{1}{2} a_{ik} + q_k - r \right) c_{kp}, \quad p = 1, \ldots, m
\]

(31)

即有

\[
\sum_{k=1}^{m} \left( r - q_k - \frac{1}{2} a_{ik} \right) s_k \frac{\partial E(s)}{\partial s_k} = -\sum_{p=1}^{m} d_p \eta_p \frac{\partial Y(\eta)}{\partial \eta_p}
\]

(32)

由(26)，(32)两式即知方程(8)与方程(22)等价。引理证毕。

引理 1.4：特征值问题 II 的特征值

\[
\lambda = \lambda_p = \sum_{k=1}^{m} \eta_k \frac{\partial}{\partial \eta_k} + \left( \frac{d_k + 1}{2} + \frac{2r}{m} \right), \quad \beta = (\beta_1, \ldots, \beta_m) \in R^n
\]

(33)
所对应的特征函数为

\[ Y(\eta) = Y_p(\eta) = \prod_{k=1}^{m} \eta_k^{1/2} \eta_k^{\alpha_k} \]  \hspace{1cm} (34)

且有

\[ X(\eta) = \prod_{k=1}^{m} \eta_k^{1/2}, \hspace{0.5cm} X(\eta) = \prod_{k=1}^{m} \eta_k^{\beta_k} \]  \hspace{1cm} (35)

证明：容易求解特征值问题 II：由分离变量法令

\[ Y(\eta) = \prod_{p=1}^{m} Y_p(\eta_p) \]  \hspace{1cm} (36)

再令

\[ Y_k(\eta_k) = \eta_k^{\alpha_k}, k \in \{1, \ldots, m\} \]

\[ Y_k'(\eta_k) = \alpha_k \eta_k^{\alpha_k-1}, Y_k''(\eta_k) = \alpha_k (\alpha_k - 1) \eta_k^{\alpha_k-2} \]

\[ \eta_k \frac{\partial Y_k(\eta_k)}{\partial \eta_k} = \alpha_k \eta_k^{\alpha_k} = \alpha_k Y_k(\eta_k) \]  \hspace{1cm} (39)

\[ \eta_k^2 \frac{\partial^2 Y_k(\eta_k)}{\partial \eta_k^2} = \alpha_k (\alpha_k - 1) \eta_k^{\alpha_k} = \alpha_k (\alpha_k - 1) Y_k(\eta_k) \]  \hspace{1cm} (40)

若(42)式成立则(41)式成立：特征函数

\[ \prod_{p=1}^{m} Y_p(\eta_p) \]

不恒为零，由(42)推出

\[ \frac{1}{2} \alpha_k^2 - \left( \frac{1}{2} + d_k \right) \alpha_k + \left( \lambda_k - \frac{r}{m} \right) = 0, \forall k \in \{1, \ldots, m\} \]  \hspace{1cm} (43)

\[ \alpha_k = \frac{1}{2} \left[ \frac{1}{2} + d_k \right] \pm \sqrt{\left( \frac{1}{2} + d_k \right)^2 - 2 \left( \lambda_k - \frac{r}{m} \right)} \]

\[ = \frac{1}{2} \left[ \frac{1}{2} + d_k \right] \pm i \sqrt{2 \left( \lambda_k - \frac{r}{m} \right) - \left( \frac{1}{2} + d_k \right)^2} \]

\[ \alpha_k = \frac{1}{2} \left[ \frac{1}{2} + d_k \right] \pm i |\beta_k| \]
\[
|\beta_k| = \sqrt{2\left(\lambda_k - \frac{r}{m} - \left(\frac{1}{2} + d_k\right)^2\right)}
\]

\[
\lambda_k = \lambda_k^0 = \frac{\beta_k^2 + \left(d_k + \frac{1}{2}\right)^2 + \frac{2r}{m}}{2}, \beta_k \in R, \forall k \in \{1, \ldots, m\}
\]

\[
\lambda = \lambda_0 = \sum_{k=1}^{m} \lambda_k = \sum_{k=1}^{m} \frac{\beta_k^2 + \left(d_k + \frac{1}{2}\right)^2 + \frac{2r}{m}}{2}, \beta \in R
\]

\[
Y_i(\eta_k) = \eta_k^{\frac{1+2d_k}{2}} \eta_i^{\frac{1}{m}} \beta_k, \beta_k \in R, \forall k \in \{1, \ldots, m\}
\]

\[
Y(\eta) = \prod_{k=1}^{m} \eta_k^{\frac{1+2d_k}{2}} \eta_i^{\frac{1}{m}} \beta_k
\]

\[
X_\eta = \prod_{k=1}^{m} \eta_k^{\frac{1+2d_k}{2}}, X_\eta Y(\eta) = \prod_{k=1}^{m} \eta_i^{\frac{1}{m}} \beta_k
\]

由(48)式即有(23)式成立。引理证毕。

由(16)和(17)式换回原变量即得特征值问题 I 的特征函数

\[
E_\rho(s) = \prod_{k=1}^{m} e^{\frac{1+2d_k}{2} \sum_{j=1}^{k} c_{ij} \ln s_j} i p_k \sum_{j=1}^{k} c_{ij} \ln s_j
\]

\[
\lambda = \lambda_0 = \sum_{k=1}^{m} \lambda_k = \sum_{k=1}^{m} \frac{\beta_k^2 + \left(d_k + \frac{1}{2}\right)^2 + \frac{2r}{m}}{2}, \beta \in R^m
\]

且

\[
X_\rho = \prod_{k=1}^{m} e^{\frac{1+2d_k}{2} \sum_{j=1}^{k} c_{ij} \ln s_j}, X_\rho E_\rho(s) = \prod_{k=1}^{m} e^{i p_k \sum_{j=1}^{k} c_{ij} \ln s_j}
\]

\[
\lim_{s \to 0} |X_\rho E_\rho(s)| = \lim_{s \to \infty} \left|e^{\frac{1+2d_k}{2} \sum_{j=1}^{k} c_{ij} \ln s_j} \right| < \infty, \lim_{s \to \infty} \left|X_\rho E_\rho(s)| = \lim_{s \to \infty} \left|e^{\frac{1+2d_k}{2} \sum_{j=1}^{k} c_{ij} \ln s_j}\right| < \infty
\]

由(51)式即有(9)式成立。于是得到

引理 1.5：特征值问题 I 的特征值

\[
\lambda = \lambda_0 = \sum_{k=1}^{m} \lambda_k = \sum_{k=1}^{m} \frac{\beta_k^2 + \left(d_k + \frac{1}{2}\right)^2 + \frac{2r}{m}}{2}, \beta \in R^m
\]

所对应的特征函数为

\[
E_\rho(s) = \prod_{k=1}^{m} e^{\frac{1+2d_k}{2} \sum_{j=1}^{k} c_{ij} \ln s_j} i p_k \sum_{j=1}^{k} c_{ij} \ln s_j}, \beta \in R^n
\]

引理 1.6：特征值问题 I 的特征函数系 \( E_\beta(s) = \prod_{k=1}^{m} e^{\frac{1+2d_k}{2} \sum_{j=1}^{k} c_{ij} \ln s_j} i p_k \sum_{j=1}^{k} c_{ij} \ln s_j}, \beta \in R^n \) 是半无界区域 \( R^n \) 带
权函数 $\rho(s) = e^{-\frac{\sum \ln s_j}{k=1} - \frac{(1+2d_k)\sum v_{k\beta j}}{j=1}}$ 的完备正交系：正交关系即

$$\int_{R^n} E_{\beta'}(s) E_{\beta}(s) \rho(s) ds = (2\pi)^n |B| \delta(\beta' - \beta), \beta', \beta \in R^n \quad (54)$$

证明：由于

$$\int_{R^n} E_{\beta'}(s) E_{\beta}(s) \rho(s) ds$$

$$= \int_{R^n} \left( \prod_{k=1}^m e^{\frac{1+2d_k}{2} \sum v_{k\beta j}} \right) \left( \prod_{k=1}^m e^{\frac{(1+2d_k)\sum v_{k\beta j}}{j=1} - \frac{v_{k\beta j}}{j=1} - \frac{(1+2d_k)\sum v_{k\beta j}}{j=1}} \right) ds$$

$$= \int_{R^n} \left( \prod_{k=1}^m e^{i(\beta_k - \beta_k') \sum v_{k\beta j}} \right) e^{-\sum \ln s_j} ds \quad (55)$$

引入变量代换

$$\sum_{j=1}^k c_{k\beta} \ln s_j = y_k, k = 1, \cdots, m \quad (56)$$

由于行列式

$$\left| \frac{\partial (y_1, \cdots, y_m)}{\partial (s_1, \cdots, s_m)} \right| = \prod_{j=1}^m \left( e^{\frac{1}{s_j}} \right) = |B|^{-1} \prod_{j=1}^m \frac{1}{s_j}$$

即有变量替换的雅可比行列式

$$|J| = \left| \frac{\partial (s_1, \cdots, s_m)}{\partial (y_1, \cdots, y_m)} \right| = \left| \frac{\partial (y_1, \cdots, y_m)}{\partial (s_1, \cdots, s_m)} \right|^{-1} = |B| \left( \prod_{j=1}^m \frac{1}{s_j} \right)^{-1} \neq 0, s \in R^n \quad (57)$$

