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Oscillate Criteria of Third Order Semi-Linear Neutral Differential Equations with Delay Argument

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Abstract

We study the oscillatory of third order semi-linear neutral differential equations with delay argument. Using a generalized Riccati substitution and inequation technique, and consulting some results in recent literature, a new oscillation criterion is established and proved, also a number of examples are given to prove their efficiency.

Keywords

Oscillation Criterion, Third Order Semi-Linear Neutral Differential Equations with Delay Argument, Generalized Riccati Substitution

一类三阶中立型半线性时滞微分方程振动准则

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摘　要

本文研究一类三阶中立型半线性时滞微分方程振动性质，利用广义Riccati变换和经典不等式技巧，参考最近论文结果，建立了一个新的振动性准则，并给出证明和例子。

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关键词

振动准则，三阶中立型半线性时滞微分方程，广义Riccati变换

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1. 引言

考虑一类三阶中立型半线性时滞微分方程

$$\left[r(t)\left|z''(t)\right|^{\alpha-1}z''(t)\right]'+q(t)\left|x(\sigma(t))\right|^{\alpha-1}x(\sigma(t))=0,\quad t>t_0 \tag{E}$$

其中 $z(t)=x(t)+p(t)x(\tau(t))$，$\alpha\ge 1$，α 为两个奇数商，$r,\sigma\in C^1([t_0,\infty),(0,\infty))$，$p,q,\tau\in([t_0,\infty),R)$ 任意 $t\ge t_0$，有 $\tau(t)\le t$，$\sigma(t)\le t$，$\sigma'(t)>0$，$\lim\limits_{t\to\infty}\sigma(t)=\lim\limits_{t\to\infty}\tau(t)=\infty$，$0\le p(t)\le 1$，$q(t)>0$。若(E)有无穷多个零点，则它为振动的；否则称它为非振动的。

最近，二阶、三阶函数微分方程的振动性受到很大关注，许多文献给出一系列振动准则如文[1]-[11]。但关于三阶中立函数微分方程的振动性准则较少。我们注意到文[3]和文[4]对方程(E_0)

$$\left[r(t)\left|z''(t)\right|^{\alpha-1}z''(t)\right]'+q(t)\left(x(\tau(t))\right)^{\alpha}=0$$

作了若干个振动性或若振动性准则。本文是研究方程(E)的振动性准则，参考了文[5]中二阶微分方程振动准则及文[1]和文[2]的引理及条件，给出了新的振动准则，并应用新的 Riccati 变换及经典不等式证明了准则。

为了方便证明引用并保留了以下引理：

引理 1 [1]：设 $x(t)$ 是方程(E)的最终正解，则 $z(t)$ 只有以下两种可能：

(I) $z(t)>0$，$z'(t)>0$，$z(t)>0$；

(II) $z(t)>0$，$z'(t)<0$，$z''(t)>0$；

引理 2 [1]：设存在函数 $A(\alpha)>0$，$B(\alpha)>0$ 且 $\alpha>0$，则 $Bu-Au^{\frac{\alpha+1}{\alpha}}\le\dfrac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\cdot\dfrac{B^{\alpha+1}}{A^\alpha}$。

2. 主要结果

定理 2.1：若 $\rho(t)\in C^1([t_0,\infty),(0,\infty))$，且 $\dfrac{\rho'(t)}{\rho(t)}\ge 0$，满足

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\rho(s)q(s)\left(1-p(\sigma(s))\right)^{\alpha}-\frac{\left(\rho'_+(s)\right)^{\alpha+1}r(\sigma(s))}{(\alpha+1)^{\alpha+1}\left(\rho(s)\sigma'(s)\right)^{\alpha}}\right]\mathrm{d}s=\infty \tag{2.1}$$

且

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\rho(s)q(s)\left(1-p(\sigma(s))\right)^{\alpha}-\frac{\left(\rho'_+(s)\right)^{\alpha+1}r(\sigma(s))}{(\alpha+1)^{\alpha+1}\left(\rho(s)\sigma'(s)\right)^{\alpha}}-\frac{\rho(s)}{\pi^{\alpha}(\sigma(s))}\right]\mathrm{d}s=\infty \tag{2.2}$$