由多重积分变量替换公式，即有

$$\int_{R^n} E_{\beta'}(s) E_{\beta}(s) \rho(s) ds = \int_{R^n} \left( \prod_{k=1}^m e^{i(\beta_k - \beta_k') \sum v_{k\beta j}} \right) e^{-\sum \ln s_j} ds$$

$$= \int_{R^n} \left( \prod_{k=1}^m e^{i(\beta_k - \beta_k') \sum v_{k\beta j}} \right) \left| \frac{\partial (s_1, \cdots, s_m)}{\partial (y_1, \cdots, y_m)} \right| e^{-\sum \ln s_j} dy$$

$$= \int_{R^n} \left( \prod_{k=1}^m e^{i(\beta_k - \beta_k') \sum v_{k\beta j}} \right) |B| \left( \prod_{j=1}^m \frac{1}{s_j} \right) dy$$

$$= |B| \int_{R^n} \left( \prod_{k=1}^m e^{i(\beta_k - \beta_k') \sum v_{k\beta j}} \right) dy$$

$$= |B| \prod_{k=1}^m \left( \int_{-\infty}^{+\infty} e^{i(\beta_k - \beta_k') \sum v_{k\beta j}} dy_k \right)$$

$$= |B|(2\pi)^n \prod_{k=1}^m \delta(\beta_k' - \beta_k)$$

$$= |B|(2\pi)^n \delta(\beta' - \beta), \beta', \beta \in R^n, \beta \in R^n$$
\[
\int_{\mathbb{R}^n} E_{\rho}(s) \overline{E_{\rho}(s)} \rho(s) \, ds = (2\pi)^n |B| \delta(\beta' - \beta), \beta', \beta \in \mathbb{R}^n \quad (58)
\]

(58) 式即 (54) 式。引理证毕。

由引理 1.5 与引理 1.6 的结论可以引入广义特征函数法 [23] [24] 求解 数学模型 1。

不妨设解 \( u \in C(\mathbb{R}^n \times [0, T]) \)，将其表为特征函数的积分形式

\[
u(s, t) = \int_{\mathbb{R}^n} U_{\rho}(t) E_{\rho}(s) \, d\beta
\]

将上式两边乘以 \( E_{\rho}(s) \rho(s) \) 再关于变量 \( s \) 在 \( \mathbb{R}^n \) 积分，利用正交关系 (54) 则有

\[
\int_{\mathbb{R}^n} u(s, t) E_{\rho}(s) \rho(s) \, ds = \int_{\mathbb{R}^n} U_{\rho}(t) \int_{\mathbb{R}^n} E_{\rho}(s) E_{\rho}(s) \rho(s) \, ds \, d\beta
\]

\[
= \int_{\mathbb{R}^n} U_{\rho}(t) |B| \prod_{\kappa=1}^m 2\pi \delta(\beta' - \beta) \, d\beta
\]

\[
= (2\pi)^n |B| U_{\rho}(t)
\]

得到

\[
U_{\rho}(t) = \frac{1}{(2\pi)^n |B|} \int_{\mathbb{R}^n} u(s, t) E_{\rho}(s) \rho(s) \, ds, \beta \in \mathbb{R}^n
\]

将方程中的自由项 \( f(s, t) \) 也表为特征函数的积分形式

\[
f(s, t) = \prod_{\kappa=1}^m \gamma_\kappa(t) s_\kappa^2(t) \delta(s_\kappa - s_\kappa(t)) = \int_{\mathbb{R}^n} f_{\rho}(t) E_{\rho}(s) \, d\beta
\]

由 (61) 即有

\[
f_{\rho}(t) = \frac{1}{(2\pi)^n |B|} \int_{\mathbb{R}^n} \prod_{\kappa=1}^m \gamma_\kappa(t) s_\kappa^2(t) \delta(s_\kappa - s_\kappa(t)) \overline{E_{\rho}(s)} \rho(s) \, ds
\]

应用 \( \delta \) - 函数的积分性质即得

\[
f_{\rho}(t) = \frac{1}{(2\pi)^n |B|} \overline{E_{\rho}(s(t))} \rho(s(t)) \prod_{\kappa=1}^m \gamma_\kappa(t) s_\kappa^2(t)
\]

含参变量积分与算子 \( L \) 的运算交换次序即有

\[
Lu(s, t) = \int_{\mathbb{R}^n} U_{\rho}(t) LE_{\rho}(s) \, d\beta = -\int_{\mathbb{R}^n} U_{\rho}(t) \lambda_{\rho} E_{\rho}(s) \, d\beta
\]

\[
\frac{\partial u}{\partial t}(s, t) = \int_{\mathbb{R}^n} U_{\rho}(t) E_{\rho}(s) \, d\beta
\]

由 (2) 即有

\[
\varphi(s) = u(s, T) = \int_{\mathbb{R}^n} U_{\rho}(T) E_{\rho}(s) \, d\beta
\]

\[
\varphi_{\rho} = U_{\rho}(T) = \frac{1}{(2\pi)^n |B|} \int_{\mathbb{R}^n} \varphi(s) \overline{E_{\rho}(s)} \rho(s) \, ds
\]

将 (62)，(65)，(66) 代入方程 (1) 即有

\[
\int_{\mathbb{R}^n} \left[ U'_{\rho}(t) - \lambda_{\rho} U_{\rho}(t) + f_{\rho}(t) \right] E_{\rho}(s) \, d\beta = 0
\]

由特征函数系的完备正交性即有

\[
U'_{\rho}(t) - \lambda_{\rho} U_{\rho}(t) + f_{\rho}(t) = 0
\]

再由 (68) 式即得
非齐次常微分方程的终值问题

\[
\begin{align*}
U'_\beta(t) - \lambda_\beta U_\beta(t) + f_\beta(t) &= 0, \quad 0 < t < T \\
U_\beta(T) &= \varphi_\beta
\end{align*}
\]

(70)

用常数变易法得到非齐次常微分方程的终值问题的解为

\[
U_\beta(t) = \varphi_\beta e^{-\lambda_\beta (t-T)} + \int_{t}^{T} f_\beta(\xi) e^{-\lambda_\beta (\xi-T)} d\xi
\]

(71)

将上式代入(59)式即得

\[
u(s, t) = \int_{\mathbb{R}^n} U_\beta(t) E_\beta(s) d\beta = \int_{\mathbb{R}^n} \varphi_\beta e^{-\lambda_\beta (t-T)} E_\beta(s) d\beta + \int_{\mathbb{R}^n} \left[ \int_{t}^{T} f_\beta(\xi) e^{-\lambda_\beta (\xi-T)} d\xi \right] E_\beta(s) d\beta
\]

(72)

将\(\lambda_\beta, E_\beta(s)\)的表达式(52), (53)代入(72), 并记

\[
u(s, t) = V(s, t) + W(s, t)
\]

(73)

\[
V(s, t) = \int_{\mathbb{R}^n} \varphi_\beta e^{-\lambda_\beta (t-T)} \prod_{k=1}^{m} \frac{1}{2} \left( 1 + \frac{i k_1 \sum_{j=1}^{n} c_{j,k} h_{n_j}}{1 + \frac{i k_1 \sum_{j=1}^{n} c_{j,k} h_{n_j}}} \right) e^{-\frac{1}{2} \sum_{k=1}^{m} \left( \frac{k_1 \sum_{j=1}^{n} c_{j,k} h_{n_j}}{1 + \frac{i k_1 \sum_{j=1}^{n} c_{j,k} h_{n_j}}} \right)^2} d\beta
\]

(74)

\[
W(s, t) = \int_{\mathbb{R}^n} \left[ \int_{t}^{T} f_\beta(\xi) e^{-\lambda_\beta (\xi-T)} \prod_{k=1}^{m} \frac{1}{2} \left( 1 + \frac{i k_1 \sum_{j=1}^{n} c_{j,k} h_{n_j}}{1 + \frac{i k_1 \sum_{j=1}^{n} c_{j,k} h_{n_j}}} \right) e^{-\frac{1}{2} \sum_{k=1}^{m} \left( \frac{k_1 \sum_{j=1}^{n} c_{j,k} h_{n_j}}{1 + \frac{i k_1 \sum_{j=1}^{n} c_{j,k} h_{n_j}}} \right)^2} d\xi \right] d\beta
\]

(75)

其中\(\varphi_\beta\)由(68), \(f_\beta(t)\)由(64)确定。

由(74)式即有

\[
V(s, t) = \int_{\mathbb{R}^n} \varphi_\beta e^{-\lambda_\beta t} \prod_{k=1}^{m} \frac{1}{2} \left( 1 + \frac{i k_1 \sum_{j=1}^{n} c_{j,k} h_{n_j}}{1 + \frac{i k_1 \sum_{j=1}^{n} c_{j,k} h_{n_j}}} \right) e^{-\frac{1}{2} \sum_{k=1}^{m} \left( \frac{k_1 \sum_{j=1}^{n} c_{j,k} h_{n_j}}{1 + \frac{i k_1 \sum_{j=1}^{n} c_{j,k} h_{n_j}}} \right)^2} d\beta
\]

(76)

\[
W(s, t) = \int_{\mathbb{R}^n} \left[ \int_{t}^{T} f_\beta(\xi) e^{-\lambda_\beta (\xi-T)} \prod_{k=1}^{m} \frac{1}{2} \left( 1 + \frac{i k_1 \sum_{j=1}^{n} c_{j,k} h_{n_j}}{1 + \frac{i k_1 \sum_{j=1}^{n} c_{j,k} h_{n_j}}} \right) e^{-\frac{1}{2} \sum_{k=1}^{m} \left( \frac{k_1 \sum_{j=1}^{n} c_{j,k} h_{n_j}}{1 + \frac{i k_1 \sum_{j=1}^{n} c_{j,k} h_{n_j}}} \right)^2} d\xi \right] d\beta
\]

(77)

其中\(\varphi_\beta\)由(68), \(f_\beta(t)\)由(64)确定。
于是有

\[
V (s,t) = \frac{e^{-(T-t)}}{(2\pi)^n |B| (T-t)^2} \int_{\mathbb{R}^n} \varphi(\xi) \frac{1}{\prod_{k=1}^n \xi_k} e^{-\frac{1}{2T} \left( \frac{\xi_k^2}{\xi_k} \right)^2} \prod_{k=1}^n \xi_k d\xi
\]

(76)

将 \( \rho(s), E(s) \) 的表达式代入(64)式，化简即得

\[
f_\beta(t) = \frac{1}{(2\pi)^n |B|} \prod_{k=1}^m \left[ e^{\frac{1}{2} \int_{\mathbb{R}^n} \varphi(\xi) \prod_{k=1}^n \gamma_k(t) e^{\frac{\xi_k^2}{\xi_k}} d\xi} \right] \prod_{k=1}^n \left[ \prod_{k=1}^m \gamma_k(t) e^{\frac{\xi_k^2}{\xi_k}} \right] d\beta
\]

(77)

将(77)代入(75)式，并化简

\[
W(s,t) = \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} \frac{1}{(2\pi)^n |B|} \left[ \prod_{k=1}^m \left[ e^{\frac{1}{2} \int_{\mathbb{R}^n} \varphi(\xi) \prod_{k=1}^n \gamma_k(t) e^{\frac{\xi_k^2}{\xi_k}} d\xi} \right] \prod_{k=1}^n \left[ \prod_{k=1}^m \gamma_k(t) e^{\frac{\xi_k^2}{\xi_k}} \right] d\beta \right] \right] d\xi
\]

\[
= \frac{1}{(2\pi)^n |B|} \prod_{k=1}^m \left[ \prod_{k=1}^m \gamma_k(t) e^{\frac{\xi_k^2}{\xi_k}} \right] \frac{1}{2} \int_{\mathbb{R}^n} \frac{1}{\xi_k^2} \prod_{k=1}^m \gamma_k(t) e^{\frac{\xi_k^2}{\xi_k}} d\xi
\]

\[
= \frac{1}{(2\pi)^n |B|} \prod_{k=1}^m \left[ \prod_{k=1}^m \gamma_k(t) e^{\frac{\xi_k^2}{\xi_k}} \right] \frac{1}{2} \int_{\mathbb{R}^n} \frac{1}{\xi_k^2} \prod_{k=1}^m \gamma_k(t) e^{\frac{\xi_k^2}{\xi_k}} d\xi
\]

\[
= \frac{1}{(2\pi)^n |B|} \prod_{k=1}^m \left[ \prod_{k=1}^m \gamma_k(t) e^{\frac{\xi_k^2}{\xi_k}} \right] \frac{1}{2} \int_{\mathbb{R}^n} \frac{1}{\xi_k^2} \prod_{k=1}^m \gamma_k(t) e^{\frac{\xi_k^2}{\xi_k}} d\xi
\]

\[
= \frac{1}{(2\pi)^n |B|} \prod_{k=1}^m \left[ \prod_{k=1}^m \gamma_k(t) e^{\frac{\xi_k^2}{\xi_k}} \right] \frac{1}{2} \int_{\mathbb{R}^n} \frac{1}{\xi_k^2} \prod_{k=1}^m \gamma_k(t) e^{\frac{\xi_k^2}{\xi_k}} d\xi
\]

\[
= \frac{1}{(2\pi)^n |B|} \prod_{k=1}^m \left[ \prod_{k=1}^m \gamma_k(t) e^{\frac{\xi_k^2}{\xi_k}} \right] \frac{1}{2} \int_{\mathbb{R}^n} \frac{1}{\xi_k^2} \prod_{k=1}^m \gamma_k(t) e^{\frac{\xi_k^2}{\xi_k}} d\xi
\]

\[
= \frac{1}{(2\pi)^n |B|} \prod_{k=1}^m \left[ \prod_{k=1}^m \gamma_k(t) e^{\frac{\xi_k^2}{\xi_k}} \right] \frac{1}{2} \int_{\mathbb{R}^n} \frac{1}{\xi_k^2} \prod_{k=1}^m \gamma_k(t) e^{\frac{\xi_k^2}{\xi_k}} d\xi
\]

\[
= \frac{1}{(2\pi)^n |B|} \prod_{k=1}^m \left[ \prod_{k=1}^m \gamma_k(t) e^{\frac{\xi_k^2}{\xi_k}} \right] \frac{1}{2} \int_{\mathbb{R}^n} \frac{1}{\xi_k^2} \prod_{k=1}^m \gamma_k(t) e^{\frac{\xi_k^2}{\xi_k}} d\xi
\]
\[
\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left[ \prod_{\alpha=1}^{m} \gamma_\alpha(x) s_\alpha(x) \right] e^{-x^T \mathbf{A}^{-1}} e^{-\frac{1}{2} \mathbf{x}^T \mathbf{B} \mathbf{x}} \, dx
\]

即得

\[
W(s,t) = \frac{1}{(2\pi)^n |B|} \int_{\mathbb{R}^n} \left[ \prod_{\alpha=1}^{m} \gamma_\alpha(x) s_\alpha(x) \right] e^{-x^T \mathbf{A}^{-1}} e^{-\frac{1}{2} \mathbf{x}^T \mathbf{B} \mathbf{x}} \, dx
\]

定理 1 (数学模型 I 解的存在定理): 若
1) \(A = (a_{ij})_{m \times m}\) 为正定对称矩阵，\(|A| = |B|^2\);
2) \(s_j(t), t \in [0,T], j \in \{1, \ldots, m\}\) 为充分光滑的单调函数；
3) \(\gamma_k(t) \in C([0,T]), k = 1, \ldots, m, \varphi(s) \in C(R^n)\)。

则数学模型 I 有精确解:

\[
u(s,t) = V(s,t) + W(s,t)
\]

\[
V(s,t) = \mathbb{E} e^{(r-c)t} \mathbb{E} \left[ \prod_{\alpha=1}^{m} \gamma_\alpha(s) s_\alpha(s) \right] e^{-s^T \mathbf{A}^{-1} \mathbf{x} + \frac{1}{2} \mathbf{x}^T \mathbf{B} \mathbf{x}} \, dx
\]

\[
W(s,t) = \frac{1}{(2\pi)^n |B|} \int_{\mathbb{R}^n} \left[ \prod_{\alpha=1}^{m} \gamma_\alpha(x) s_\alpha(x) \right] e^{-x^T \mathbf{A}^{-1} \mathbf{x} + \frac{1}{2} \mathbf{x}^T \mathbf{B} \mathbf{x}} \, dx
\]

且数学模型 I.1 的解 \(V(s,t)\) 由(80)式给出，数学模型 I.2 的解 \(W(s,t)\) 由(81)式给出。

2.1.2. 多维 Black-Scholes 方程确定奇异内边界的终值问题
数学模型 II (多维 Black-Scholes 方程确定奇异内边界的终值问题):
求 \(w(s,t), s(t) = (s_1(t), \ldots, s_k(t))\) 的允许解

\[
\mathbb{E} \mathbb{E} \left[ \prod_{\alpha=1}^{m} \gamma_\alpha(s) s_\alpha(s) \right] e^{-X^T \mathbf{A}^{-1} \mathbf{x} + \frac{1}{2} \mathbf{x}^T \mathbf{B} \mathbf{x}} \, dx
\]

定理 2 (数学模型 II 解的存在定理): 若
1) \(A = (a_{ij})_{m \times m}\) 为正定对称矩阵；
2) \(s_j(t), t \in [0,T], j \in \{1, \ldots, m\}\) 为充分光滑的单调函数；

\[
\lim_{s \to a^+} |w| < \infty, \lim_{s \to b^-} |w| < \infty
\]
3) \( \gamma_k(t) \in C([0,T]), k = 1, \ldots, m \)。

则数学模型 II 有连续有界的精确解

\[
\begin{align*}
\psi(s,t) &= \frac{e^{\alpha t}e^x}{(2\pi)^{\frac{n}{2}} |B|} \int_{B} \frac{Y(\xi)}{\nu \tau} e^{i(y-s)\tau} e^{\frac{\xi^2}{2} - i\frac{y^2}{2}} d\xi, \\
\sigma_k(t) &= \sigma_k(T)e^{\nu(T-t)}, k = 1, \ldots, m
\end{align*}
\] (87)

数学模型 II 有解的相容性条件是

\[
\frac{e^{\alpha t}e^x}{(2\pi)^{\frac{n}{2}} |B|} \int_{B} \frac{Y(\xi)}{\nu \tau} e^{i(y-s)\tau} d\xi = \mu(t), 0 < t < T
\] (88)

其中

\[
\prod_{k=1}^{m} s_k(T) \gamma_k(t) \equiv Y(t)
\] (90)

\[
\sum_{k=1}^{m} \omega_k = \sigma
\] (91)

\[
\omega_k = \frac{1}{2} a_{x_k} + q - \tau
\] (92)

\[
d_j = \sum_{i=1}^{l} \left( \frac{1}{2} a_{x_i} + q - \tau \right) c_{x_i}, j = 1, \ldots, m
\] (93)

证明：由定理 I(数学模型 I 解的存在定理)的结论，(81)式给出的 \( \psi(s,t) \) 已满足条件(82) (83) (85)三式，让 \( \psi(s,t) \) 满足条件(84)式去确定奇异内边界 \( s(t) \equiv (s_1(t), \ldots, s_k(t)) \)。

将(81)式 \( \psi(s,t) \) 记为

\[
\omega(s,t) = \frac{1}{(2\pi)^{\frac{n}{2}} |B|} \int_{B} \prod_{n=1}^{m} \gamma_n(\xi) s_n(\xi) e^{i(y-s)\tau} e^{\frac{\xi^2}{2} - i\frac{y^2}{2}} d\xi
\] (94)