其中

$$\pi(t)=\int_t^{\infty} r^{-\frac{1}{\alpha}}(s)\mathrm{d}s<\infty,\quad \rho'_+=\max\{0,\rho'(t)\}$$

则方程(E)是振动的。

证明：设方程有一个非振动解 $x(t)$，且

$$x(t)>0,\ x(\sigma(t))>0,\ x(\tau(t))>0,\ t\ge t_1\ge t_0,$$

因为

$$\left[r(t)\left|z''(t)\right|^{\alpha-1}z''(t)\right]'\le 0$$

则 $\left(r(t)\left|z''(t)\right|^{\alpha-1}z''(t)\right)'$ 是非增函数，且满足引理 1 [1]，故分两种情况讨论：

(I) 假设 $z(t)>0$， $z'(t)>0$， $z''(t)>0$， $z(t)\le t$， $z(t)\ge z(\tau(t))$

即有

$$x(t)=z(t)-p(t)x(\tau(t))\ge z(t)-p(t)z(\tau(t))\ge z(t)-p(t)z(t)=(1-p(t))z(t)$$

即

$$x^{\alpha}(\sigma(t))\ge\left[(1-p(\sigma(t)))z(\sigma(t))\right]^{\alpha}$$

方程(E)去绝对值，则变成

$$\left(r(t)(z''(t))^{\alpha}\right)'+q(t)\left[(1-p(\sigma(t)))z(\sigma(t))\right]^{\alpha}\le 0$$

令 $Q(t)=q(t)(1-p(\sigma(t)))^{\alpha}$，则有

$$\left(r(t)(z''(t))^{\alpha}\right)'\le -Q(t)(z(\sigma(t)))^{\alpha}$$

由广义 Riccati 变换得

$$u(t)=\frac{\rho(t)r(t)(z''(t))^{\alpha}}{(z'(\sigma(t)))^{\alpha}}>0,\ t\ge t_2 \tag{2.3}$$

$$u^{\frac{1}{\alpha}+1}(t)=\frac{(\rho(t)r(t))^{\frac{1}{\alpha}+1}(z''(t))^{\alpha+1}}{(z'(\sigma(t)))^{\alpha+1}} \tag{2.4}$$

由于 $\left(r(t)(z''(t))^{\alpha}\right)'\le 0$，且 $\sigma(t)\le t$，即得

$$\frac{r^{\frac{1}{\alpha}}(t)}{r^{\frac{1}{\alpha}}(\sigma(t))}\le\frac{z''(\sigma(t))}{z''(t)} \tag{2.5}$$

对(2.3)两边对 t 求导，由式(2.4)和式(2.5)得到下式

$$u'(t)\le\frac{\rho'(t)}{\rho(t)}u(t)-\rho(t)Q(t)-\frac{\alpha\sigma'(t)}{(\rho(t)r(\sigma(t)))^{\frac{1}{\alpha}}}u^{1+\frac{1}{\alpha}}(t) \tag{2.6}$$

由经典不等式 $By - Ay^{\frac{\alpha+1}{\alpha}} \le \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \cdot \frac{B^{\alpha+1}}{A^{\alpha}}$，令 $B = \frac{\rho'(t)}{\rho(t)}$， $A = \frac{\alpha\sigma'(t)}{\left(\rho(t)r(\sigma(t))\right)^{\frac{1}{\alpha}}}$

式(2.6)变为

$$u'(t) \le -\rho(t)Q(t) + \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \cdot \left(\frac{\rho'(t)}{\rho(t)}\right)^{\alpha+1} \cdot \frac{\rho(t)r(\sigma(t))}{(\alpha\sigma'(t))^{\alpha}}$$
$$\le -\rho(t)Q(t) + \frac{1}{(\alpha+1)^{\alpha+1}} \cdot \frac{\left(\rho'_+(t)\right)^{\alpha+1} r(\sigma(t))}{\left(\rho(t)\sigma'(t)\right)^{\alpha}}$$

对上式在 $[t_2, t]$ 上积分，即有

$$\int_{t_2}^{t} u'(t)\mathrm{d}t = u(t) - u(t_2) \le \int_{t_2}^{t}\left[-\rho(t)Q(t) + \frac{1}{(\alpha+1)^{\alpha+1}} \cdot \frac{\left(\rho'_+(t)\right)^{\alpha+1} r(\sigma(t))}{\left(\rho(t)\sigma'(t)\right)^{\alpha}}\right]\mathrm{d}t$$
$$= -\int_{t_2}^{t}\left[\rho(t)Q(t) - \frac{1}{(\alpha+1)^{\alpha+1}} \cdot \frac{\left(\rho'_+(t)\right)^{\alpha+1} r(\sigma(t))}{\left(\rho(t)\sigma'(t)\right)^{\alpha}}\right]\mathrm{d}t$$