由(94)式对 \( \omega(s,t) \) 关于自变量 \( s_k, k = 1, \ldots, m \) 求偏导，由复合函数的求导法则有

\[
\frac{\partial \omega(s,t)}{\partial s_k} = \frac{1}{(2\pi)^{\frac{n}{2}} |B|} \int_{B} \prod_{n=1}^{m} \gamma_n(\xi) s_n(\xi) e^{i(y-s)\tau} e^{\frac{\xi^2}{2} - i\frac{y^2}{2}} d\xi
\] (95)

若令

\[
\frac{1}{2} \sum_{j=1}^{n} c_{x_j} \ln s_j(t) = 0, \forall \xi, t \in (0,T), k = 1, \ldots, m
\] (96)

则
\[
\frac{(1+2d_j)(\xi-t)}{2} - \sum_{j=1}^{k_c} c_{jk} \ln \frac{s_j(t)}{s_j(\xi)} = \frac{(1+2d_j)(\xi-t)}{2} - \sum_{j=1}^{k_c} c_{jk} \ln \frac{s_j(t)}{s_j(\xi)} - \sum_{j=1}^{k_c} c_{jk} \ln \frac{s_j(t)}{s_j(\xi)} - \sum_{j=1}^{k_c} c_{jk} \ln \frac{s_j(t)}{s_j(\xi)}
\]

\[
= \frac{(1+2d_j)(\xi-t)}{2} - \sum_{j=1}^{k_c} c_{jk} \ln \frac{s_j(t)}{s_j(\xi)} - \sum_{j=1}^{k_c} c_{jk} \ln \frac{s_j(t)}{s_j(\xi)}\]

即有

\[
\frac{(1+2d_j)(\xi-t)}{2} - \sum_{j=1}^{k_c} c_{jk} \ln \frac{s_j(t)}{s_j(\xi)} - \sum_{j=1}^{k_c} c_{jk} \ln \frac{s_j(t)}{s_j(\xi)} \quad (97)
\]

将(97)式代入(95)式即有

\[
\frac{\partial w(s,t)}{\partial s_k} = -\frac{1}{(2\pi)^2 |B|} \left[ \int_{\mathbb{R}^m} \prod_{a=1}^{m} \phi_a(\xi) s_a(\xi) e^{-|\xi|^2/2} \sum_{n=1}^{m} \left[ \sum_{j=1}^{k_c} c_{jk} \ln \frac{s_j(t)}{s_j(\xi)} \right] c_{kn} d\xi \right] (\xi-t) (98)
\]

下面建立引理2.1~引理2.4来完成定理2的证明。

引理2.1：条件(96)成立，有

\[
w(s(t),t) = \max_{s \in \mathbb{R}_+^m} w(s(t),t) \in (0,T)
\]

和

\[
\frac{\partial w(s(t),t)}{\partial s_k} = 0, k = 1, \ldots, m
\]

证明：若条件(96)成立，则有(98)成立。由(98)式易知(100)式成立。

由(98)式即得到对任何\( t \in (0,T) \)有

1) 当\( s \in E_+(t) \) 有\( \ln \frac{s_j}{s_j(t)} < 0, j = 1, \ldots, m \)

由引理1.1中结论4)即有：当\( \ln \frac{s_j}{s_j(t)} < 0, j = 1, \ldots, m \) 有

\[
\frac{\partial w(s(t),t)}{\partial s_k} > 0, k = 1, \ldots, m
\]

\[
w(s(t),t) \leq w(s(t),t), s \in E_+(t)
\]

2) 当\( s \in E_-(t) \) 有\( \ln \frac{s_j}{s_j(t)} > 0, j = 1, \ldots, m \)

由引理1.1中结论4)即有：当\( \ln \frac{s_j}{s_j(t)} > 0, j = 1, \ldots, m \) 有

\[
\frac{\partial w(s(t),t)}{\partial s_k} < 0, k = 1, \ldots, m
\]

\[
w(s(t),t) \leq w(s(t),t), s \in E_-(t)
\]

从而(99)式成立。引理证毕。
引理 2.2：条件

\[ \ln \frac{s_j(t)}{s_j(\xi)} = \omega_j(\xi - t), \forall \xi, t \in (0, T), j \in \{1, \ldots, m\} \]

(103)

成立的充要条件为

\[ s_j(t) = s_j(T)e^{\omega_j(T-t)}, t \in (0, T), j \in \{1, \ldots, m\} \]

(104)

证明：1) 必要性，若(103)成立，由

\[ \frac{s_j(t)}{s_j(\xi)} = e^{\omega_j(\xi - t)} = \frac{e^{\omega_j T}}{e^{\omega_j T}} \]

(105)

即

\[ I_j(t) \equiv s_j(t)e^{\omega_j T} \]

(106)

由(105)式有

\[ I_j(t) = I_j(\xi), \forall \xi \in (t, T], t \in (0, T], j \in \{1, \ldots, N\} \]

(107)

让 \( \xi = T \) 即有

\[ s_j(t)e^{\omega_j T} = s_j(T)e^{\omega_j T} \]

(108)

于是有(104)式成立。

2) 充分性，若(104)式成立，即有

\[ \ln \frac{s_j(t)}{s_j(\xi)} - \omega_j(\xi - t) \]

\[ = \ln \frac{s_j(T)e^{\omega_j(T-t)}}{s_j(T)e^{\omega_j(T-\xi)}} - \omega_j(\xi - t) \]

\[ = \ln \frac{e^{\omega_j(T-t)}}{e^{\omega_j(T-\xi)}} - \omega_j(\xi - t) \]

\[ = \omega_j(T-t) - \omega_j(T - \xi) - \omega_j(\xi - t) \]

\[ = \omega_j(\xi - t) - \omega_j(\xi - t) = 0 \]

(109)

(110)

即(103)成立。引理证毕。

引理 2.3：当条件

\[ \begin{aligned}
  s_j(t) &= s_j(T)e^{\omega_j(T-t)}, t \in (0, T), j \in \{1, \ldots, m\} \\
  \sum_{j=1}^{k} c_j \omega_j - \frac{(1 + 2d_1)}{2} &= 0
\end{aligned} \]

(109)

成立时，则条件(96)成立，从而有

\[ w(s(t), t) = \max_{s \in \mathbb{R}} w(s, t), t \in (0, T), \frac{\partial w}{\partial s_k}(s(t), t) = 0, k = 1, \ldots, m \]

证明：由(109)式则有
再由(110)式即有
\[
\frac{(1+2d_j)(\xi-t)}{2} - \sum_{j=1}^{k} c_{j\xi} \ln \frac{s_j(t)}{s_j(\xi)} = \frac{(1+2d_k)(\xi-t)}{2} - \sum_{j=1}^{k} c_{j\xi} \omega_j(\xi-t) \equiv (\xi-t) \left[ \frac{(1+2d_k)}{2} - \sum_{j=1}^{k} c_{j\xi} \omega_j \right]
\] (111)

即条件(96)成立，由引理2.1即有
\[
w(s(t), t) = \max_{s \in K^n} w(s(t), t) \in (0, T)
\]
成立。引理证毕。

引理2.4：未知数$\omega_k, k = 1, \ldots, m$的线性方程组(110)的解为
\[
\omega_k = \sum_{j=1}^{k} b_{kj} \left( \frac{1+2d_j}{2} \right), k = 1, \ldots, m
\] (113)

证明：线性方程组(110)写成矩阵形式即为
\[
C^T \begin{bmatrix}
\omega_1 \\
\omega_2 \\
\vdots \\
\omega_m
\end{bmatrix} = \begin{bmatrix}
\frac{(1+2d_1)}{2} \\
\frac{(1+2d_2)}{2} \\
\vdots \\
\frac{(1+2d_m)}{2}
\end{bmatrix}
\] (114)

由$C$矩阵的定义即有$C^T = B^{-1}$，从而
\[
B^{-1} \begin{bmatrix}
\omega_1 \\
\omega_2 \\
\vdots \\
\omega_m
\end{bmatrix} = \begin{bmatrix}
\frac{(1+2d_1)}{2} \\
\frac{(1+2d_2)}{2} \\
\vdots \\
\frac{(1+2d_m)}{2}
\end{bmatrix}
\] (115)

由矩阵乘法即得线性方程组(110)的解由(113)式给出。引理证毕。

记
\[
s_j(t) = s_j(T)e^{\omega_j(t-T)}, t \in (0, T), \omega_j = \sum_{k=1}^{j} b_{kj} \frac{(1+2d_k)}{2}, j \in \{1, \ldots, m\}
\] (117)
由引理 2.1~引理 2.4 即知：当 (117) 成立时，有解 \( \{w(s,t), s(t)\} \) 其中 \( w(s,t) \) 由 (94) 给出，
\[
s(t) = \left( s_1(t), \ldots, s_m(t) \right) = \left( s_1(T) e^{\omega(T-t)}, \ldots, s_m(T) e^{m\omega(T-t)} \right).
\]
由 (117) 式即有 (96) 成立，由引理 2.1 即有
\[
w(s(t), t) = \max_{x \in \mathbb{R}^m} w(s(t), t) \quad \text{和} \quad \frac{\partial w}{\partial x_j}(s(t), t) = 0, k = 1, \ldots, m \quad \text{成立}.
\]

由 (117) 式即有 (97) 式成立，将 (97) 代入 (94) 即有:
\[
\left( \begin{array}{ll}
1 & 1
\end{array} \right) \mathbf{T}_1 \mathbf{T}_1 e^{\omega(T-t)} \leq \mathbf{T}_1 \mathbf{T}_1 e^{m\omega(T-t)} \quad \text{成立}.
\]