即

$$\int_{t_2}^{t}\left[\rho(t)Q(t) - \frac{1}{(\alpha+1)^{\alpha+1}} \cdot \frac{\left(\rho'_+(t)\right)^{\alpha+1} r(\sigma(t))}{(\alpha+1)^{\alpha+1}\left(\rho(s)\sigma'(t)\right)^{\alpha}}\right]\mathrm{d}t \le u(t_2) - u(t) < u(t_2)$$

即

$$\limsup_{t\to\infty} \int_{t_0}^{t}\left[\rho(s)Q(s) - \frac{\left(\rho'_+(s)\right)^{\alpha+1} r(\sigma(s))}{(\alpha+1)^{\alpha+1}\left(\rho(s)\sigma'(s)\right)^{\alpha}}\right]\mathrm{d}s$$
$$= \int_{t_2}^{t}\left[\rho(t)Q(t) - \frac{1}{(\alpha+1)^{\alpha+1}} \cdot \frac{\left(\rho'_+(t)\right)^{\alpha+1} r(\sigma(t))}{\left(\rho(t)\sigma'(t)\right)^{\alpha}}\right]\mathrm{d}t < u(t_2) \tag{2.7}$$

显然，式(2.7)与条件(2.1)矛盾

(II) 假设 $z(t) > 0$， $z'(t) < 0$， $z''(t) > 0$；

易知 $\left(r(t)\left(z''(t)\right)^{\alpha}\right)' \le 0$， $1 - p(t) > 0$， $q(t) > 0$， $z'(\sigma(t)) < 0$

则有 $\left(r(t)\left(z''(t)\right)^{\alpha}\right)' \le -q(t)\left(1-p(t)\right)^{\alpha}\left[z'(\sigma(t))\right]^{\alpha}$

由广义 Riccati 变换得

$$w(t) = \rho(t)\frac{r(t)\left(z''(t)\right)^{\alpha}}{\left(z'(\sigma(t))\right)^{\alpha}} < 0 \tag{2.8}$$

$$w^{\frac{1}{\alpha}+1}(t) = \frac{\left(\rho(t)r(t)\right)^{1+\frac{1}{\alpha}}\left(z''(t)\right)^{\alpha+1}}{\left(z'(\sigma(t))\right)^{\alpha+1}} \tag{2.9}$$

对(2.8)两边对 t 求导，由式(2.5)和式(2.9)得到下式

$$w'(t)\le\frac{\rho'(t)}{\rho(t)}w(t)-\rho(t)q(t)(1-p(t))^{\alpha}-\frac{\alpha\sigma'(t)z'(\sigma(t))}{(r(\sigma(t))\rho(t))^{\frac{1}{\alpha}}}w^{1+\frac{1}{\alpha}}(t)$$

由经典不等式 $By-Ay^{\frac{\alpha+1}{\alpha}}\le\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\cdot\frac{B^{\alpha+1}}{A^{\alpha}}$，令 $B=\frac{\rho'(t)}{\rho(t)}$，$A=\frac{\alpha\sigma'(t)z'(\sigma(t))}{(\rho(t)r(\sigma(t)))^{\frac{1}{\alpha}}}$

$$\begin{aligned}w'(t)&\le\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\cdot\left(\frac{\rho'(t)}{\rho(t)}\right)^{\alpha+1}\left(\frac{(\rho(t)r(\sigma(t)))^{\frac{1}{\alpha}}}{\alpha\sigma'(t)}\right)^{\alpha}-\rho(t)Q(t)\\&=\frac{(\rho'(t))^{\alpha+1}r(\sigma(t))}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(t)(\sigma'(t))^{\alpha}}-\rho(t)Q(t)\\&\le\frac{(\rho'_{+}(t))^{\alpha+1}r(\sigma(t))}{(\alpha+1)^{\alpha+1}(\rho(t)\sigma'(t))^{\alpha}}-\rho(t)Q(t)\end{aligned}\tag{2.10}$$