由 (117) 式即有 (98) 式成立，再将 (117) 式代入 (118) 式，即有
\[
\left( \begin{array}{ll}
1 & 1
\end{array} \right) \mathbf{T}_1 \mathbf{T}_1 e^{\omega(T-t)} \leq \mathbf{T}_1 \mathbf{T}_1 e^{m\omega(T-t)} \quad \text{成立}.
\]

引入记号：
\[
\left( \begin{array}{ll}
1 & 1
\end{array} \right) \mathbf{T}_1 \mathbf{T}_1 e^{\omega(T-t)} \leq \mathbf{T}_1 \mathbf{T}_1 e^{m\omega(T-t)} \quad \text{成立}.
\]

由 (117) 式即有 (99) 式成立，再由 (119) 即得到 \( w(s, t) \) 由 (87) 给出。由 (87) 式给出的解
\[
\left( \begin{array}{ll}
1 & 1
\end{array} \right) \mathbf{T}_1 \mathbf{T}_1 e^{\omega(T-t)} \leq \mathbf{T}_1 \mathbf{T}_1 e^{m\omega(T-t)} \quad \text{成立}.
\]

引入记号：
\[
\left( \begin{array}{ll}
1 & 1
\end{array} \right) \mathbf{T}_1 \mathbf{T}_1 e^{\omega(T-t)} \leq \mathbf{T}_1 \mathbf{T}_1 e^{m\omega(T-t)} \quad \text{成立}.
\]

引入记号：
\[
\left( \begin{array}{ll}
1 & 1
\end{array} \right) \mathbf{T}_1 \mathbf{T}_1 e^{\omega(T-t)} \leq \mathbf{T}_1 \mathbf{T}_1 e^{m\omega(T-t)} \quad \text{成立}.
\]

记
\[
\left( \begin{array}{ll}
1 & 1
\end{array} \right) \mathbf{T}_1 \mathbf{T}_1 e^{\omega(T-t)} \leq \mathbf{T}_1 \mathbf{T}_1 e^{m\omega(T-t)} \quad \text{成立}.
\]

定理 3(数学模型 I 1 解的性质定理)：若 \( A = \left( a_{ij} \right)_{m \times m} \) 为正定矩阵，
\[
\left( \begin{array}{ll}
1 & 1
\end{array} \right) \mathbf{T}_1 \mathbf{T}_1 e^{\omega(T-t)} \leq \mathbf{T}_1 \mathbf{T}_1 e^{m\omega(T-t)} \quad \text{成立}.
\]

1) 当 \( \varphi(s) = \delta(s - \theta) \)：数学模型 I 1 的解
\[
V(s, t) = \frac{c_s e^{-\omega(T-t)}}{(T-t)^{\frac{1}{2}}} e^{\frac{\omega(T-t)}{2}} \quad (T-t)^{\frac{1}{2}}
\]

满足
和
\[
\frac{\partial V}{\partial s}(s(t), t) = 0, k = 1, \ldots, m
\] (124)

其中
\[
c_{\theta} \equiv \frac{1}{(2\pi)^{\frac{m}{2}} |B| \prod_{j=1}^{m} \theta_j}
\] (125)

2) 当 \( \varphi(s) \in \Lambda_{(\omega, \alpha)} \)，则数学模型 I.1 的解
\[
V_{a}(s, t) = \frac{e^{-(T-t)r}}{(2\pi)^{\frac{m}{2}} |B| (T-t)^{\frac{m}{2}}} \int_{(\theta, \alpha)} \varphi(\xi) \frac{1}{\prod_{k=1}^{m} \xi_k} e^{-\frac{\sum_{j=1}^{m} (T-(\xi_j+\frac{1}{2}))^2 \xi_j^2}{2(T-t)}} d\xi
\] (126)

且解 \( V_{a}(s, t) \) 满足
\[
\max_{s \in \Omega_{(\omega, \alpha)}} V_{a}(s, t) = V_{a}(s(t), t)
\] (127)

3) 当 \( \varphi(s) \in \Lambda_{(0, \alpha)} \)，数学模型 I.1 的解
\[
V_{a}(s, t) = \frac{e^{-(T-t)r}}{(2\pi)^{\frac{m}{2}} |B| (T-t)^{\frac{m}{2}}} \int_{(0, \alpha)} \varphi(\xi) \frac{1}{\prod_{k=1}^{m} \xi_k} e^{-\frac{\sum_{j=1}^{m} (T-(\xi_j+\frac{1}{2}))^2 \xi_j^2}{2(T-t)}} d\xi
\] (128)

且解 \( V_{a}(s, t) \) 满足
\[
\max_{s \in \Omega_{(0, \alpha)}} V_{a}(s, t) = V_{a}(s(t), t)
\] (129)

证 1): 数学模型 I.1 的解由定理 1 的(74)式给出，由 \( \varphi(s) = \delta(s - \theta) \) 有
\[
V(s, t) = \frac{e^{-(T-t)r}}{(2\pi)^{\frac{m}{2}} |B| (T-t)^{\frac{m}{2}}} \int \delta(\xi - \theta) \frac{1}{\prod_{k=1}^{m} \xi_k} e^{-\frac{\sum_{j=1}^{m} (T-(\xi_j+\frac{1}{2}))^2 \xi_j^2}{2(T-t)}} d\xi
\] (130)

应用多维狄拉克 \( \delta \) 函数的积分性质即得
\[
V(s, t) = \frac{e^{-(T-t)r}}{(2\pi)^{\frac{m}{2}} (T-t)^{\frac{m}{2}}} \frac{1}{\prod_{k=1}^{m} \theta_k} e^{-\frac{\sum_{j=1}^{m} (T-(\theta_j+\frac{1}{2}))^2 \theta_j^2}{2(T-t)}}
\] (131)

引入记号 \( c_{\theta} \equiv \frac{1}{(2\pi)^{\frac{m}{2}} |B| \prod_{j=1}^{m} \theta_j} \) 即有
\begin{align}
V(s, t) &= c_0 e^{-\gamma t} \frac{\sum_{j=1}^{m} \left( T - t \left( d_s + 1/2 \right) \sum_{j=1}^{m} c_{jk} \ln \frac{s_j(t)}{\theta_j} \right)^2}{(T-t)^2} \tag{132}
\end{align}

由于 \( \omega_k, k = 1, \ldots, m \) 线性方程组(110)的解，由(110)即有

\begin{align}
\sum_{j=1}^{m} c_{jk} \omega_j - \frac{1 + 2d_s}{2} = 0, k = 1, 2, \ldots, m \tag{133}
\end{align}

由(122)式即有

即有

\begin{align}
(T-t) \left( d_s + 1/2 \right) - \sum_{j=1}^{m} c_{jk} \ln \frac{s_j(t)}{\theta_j} = 0, t \in (0, T), k = 1, \ldots, m \tag{134}
\end{align}

由(132)和(134)两式即有

\begin{align}
V(s(t), t) &= c_0 e^{-\gamma t} \frac{\sum_{j=1}^{m} \left( T - t \left( d_s + 1/2 \right) \sum_{j=1}^{m} c_{jk} \ln \frac{s_j(t)}{\theta_j} \right)^2}{(T-t)^2}, t \in (0, T) \tag{135}
\end{align}

由于

\begin{align}
\sum_{j=1}^{m} \left( T - t \left( d_s + 1/2 \right) \sum_{j=1}^{m} c_{jk} \ln \frac{s_j(t)}{\theta_j} \right)^2 \leq 1, s \in R^m, t \in (0, T) \tag{136}
\end{align}

从而

\begin{align}
V(s, t) \leq c_0 e^{-\gamma t} \frac{\sum_{j=1}^{m} \left( T - t \left( d_s + 1/2 \right) \sum_{j=1}^{m} c_{jk} \ln \frac{s_j(t)}{\theta_j} \right)^2}{(T-t)^2}, s \in R^m, t \in (0, T) \tag{137}
\end{align}

故 \( V(s(t), t) = \max_{s \in R^m} V(s, t) = c_0 e^{-\gamma t} \frac{\sum_{j=1}^{m} \left( T - t \left( d_s + 1/2 \right) \sum_{j=1}^{m} c_{jk} \ln \frac{s_j(t)}{\theta_j} \right)^2}{(T-t)^2} \)

由(132)对 \( V(s, t) \) 关于 \( s_k, k = 1, \ldots, m \) 求偏导得到

\begin{align}
\frac{\partial V}{\partial s_k}(s, t) &= c_0 e^{-\gamma t} \frac{\sum_{j=1}^{m} \left( T - t \left( d_s + 1/2 \right) \sum_{j=1}^{m} c_{jk} \ln \frac{s_j(t)}{\theta_j} \right)^2}{(T-t)^2 s_k} \tag{138}
\end{align}

\begin{align}
\frac{\partial V}{\partial s_k}(s(t), t) &= c_0 e^{-\gamma t} \frac{\sum_{j=1}^{m} \left( T - t \left( d_s + 1/2 \right) \sum_{j=1}^{m} c_{jk} \ln \frac{s_j(t)}{\theta_j} \right)^2}{(T-t)^2 s_k} \tag{139}
\end{align}

由(134), (139)即得(124)。

证 2) 数学模型 1.1 的解由定理 1 的(74)式给出。任意

\( \varphi \in \Lambda_{(\beta, \varnothing)} \equiv \{ \varphi | \varphi \in C(R^m), \varphi(s) \geq 0, s \in R^m, \sup \varphi \subset (\beta, \varnothing) \subset R^m \} \) 即有 \( \sup \varphi \subset (\beta, \varnothing) \)。

\( \varphi(s) > 0, s \in \sup \varphi \subset (\beta, \varnothing); \varphi(s) = 0, s \in R^m - \sup \varphi, \) 故 \( \varphi(s) \equiv 0, s \in (0, \beta) \)。于是(74)式中积分的积分区域由 \( R^m \) 变为 \( (\beta, \varnothing) \)。则数学模型 1.1 的解

\begin{align}
V_s(s(t), t) &= \int_{(\beta, \varnothing)} \varphi(\xi) \frac{1}{\prod_{k=1}^{m} \xi_k} e^{-\gamma t} \frac{\sum_{j=1}^{m} \left( T - t \left( d_s + 1/2 \right) \sum_{j=1}^{m} c_{jk} \ln \frac{s_j(t)}{\theta_j} \right)^2}{(T-t)^2} \frac{d^m \xi}{(2\pi)^m B(T-t)^{m/2}} \tag{140}
\end{align}
由(140)关于 $s, k = 1, \cdots, m$ 求偏导
$$
\frac{\partial V_A(s, t)}{\partial s_k} =
$$
$$
= \frac{e^{(T-t)\nu}}{(2\pi)^{\frac{n}{2}} |B| (T-t)^{\frac{n}{2}}} \int \varphi(\xi) \prod_{k=1}^{m} \frac{1}{\xi_k} e^{-\frac{\xi^T (T-t) \varphi^{(2)}}{2(T-t)}} \left( \sum_{n=1}^{\infty} \left( T-t \left( d_n + \frac{1}{2} \right) - \sum_{j=1}^{\infty} c_{jn} \frac{s_j}{\xi_j} \right)^2 \right) \xi_k \sum_{n=1}^{\infty} \left( T-t \left( d_n + \frac{1}{2} \right) - \sum_{j=1}^{\infty} c_{jn} \frac{s_j}{\xi_j} \right) c_{kn} d\xi
$$
$$
= \frac{e^{(T-t)\nu}}{(2\pi)^{\frac{n}{2}} |B| (T-t)^{\frac{n}{2}}} \int \varphi(\xi) \prod_{k=1}^{m} \frac{1}{\xi_k} e^{-\frac{\xi^T (T-t) \varphi^{(2)}}{2(T-t)}} \left( \sum_{n=1}^{\infty} \left( T-t \left( d_n + \frac{1}{2} \right) - \sum_{j=1}^{\infty} c_{jn} \frac{s_j}{\xi_j} \right) c_{kn} d\xi \right) \tag{141}
$$
当 $s \in \mathbb{E}_-(t)$ 有 $0 < s_j < \xi_j, j = 1, \cdots, m$；积分变量 $\xi \in (\vartheta, \infty)$，有 $\vartheta \leq \xi_j, j = 1, \cdots, m$；
$$
\frac{s_j(t) \xi_j}{\vartheta s_j} > 1, \ln \frac{s_j(t) \xi_j}{\vartheta s_j} > 0, j = 1, \cdots, m \tag{142}
$$
又
$$
\sum_{n=1}^{\infty} \left( T-t \left( d_n + \frac{1}{2} \right) - \sum_{j=1}^{\infty} c_{jn} \frac{s_j}{\xi_j} \right) c_{kn} = \sum_{n=1}^{\infty} \left( T-t \left( d_n + \frac{1}{2} \right) - \sum_{j=1}^{\infty} c_{jn} \vartheta \frac{s_j}{\xi_j} + \sum_{j=1}^{\infty} c_{jn} \frac{s_j(t) \xi_j}{\vartheta s_j} - \sum_{j=1}^{\infty} c_{jn} \frac{s_j}{\xi_j} \right) c_{kn}
$$
$$
= \sum_{n=1}^{\infty} \left( \sum_{j=1}^{\infty} c_{jn} c_{kn} \ln \frac{s_j(t) \xi_j}{\vartheta s_j} \right) \tag{143}
$$
记
$$
\ln \frac{s_j(t) \xi_j}{\vartheta s_j} \equiv x_j, j = 1, \cdots, m; \sum_{n=1}^{\infty} \sum_{k=1}^{m} c_{jn} c_{kn} \ln \frac{s_j(t) \xi_j}{\vartheta s_j} = \sum_{n=1}^{\infty} c_{kn} \sum_{j=1}^{\infty} c_{jn} x_j \equiv I(k; X), k = 1, \cdots, m; \text{ 由(142)，有}
$$
$$
\ln \frac{s_j(t) \xi_j}{\vartheta s_j} \equiv x_j > 0, j = 1, \cdots, m, \text{ 引理 1.1 的结论 4) 即有 } I(k; X) > 0, k = 1, \cdots, m; \text{ 从而由(143) 有}
$$
$$
\sum_{n=1}^{\infty} \left( T-t \left( d_n + \frac{1}{2} \right) - \sum_{j=1}^{\infty} c_{jn} \frac{s_j}{\xi_j} \right) > 0 \tag{144}
$$
再由(141) 即有当 $s \in \mathbb{E}_-(t)$，
$$
\frac{\partial V_A(s, t)}{\partial s_n} > 0, n = 1, \cdots, m \tag{145}
$$
当 $s \in \mathbb{E}_-(t)$，$V_A(s, t) \leq V_A(s(t), t)$，从而有 $\max_{s \in \mathbb{E}_-(t)} V_A(s, t) = V_A(s(t), t)$ 成立。
证 3): 当 \( \varphi(s) \in \Lambda_{[0, \theta]} \)，由数学模型 1.1 的解（74）式即有（128）成立。由（128）式对 \( V_{s}(s, t) \) 关于

\[
\frac{\partial V}{\partial s_{k}}(s, t) = \frac{e^{(r-t)s}}{(2\pi)^{m/2}|B|} \int_{\mathbb{R}^{m}} \varphi(x) \prod_{k=1}^{m} e^{-|x_{k}|^{2}/2} \left( \sum_{j=1}^{n} \left( T-t \right) \left( d_{n} + \frac{1}{2} \right) - \sum_{j=1}^{n} \ln \frac{s}{x_{j}} \right) \frac{c_{n} d_{k}}{s_{k}} dx
\]

(146)

当 \( s \in E_{s}(t) \)，有 \( \frac{s}{\varphi_{j} s_{j}} < 1, \ln \frac{s}{\varphi_{j} s_{j}} < 0, j = 1, \ldots, m \).

引理 1.1 的结论 4) 即有

\[
\sum_{j=1}^{n} \left( T-t \right) \left( d_{n} + \frac{1}{2} \right) - \sum_{j=1}^{n} \ln \frac{s}{x_{j}} < 0
\]

由（146）即有：当 \( s \in E_{s}(t) \)，有 \( \frac{\partial V}{\partial s_{k}}(s, t) < 0, n = 1, \ldots, m \)。当 \( s \in E_{s}(t) \)，有 \( V_{s}(s, t) \leq V_{s}(s(t), t) \)，从而有

\[
\max_{s \in E_{s}(t)} V_{s}(s, t) = V_{s}(s(t), t)
\]

定理证毕。

2.2. 关于多维 Black-Scholes 方程的自由边界问题的研究

由 \( E_{s}(t) \): 0 < \( s_{j} < s_{j}(t), j = 1, \ldots, m; E_{s}(t) \): \( s_{j}(t) < s_{j} < \infty, j = 1, \ldots, m \)

即有 \( E_{s}(T) \): 0 < \( s_{j} < s_{j}(T), j = 1, \ldots, m; E_{s}(T) \): \( s_{j}(T) < s_{j} < \infty, j = 1, \ldots, m \)

即 \( E_{s}(T) \): 0 < \( s_{j} < \varphi_{j}, j = 1, \ldots, m; E_{s}(T) \): \( \varphi_{j} < s_{j} < \infty, j = 1, \ldots, m \)

即 \( E_{s}(T) \equiv (0, \varphi), E_{s}(t) \equiv (\varphi, \infty) \)

下面分别讨论关于多维 Black-Scholes 方程在 \( \Omega_{s} \) 的自由边界问题 A 和在 \( \Omega_{s} \) 的自由边界问题 B。

自由边界问题 A (关于多维 Black-Scholes 方程在 \( \Omega_{s} \) = \{ (s, t) | s \in E_{s}(t), t \in (0, T) \} \) 的自由边界问题):

求 \( \{u(s, t), s(t)\} \)，使其满足

\[
\frac{\partial u}{\partial t} + \frac{1}{2} \sum_{i=1}^{m} a_{ij} \left( s_{i} \frac{\partial}{\partial s_{i}} \right) u + \sum_{i=1}^{m} \left( r - q_{i} - \frac{1}{2} a_{ii} \right) s_{i} \frac{\partial u}{\partial s_{i}} - ru = 0, s \in E_{s}(t), 0 < t < T
\]

(147)

\[
u(s, T) = 0, s \in (0, \varphi)
\]

(148)

\[
u(s(t), t) = \max_{s \in E_{s}(t)} u(s, t) = \mu_{k}(t), t \in (0, T)
\]

(149)

\[
\frac{\partial u}{\partial s_{k}}(s(t), t) = \psi_{st}(t), k = 1, \ldots, m
\]

(150)

\[
\lim_{s \to \varphi} |u| < \infty, \lim_{s \to 0} |u| < \infty
\]

(151)

定理 4 (自由边界问题 A 多解性定理): 若

1) \( A = \{a_{ij}\}_{i=1}^{m} \) 为正定矩阵；

2) \( \gamma_{k}(t) \in C([0, T]), \gamma_{k}(t) \geq 0, t \in (0, T), k = 1, \ldots, m \):

则自由边界问题 A 的有解 \( \{u_{k}(s, t), s(t)\} \)，自由边界为

\[
s(t) = \left( \varphi \epsilon_{k}(t), \varphi \epsilon_{m}(t) \right)
\]

(152)