对式(2.10)从 $[t_2,t]$ 上积分有

$$\begin{aligned}w(t)-w(t_2)&\le\int_{t_2}^{t}\left[-\rho(t)Q(t)+\frac{(\rho'_{+}(t))^{\alpha+1}r(\sigma(t))}{(\alpha+1)^{\alpha+1}(\rho(t)\sigma'(t))^{\alpha}}\right]\mathrm{d}t\\&=-\int_{t_2}^{t}\left[\rho(t)Q(t)-\frac{(\rho'_{+}(t))^{\alpha+1}r(\sigma(t))}{(\alpha+1)^{\alpha+1}(\rho(t)\sigma'(t))^{\alpha}}\right]\mathrm{d}t\end{aligned}$$

即

$$\int_{t_2}^{t}\left[\rho(t)Q(t)-\frac{1}{(\alpha+1)^{\alpha+1}}\cdot\frac{(\rho'_{+}(t))^{\alpha+1}r(\sigma(t))}{(\rho(t)\sigma'(t))^{\alpha}}\right]\mathrm{d}t\le w(t_2)-w(t)\tag{2.11}$$

又因为 $\left(r(t)(z''(t))^{\alpha}\right)'\le 0$，则 $\left(-r(t)(z''(t))^{\alpha}\right)'\ge 0$

当 $s>t$ 时，有 $r^{\frac{1}{\alpha}}(s)(-z''(s))\ge r^{\frac{1}{\alpha}}(t)(-z''(t))$

即

$$(-z''(s))\ge r^{\frac{1}{\alpha}}(t)(-z''(t))\left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}}\tag{2.12}$$

对式(2.12)从 $[t,l]$ 上对 s 积分有

$$-(z'(l)-z'(t))\le -r^{\frac{1}{\alpha}}(t)z''(t)\int_{t}^{l}\left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}}\mathrm{d}s$$

即

$$(z'(t)-z'(l))\le -r^{\frac{1}{\alpha}}(t)z''(t)\int_{t}^{l}\left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}}\mathrm{d}s$$

即有

$$0 < -z'(l) \le -r^{\frac{1}{\alpha}}(t) z''(t) \int_t^l \left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}} \mathrm{d}s - z'(t)$$

得到 $z'(t) < -r^{\frac{1}{\alpha}}(t) z''(t) \int_t^l \left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}} \mathrm{d}s$

令 $l \to \infty$，得

$$z'(t) < -r^{\frac{1}{\alpha}}(t) z''(t) \pi(t)$$

那么

$$-\frac{z''(t)}{z'(t)} < \frac{1}{r^{\frac{1}{\alpha}}(t)\pi(t)} \tag{2.13}$$

因为 $\left(r(t)\left(z''(t)\right)^{\alpha}\right)' \le 0$，且 $\sigma(t) \le t$，得

$$r(t)\left(z''(t)\right)^{\alpha} \le r(\sigma(t))\left(z''(\sigma(t))\right)^{\alpha} \tag{2.14}$$

由式(2.13)及(2.14)得

$$-w(t) = -\rho(t)\frac{r(t)\left(z''(t)\right)^{\alpha}}{\left(z'(\sigma(t))\right)^{\alpha}} \le -\rho(t)\frac{r(\sigma(t))\left(z''(\sigma(t))\right)^{\alpha}}{\left(z'(\sigma(t))\right)^{\alpha}} < \frac{\rho(t)}{\pi^{\alpha}(\sigma(t))} \tag{2.15}$$

则由式(2.15)，式(2.11)变成下式

$$\int_{t_2}^{t}\left[\rho(t)Q(t) - \frac{1}{(\alpha+1)^{\alpha+1}} \cdot \frac{\left(\rho'_+(t)\right)^{\alpha+1} r(\sigma(t))}{\left(\rho(t)\sigma'(t)\right)^{\alpha}}\right]\mathrm{d}t - \frac{\rho(t)}{\pi^{\alpha}(\sigma(t))} < w(t_2) \tag{2.16}$$