\( u_{k}(s, t) \) 具有多解性，第一解:
有解的相容性条件

\[
\begin{cases}
\mu(t) = \frac{\c_0 e^{-\sigma(t)\eta}}{(T-t)^\frac{m}{2}} \int_{\mathcal{B}(B)} \phi(\xi) \frac{1}{\xi_i} \prod_{k=1}^{m} \xi_i \left( \frac{\xi_j^2}{2(T-t)^\frac{m}{2}} \right) \, d\xi + \frac{e^{\sigma(t)\vartheta}}{2(T-t)^\frac{m}{2}} \int_{\mathcal{B}(B)} \frac{\vartheta}{\xi_i} \, d\xi, \forall \varphi(s) \in \Lambda_{(s,\alpha)}
\end{cases}
\]

(157)

第二解:

\[
u_{m}(t) = \frac{e^{\sigma(t)\vartheta}}{(2\pi)^{\frac{m}{2}}} \int_{\mathcal{B}(B)} \vartheta, \varphi(s) \in \Lambda_{(s,\alpha)}
\]

(158)

有解的相容性条件

\[
u_{m}(t) = \frac{e^{\sigma(t)\vartheta}}{(2\pi)^{\frac{m}{2}}} \int_{\mathcal{B}(B)} \vartheta, \varphi(s) \in \Lambda_{(s,\alpha)}
\]

(159)

证明：当 \( s \in \mathcal{E}_1(t), 0 < T < T_s \neq s, \delta(s_t - s_t) = 0, f(s, t) = \prod_{k=1}^{m} \gamma_k(t) s_t^2(t) \delta(s_t - s_t) = 0 \) 由定理 2 的(87)式给出的解满足齐次方程(147)，故由(153)式和(156)式给出的解满足齐次方程(147)。再由定理 2，定理 3 的结论即知定理 4 成立。由(112)，(152)两式即有

(160)

推证相容性条件(158)，由(141)，(160)两式即得。定理证毕。

附注 1：定理 4 中的第二解对在函数集合 \( \Lambda_{(s,\alpha)} \) 任意给定的 \( \varphi(s) \) 都有由(156)式给出的解 \( u_1(s,t) \) 与之对应，即得到了一个解族 \( \Phi_1 \equiv \{ u_1(s,t) | u_1(s,t) = \varphi(s) \in \Lambda_{(s,\alpha)} \} \)。
自由边界问题 B (关于多维 Black-Scholes 方程在 \( \Omega = \{(s,t) \mid s \in E_s(t), t \in (0,T)\} \) 的自由边界问题):

求 \( \{u(s,t), s(t)\} \)，使其满足

\[
\frac{\partial u}{\partial t} + \frac{1}{2} \sum_{k,j=1}^{m} a_{kj} \left( \frac{\partial}{\partial s_k} \right) \left( s_j \frac{\partial}{\partial s_j} \right) u + \sum_{i=1}^{m} \left( r - q_i - \frac{1}{2} a_{ii} \right) s_i \frac{\partial u}{\partial s_i} - ru = 0, s \in E_s(t), t \in (0,T) \tag{161}
\]

\[
u(s,T) = 0, s \in (\mathcal{B}, \infty) \tag{162}
\]

\[
u(s(t), t) = \max_{s \in E_s(t)} u(s, t) = \mu_B(t), t \in (0,T) \tag{163}
\]

\[
\left. \frac{\partial u}{\partial s_k}(s(t), t) = \psi_{\mathbb{M}}(t), k = 1, \ldots, m \right|_{s \to \omega} \tag{164}
\]

\[
\lim_{k \to \infty} \left| \left| \mu_B \right| \right| < \infty, \lim_{s \to \omega} \left| \left| u \right| \right| < \infty \tag{165}
\]

**定理 5**（自由边界问题 B 多解性定理）：若

1) \( A = (a_{kj})_{m \times m} \) 为正定矩阵；

2) \( \gamma_k(t) \in C([0,T]), \gamma_k(t) \geq 0, t \in (0,T), k = 1, \ldots, m \):

则自由边界问题 B 有解 \( \{u_B(s(t), s(t))\} \)，自由边界为

\[
s(t) = \left( \partial_t e^{\mu(t)}, \ldots, \partial_m e^{\mu_m(t)} \right) \tag{166}
\]

\[
u_B(s(t)) = \mu_B(t), t \in (0,T)
\]

\[
\text{有解的相容性条件}
\]

**第一解**:

\[
u_B(s(t)) = \mu_B(t) = \frac{c e^{-\psi(\theta)}}{(T-t)^{\frac{m}{2}}} \left( \frac{\sum_{j=1}^{m} \left( a_{ik} + \frac{1}{2} \sigma_i \sigma_j \right) - \sum_{i,j=1}^{m} a_{ij} \theta}{2(T-t)} \right) + \frac{e^{\theta t} e^{\psi}}{(2\pi)^{\frac{m}{2}} |\theta|} \int_{\mathbb{R}^{m}} Y(\xi) e^{-\xi^T \theta} d\xi
\]

**第二解**:

\[
u_B(s(t)) = \mu_B(t) = \frac{c e^{-\psi(\theta)}}{(T-t)^{\frac{m}{2}}} \left( \frac{\sum_{j=1}^{m} \left( a_{ik} + \frac{1}{2} \sigma_i \sigma_j \right) - \sum_{i,j=1}^{m} a_{ij} \theta}{2(T-t)} \right) + \frac{e^{\theta t} e^{\psi}}{(2\pi)^{\frac{m}{2}} |\theta|} \int_{\mathbb{R}^{m}} Y(\xi) e^{-\xi^T \theta} d\xi, \forall \varphi(s) \in \Lambda_{(0,\theta)}
\]
\[ \psi_{ab}(t) = \frac{e^{-(T-t)r}}{(2\pi)^{\frac{n}{2}} |\beta| (T-t)^{\frac{n+1}{2}}} \int_{(0, \delta^2)} e^{-\sum_{k=1}^{n} \frac{c_k^2}{2(T-t)}} \sum_{j=1}^{m} c_{jm} \ln \frac{\xi_j}{\beta_j} \, d\xi, \phi(s) \in \Lambda_{(0, \delta)}, k = 1, \ldots, m \] (172)

其中

\[ c_{ab} \equiv \frac{1}{(2\pi)^{\frac{n}{2}} |\beta|^{\frac{n}{2}}} \gamma_j(t), \sigma = \sum_{k=1}^{n} \alpha_k, \alpha_k = \frac{b_k}{2} \left( 1 + 2d_j \right), d_k = \frac{1}{2} \left( a_m + q_n - r \right) c_{ab} \] (173)

证明：当 \( s \in E, (t, r) \in (0, T), s_k \neq s_k(t), \delta(s_k - s_k(t)) \equiv 0 \), \( f(s, t) = \sum_{k=1}^{m} \gamma_j(t) s_j(t) \delta(s_k - s_k(t)) \) 由定理 2 的(87)式给出的解满足齐次方程(161)，故由(167)式和(170)式给出的解满足齐次方程(161)。再由定理 2，定理 3 的结论即知定理 5 成立。推证相容性条件(172)由(146) (160)两式即得。定理证毕。

附注 2：定理 5 中的第二解对在函数集合 \( \Lambda_{(0, \delta)} \) 任意给定的 \( \phi(s) \) 都有由(170)式给出的解 \( u_{ab}(s, t) \) 与之对应，即得到了一个解族 \( \Phi_{ab} \equiv \{ u_{ab}(s, t) | u_{ab}(s, T) = \phi(s) \in \Lambda_{(0, \delta)} \} \)。

2.3. 数学模型 III 与自由边界问题 A 和问题 B 的关系

数学模型 III（多维 Black-Scholes 方程确定奇异内边界的终值问题），求 \( \{ u(s, t), s(t) \} \)，使其满足

\[ \frac{\partial u}{\partial t} + \frac{1}{2} \sum_{j=1}^{n} a_j \left( s^j \frac{\partial}{\partial s_k} \right) \left( s_j \frac{\partial}{\partial s_k} \right) u + \sum_{k=1}^{n} \left( r - q_k - \frac{1}{2} a_k \right) s_k \frac{\partial u}{\partial s_k} = ru \]

\[ u(s, T) = \phi(s) \]

\[ u(s(t), t) = \max_{s \in \mathbb{R}^n} u(s, t) = \mu(t), t \in (0, T) \]

\[ \frac{\partial u}{\partial s_k}(s(t), t), k = 1, \ldots, m \]

\[ \lim_{s \to \infty} |s| < \infty, \lim_{s \to \infty} |s| < \infty \]

定理 6（奇异内边界的终值问题 A 和问题 B 的自由边界三线合一定理一）：若
1) \( A = (a_{ij})_{n \times m} \) 为正定矩阵；
2) \( \gamma_k(t) \in C([0, T]), \gamma_k(t) \geq 0, k = 1, \ldots, m \); 
3) \( \phi(s) = \delta(s - \theta) \);
4) \( \mu(t) = \mu_k(t) = \mu_k(t) \);
5) \( \psi_{ab}(t) = \psi_{ab}(t) \equiv 0, k = 1, \ldots, m \)。

则数学模型 III 与问题 A 和问题 B 有相同表达式的解

\[ u(s, t) = \frac{e^{-\frac{c^2(T-t)}{2 \beta^2}}} {2\pi |\beta|} \int_{(0, \delta^2)} e^{-\frac{c^2 x^2}{2(T-t)}} \sum_{j=1}^{m} \gamma_j(t) e^{\frac{c_j x_j}{y_j}} \frac{x^2}{2(T-t)} \, d\xi \]

\[ s(t) = (\xi_k e^{c(t)}, \ldots, \xi_m e^{c(t)}), t \in (0, T) \] (174) (175) (176) (177) (178)

其中

\[ c_{ab} \equiv \frac{1}{(2\pi)^{\frac{n}{2}} |\beta|^{\frac{n}{2}}} \gamma_j(t), \sigma = \sum_{k=1}^{n} \alpha_k, \alpha_k = \frac{b_k}{2} \left( 1 + 2d_j \right), d_k = \frac{1}{2} \left( a_m + q_n - r \right) c_{ab} \] (173)

证明：当 \( s \in E, (t, r) \in (0, T), s_k \neq s_k(t), \delta(s_k - s_k(t)) \equiv 0 \), \( f(s, t) = \sum_{k=1}^{m} \gamma_j(t) s_j(t) \delta(s_k - s_k(t)) \) 由定理 2 的(87)式给出的解满足齐次方程(161)，故由(167)式和(170)式给出的解满足齐次方程(161)。再由定理 2，定理 3 的结论即知定理 5 成立。推证相容性条件(172)由(146) (160)两式即得。定理证毕。

附注 2：定理 5 中的第二解对在函数集合 \( \Lambda_{(0, \delta)} \) 任意给定的 \( \phi(s) \) 都有由(170)式给出的解 \( u_{ab}(s, t) \) 与之对应，即得到了一个解族 \( \Phi_{ab} \equiv \{ u_{ab}(s, t) | u_{ab}(s, T) = \phi(s) \in \Lambda_{(0, \delta)} \} \)。
有解的相容性条件

$$
\mu(t) = \frac{c_{\beta}e^{-(T-t)p}}{(T-t)^{\frac{m}{2}}} + \frac{e^{\sigma T}e^{
u} + \int_{x}^{T} e^{-(\sigma + \nu) t} d\xi}{(2\pi)^{\frac{m}{2}}|B|} \frac{1}{(\xi - t)^{\frac{m}{2}}}
$$

(181)

其中

$$
c_{\beta} \equiv \frac{1}{(2\pi)^{\frac{m}{2}}|B|\prod_{j=1}^{n}\partial_j}, Y(t) = \prod_{k=1}^{n}g_k(t), \sigma = \sum_{k=1}^{n}y_k, \\
d_k = \sum_{n=1}^{m} \left( \frac{1}{2}a_{mn} + q_n - r \right) c_{nk}
$$

(182)

数学模型 III 的奇异内边界与问题 A 和问题 B 的自由边界三曲线重合成同一指数函数向量

$$
s(t) = \left( \partial_1 e^{na(T-t)}, \ldots, \partial_m e^{na(T-t)} \right), t \in (0,T)$$

数学模型 III 的解函数是问题 A 和 B 的解函数的共同连续开拓，即

$$
\begin{align*}
\begin{cases}
u(s,t), s \in E\left(s, t \right), t \in (0, T) \\
u(s,t) = \nu(s,t), t \in (0, T) \\
u(s,t), s \in E\left(s, t \right), t \in (0, T)
\end{cases}
\end{align*}
$$

(183)

定义 2: 若

$$
s(s) = \left( \partial_1 e^{na(T-t)}, \ldots, \partial_m e^{na(T-t)} \right), t \in (0,T), u(s,t) \text{ 由}(183) \text{ 定义，称} \{ u(s,t), s(t) \} \text{ 为数学模型 III 与问题 A 和问题 B 的一致相容解。}
$$

定理 7 (奇异内边界与问题 A 和 B 的自由边界三线合定理二): 若

1) $A = \left( a_{ij} \right)_{m \times m}$ 为正定矩阵；
2) $\gamma_k(t) = \delta(t - \partial), t \in (0, T), k = 1, \ldots, m$；
3) $\varphi(s) = \delta(s - \partial)$；
4) $\mu(t) = \mu(s, t) = \mu_B(t)$；
5) $\psi_{Ak}(t) = \psi_{Bl}(t) = 0, k = 1, \ldots, m$.

则数学模型 III 与问题 A 和问题 B 的一致相容解

$$
\begin{align*}
\begin{cases}
u(s,t) = \frac{c_{\beta}e^{-(T-t)p}}{(T-t)^{\frac{m}{2}}}e^{\sigma T}e^{\nu} + \int_{x}^{T} e^{-(\sigma + \nu) t} d\xi \\
u(s,t) = \left( \partial_1 e^{na(T-t)}, \ldots, \partial_m e^{na(T-t)} \right), t \in (0,T)
\end{cases}
\end{align*}
$$

(184)

(185)

有解的相容性条件

$$
\mu(t) = \frac{2c_{\beta}e^{-(T-t)p}}{(T-t)^{\frac{m}{2)}}
$$

(186)

$$
c_{\beta} \equiv \frac{1}{(2\pi)^{\frac{m}{2}}|B|\prod_{j=1}^{n}\partial_j}, \sigma_k = \sum_{j=1}^{n} \frac{b_j \left( 1+2d \right)}{2} d_k = \sum_{n=1}^{m} \left( \frac{1}{2}a_{mn} + q_n - r \right) c_{nk}
$$

(187)
定理 8 (奇异内边界与问题 A 和问题 B 的自由边界三线合一定理三)：若
1) \( A = \{a_j\}_{j=1}^m \) 为正定矩阵；
2) \( \gamma_k(t) = 0, t \in (0,T), k = 1,\ldots,m \); 
3) \( \varphi(s) = \delta(s - \beta) \); 
4) \( \mu(t) = \mu_n(t) = \mu_B(t) \); 
5) \( \psi_{nk}(t) = \psi_{nk}(t) = 0, k = 1,\ldots,m \)。
则数学模型 III 与问题 A 和问题 B 的一致相容解

\[
\begin{align*}
& u(s,t) = c_0 e^{(T-t)\mu} e^{\int_0^t (T-t') A^{-1} d\tau} e^{\frac{1}{2} \int_0^t (T-t') A^{-1} A^{-1} d\tau} \\
& s(t) = \left(\partial_t e^{\sigma(t)}, \ldots, \partial_t e^{\sigma(t)}\right), t \in (0,T)
\end{align*}
\] (188)

有解的相容性条件

\[
\mu(t) = \frac{c_0 e^{(T-t)\mu}}{(T-t)^{\mu}}
\] (190)

其中

\[
c_0 = \frac{1}{(2\pi)^m B} \prod_{j=1}^m \sigma_j, \quad \alpha_k = \sum_{j=1}^k b_j \left(1 + 2d_j\right), \\
d_k = \frac{1}{2} \sum_{n=1}^k \left(a_{nn} + q_n - r\right) c_{nk}
\] (191)

由定理 6，定理 7，定理 8 给出的一致相容解 \( u(s,t) \) 满足条件

\[
u(s(t),t) = \max_{s \in s(t)} u(s,t), t \in (0,T)
\] (192)

即一致相容解 \( u(s,t) \) 在任意时刻 \( t \in (0,T) \) 时，取 \( s(t) = \left(\partial_t e^{\sigma(t)}, \ldots, \partial_t e^{\sigma(t)}\right) \) 取 \( R^m \) 中的最大值

\[
u(s(t),t) = \max_{s \in s(t)} u(s,t), \quad s(t) = \left(\partial_t e^{\sigma(t)}, \ldots, \partial_t e^{\sigma(t)}\right)
\] (193)

定理 9（多资产期权最佳实施边界定理）：\( A = \{a_j\}_{j=1}^m \) 为正定矩阵，则期权价格函数 \( u(s,t) \) 在任意时刻 \( t \in (0,T) \) 在 \( s(t) = \left(\partial_t e^{\sigma(t)}, \ldots, \partial_t e^{\sigma(t)}\right) \) 取 \( R^m \) 中的最大值 \( u(s(t),t) = \max_{s \in s(t)} u(s,t) \)，多资产期权最佳实施边界为指数函数向量

\[
s(t) = \left(\partial_t e^{\sigma(t)}, \ldots, \partial_t e^{\sigma(t)}\right), t \in (0,T)
\] (193)

满足

\[
\omega_k = -\frac{1}{s_k(t)} \frac{d s_k(t)}{dt}, k = 1,\ldots,m
\] (194)

且有 \( \omega_k \) 的计算公式

\[
\omega_k = \sum_{j=1}^k b_j \left[1 + \sum_{n=1}^k \left(a_{nn} + q_n - r\right) c_{nj}\right]
\] (195)
公式(195)表明 \( \omega_k, k = 1, \cdots, m \) 由多维 Black-Scholes 方程中出现的所有参数 \( a_{ij}, q_j, r \) 唯一确定。

**证明**：由定理 6，定理 7，定理 8 即知期权价格函数 \( u(s, t) \) 在任意时刻 \( t \in (0, T) \) 为

\[
s(t) = \left(\partial_t e^{(T-t)}, \cdots, \partial_m e^{(T-t)}\right)
\]

取 \( R^n \) 中的最大值

\[
u(t) = \max_{s \in R^n} u(s, t).
\]

从而多资产期权最佳实施边界为

\[
\omega = \frac{-1}{s_t(t)} d_1(t), \quad \omega_k, k = 1, \cdots, m \] 由多维 Black-Scholes 方程中出现的所有参数 \( a_{ij}, q_j, r \) 唯一确定。定理证毕。

### 3. 结论

指数函数向量

\[
s(t) = \left(\partial_t e^{(T-t)}, \cdots, \partial_m e^{(T-t)}\right), t \in (0, T)
\]

为多资产期权的最佳实施边界，满足条件

\[
\omega = \frac{-1}{s_t(t)} d_1(t) \quad \omega_k, k = 1, \cdots, m \] 由多维 Black-Scholes 方程中出现的所有参数 \( a_{ij}, q_j, r \) 唯一确定。

### 参考文献 (References)