显然，式(2.16)与条件(2.2)相矛盾，因此，我们说满足定理 2.1，方程(E)是振动的。

3. 例子

考虑三阶微分方程

$$\left(t^6\left[\left(x(t) + \frac{1}{6}x\left(\frac{t}{3}\right)\right)''\right]^3\right)' + kt^3x^3\left(\frac{t}{2}\right) = 0,\quad t \ge t_0 > 0 \tag{3.1}$$

其中，$r(t) = t^6, p(t) = \frac{1}{6}, q(t) = kt^3, \tau(t) = \frac{t}{3}, \sigma(t) = \frac{t}{2}, \rho(t) = 1$，

显然 $\pi(t) = \int_t^{\infty}\left(\frac{1}{t^6}\right)^{\frac{1}{3}} \mathrm{d}s = \frac{1}{t}$，

$$\frac{\rho(t)}{\pi^{\alpha}(\tau(s))} = \frac{t^3}{216}, \rho(t)q(t)\left(1 - p(\sigma(t))\right)^{\alpha} = kt^3\left(1 - \frac{1}{6}\right)^3 = \frac{125}{216}kt^3,$$

$$\int_{t_0}^{t}\frac{125}{216}ks^3ds=\frac{125}{4\times216}k\left(t^4-t_0^4\right),$$

令 $t\to\infty$， $\lim\limits_{t\to\infty}\frac{125}{4\times216}k\left(t^4-t_0^4\right)=\infty$，

$$\int_{t_0}^{t}\left(\frac{125}{216}ks^3-\frac{s^3}{216}\right)\mathrm{d}s=\frac{1}{4\times216}t^4\left(125k-1\right),$$

当 $k>\frac{1}{125}$ 时， $t\to\infty$，

则 $\lim\limits_{t\to\infty}\left(125k-1\right)\frac{1}{4\times216}t^4=\infty$， 式(3.1)满足定理 2.1，故方程(3.1)是振动。

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参考文献 (References)

[1] 惠远先，王俊杰. 三阶中立型半线性时滞微分方程的振动性[J]. 井冈山大学学报, 2017, 38(1): 8-13.

[2] 仉志余，王晓霞，俞元洪. 三阶半线性中立型分布时滞微分方程的振动性[J]. 应用数学学报，2015，38(3): 450-459.

[3] Baculikova, B. and Dzurina, J. (2010) Oscillation of Third-Order Neutral Differential Equations. *Mathematical and Computer Modelling*, **52**, 215-222. https://doi.org/10.1016/j.mcm.2010.02.011

[4] Yuan, G.S. and Fan, W.M. (2006) Note on the Paper of Dzurina and Stavroulakis. *Applied Mathematics and Computation*, **174**, 1634-1641. https://doi.org/10.1016/j.amc.2005.07.008

[5] Agarwal, R.P., Zhang, C.H. and Li, T.X. (2016) Some Remarks on Oscillantion of Second Order Neutral Differential Equations. *Applied Mathematics and Computation*, **274**, 178-181. https://doi.org/10.1016/j.amc.2015.10.089

[6] Ye, L. and Xu, Z. (2009) Oscillation Criteria for Second Order Quasilinear Neutral Delay Differential Equations. *Applied Mathematics and Computation*, **207**, 388-396. https://doi.org/10.1016/j.amc.2008.10.051

[7] Philos, C.G. (1983) On a Kamenev's Integral Criterion for Oscillation of Linear Differential Equations of Second Order. *Utilitas Mathematica*, **24**, 277-289.

[8] Han, Z., Li, T., Sun, S. and Sun, Y. (2010) Remarks on the Paper [Appl. Math. Comput. 207 (2009) 388-396]. *Applied Mathematics and Computation*, **215**, 3998-4007. https://doi.org/10.1016/j.amc.2009.12.006

[9] Dzurina, J. and Kotorova, R. (2009) Properties of the Third Order Trinomial Differential Equations with Delay Argument. *Nonlinear Analysis*, **71**, 1995-2002. https://doi.org/10.1016/j.na.2009.01.070

[10] Aktas, M.F., TIryaki, A. and Zafer, A. (2010) Oscillation Criteria for Third-Order Nonlinear Functional Differential Equations. *Applied Mathematics Letters*, **23**, 756-762. https://doi.org/10.1016/j.aml.2010.03.003

[11] 林全文，俞元洪. 三阶半线性时滞微分方程的振动性和渐进性[J]. 系统科学与数学, 2015, 35(2): 233-244.

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